COLLOQUIUM MATHEMATICUM

VOL. 114 2009 NO. 2

AFFINE STRUCTURES ON JET AND WEIL BUNDLES

BY

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Abstract. Weil algebra morphisms induce natural transformations between Weil bundles. In some well known cases, a natural transformation is endowed with a canonical structure of affine bundle. We show that this structure arises only when the Weil algebra morphism is surjective and its kernel has null square. Moreover, in some cases, this structure of affine bundle passes to jet spaces. We give a characterization of this fact in algebraic terms. This algebraic condition also determines an affine structure on the groups of automorphisms of related Weil algebras.

Introduction. The theory of Weil bundles and jet spaces is developed in order to understand the geometry of PDE systems. C. Ehresmann formalized contact elements of S. Lie, introducing the spaces of jets of sections; simultaneously A. Weil showed in [8] that the theory of S. Lie could be formalized easily by replacing the spaces of contact elements by the more formal spaces of "points proches", known as Weil bundles. The general theory of jet spaces [6] recovers the classical spaces of contact elements $J^l_m M$ of S. Lie applying the ideas and methodology of A. Weil.

In the theory of Weil bundles, morphisms $A \to B$ of Weil algebras induce *natural transformations* [5] between Weil bundles. There are well known cases in which these natural transformations are affine bundles that often appear in differential geometry [5]. In [4] I. Kolář showed that this is the behaviour of $M^{A_l} \to M^{A_{l-1}}$. In this paper *we characterize the natural transformations that are affine bundles.* It is done easily by adopting a different point of view on the tangent space of M^A than in [6]. Our result is as follows: *there is a canonical affine structure for natural transformations $M^A \to M^B$ induced by a surjective morphism $A \to B$ whose kernel has null square.* This is true for $M^{A_l} \to M^{A_k}$ with $2k+1 \geq l > k \geq 0$.

In some cases the natural transformations induce maps between jet spaces. This holds in the cases studied in [4]. We characterize this situation, and moreover, *we determine when an affine structure on the Weil bundle morphism passes to the jet space morphism.* In addition, we prove that in this case there also exists an affine structure in the morphisms be-

2000 *Mathematics Subject Classification*: 58A20, 58A32.

Key words and phrases: jet, Weil bundles, affine structure, natural transformations.

DOI: 10.4064/cm114-2-10

tween the groups of automorphisms of the Weil algebras. This is true for spaces $J^l_m M \to J^k_m$ with $l > k > 0$ and $3k + 1 \geq 2l$.

Notation and conventions. All manifolds and maps are assumed to be infinitely differentiable. All results involving a manifold M assume that it is not empty and all results involving jet spaces $J^A M$ assume that $J^A M$ is also not empty (if it is empty, the algebraic conditions for the existence of affine structure may be satisfied, but no structure exists).

1. Weil bundles. By a *Weil algebra* we mean a finite-dimensional, local, commutative $\mathbb{R}$-algebra with unit. If A is a Weil algebra, let $\mathfrak{m}_A$ be its maximal ideal. If A and B are Weil algebras, by a *morphism* $A \to B$ we mean an $\mathbb{R}$-algebra morphism.

EXAMPLE 1. Let $\mathbb{R}[[\xi_1, \ldots, \xi_m]]$ be the ring of formal series with real coefficients and free variables $\xi_1, \ldots, \xi_m$. Let $\mathfrak{m}$ be the maximal ideal spanned by $\xi_1, \ldots, \xi_m$. Then, for any non-negative integer l, the ring

$$\mathbb{R}^l_m = \mathbb{R}[[\xi_1, \ldots, \xi_m]]/\mathfrak{m}^{l+1}$$

is a Weil algebra.

For every Weil algebra A, there is a non-negative integer l such that $\mathfrak{m}^l_A \neq 0$ but $\mathfrak{m}^{l+1}_A = 0$; we say that l is the *height* of A. The *width* of A is the dimension of the vector space $\mathfrak{m}_A/\mathfrak{m}^2_A$. Thus, $\mathbb{R}^l_m$ is a Weil algebra of height l and width m. If A is of height l and width m there exists a surjective morphism $\mathbb{R}^l_m \to A$ (see [4], [6]).

DEFINITION 1. Let M be a smooth manifold and A a Weil algebra. The set M^A of $\mathbb{R}$-algebra morphisms

$$p^A : C^\infty(M) \to A$$

is called the *space of near-points of type* A of M, or A-*points* of M.

Let $\{a_k\}$ be a basis of A. For each $f \in C^\infty(M)$ we define real-valued functions $\{f_k\}$ on M^A by setting

$$p^A(f) = \sum_k f_k(p^A) a_k.$$

We say that the $\{f_k\}$ are the *real components* of f relative to the basis $\{a_k\}$.

THEOREM 1 ([6]). *The space M^A is endowed with a unique structure of a smooth manifold such that the real components of smooth functions on M are smooth functions on M^A.*

EXAMPLE 2. It is well known that each morphism $C^\infty(M) \to \mathbb{R}$ is a point of M. Since the real components on $M^{\mathbb{R}}$ of smooth functions coincide with the functions themselves we know that $M^{\mathbb{R}} = M$.

EXAMPLE 3. For each Weil algebra $\mathbb{R}^l_m$ let M^l_m be the space of near-points of type $\mathbb{R}^l_m$. Then M^1_1 is the tangent bundle TM. In general M^l_m is the space of germs at the origin of smooth maps $\mathbb{R}^m \to M$ up to order l (see [8]).

A Weil algebra morphism $\phi\colon A \to B$ induces by composition a smooth map $\hat{\phi}\colon M^A \to M^B$ [5, 6] which is called a *natural transformation*,

$$\hat{\phi}(p^A) = \phi \circ p^A.$$

EXAMPLE 4. Notice that a Weil algebra is provided with a unique morphism $A \to \mathbb{R}$. It induces a canonical map $M^A \to M$ which is a fibre bundle. This bundle is the so-called *Weil bundle* of type A over M. Let p^A be in M^A and p its projection to M. Then we will say that p^A is an A-point *near* p. For each smooth function f the value $p^A(f)$ depends only on the germ of f at p.

A smooth map $f\colon M \to N$ of smooth manifolds induces by composition an $\mathbb{R}$-algebra morphism

$$f^*\colon C^\infty(N) \to C^\infty(M), \quad f^*(g) = g \circ f.$$

We can compose this morphism with A-points of M obtaining A-points of N. Thus, Weil algebra morphisms and smooth maps transform near-points by composition, and this implies a functorial behaviour of Weil bundles with respect to those transformations.

We can formalize this situation in the following way. Let $\mathcal{M}$ be the category of smooth manifolds, and $\mathcal{W}$ the category of Weil algebras. In the direct product category $\mathcal{M}\times\mathcal{W}$, objects are pairs (M, A) and morphisms are pairs (f, ϕ). We define $w(M, A) = M^A$. Thus, the natural way of defining the natural image of the morphism (f, ϕ),

$$f\colon M \to N, \qquad \phi\colon A \to B,$$

is

$$w(f, \phi)\colon M^A \to N^B, \quad p^A \mapsto \phi \circ p^A \circ f^*.$$

The following result follows easily:

PROPOSITION 1. *The assignment*

$$w\colon \mathcal{M} \times \mathcal{W} \rightsquigarrow \mathcal{M}, \quad (M, A) \rightsquigarrow M^A,$$

is a covariant functor.

REMARK 1. There are two remarkable cases of induced maps:

- If $X \subset M$ is an embedding then for each Weil algebra A the induced map $X^A \to M^A$ is also an embedding.
- If $A \to B$ is a surjective morphism then for all M the induced natural transformation $M^A \to M^B$ is a fibre bundle.

Example 5. Let A be a Weil algebra of height l. For each $k \leq l$ define $A_k = A/\mathfrak{m}_A^{k+1}$. Then A_k is a Weil algebra of height k and $A_l = A$. For $k \geq r$ we have a natural projection $M^{A_k} \to M^{A_r}$ which is a bundle. In particular, we have the canonical bundles $M_m^k \to M_m^r$.

1.1. *Tangent structure.* Given $p^A \in M^A$, denote by $\mathrm{Der}_{p^A}(C^\infty(M), A)$ the space of *derivations* of the ring $C^\infty(M)$ into the module A, where the structure of $C^\infty(M)$-module on A is induced by the point p^A itself. By derivations we mean $\mathbb{R}$-linear maps $\delta\colon C^\infty(M) \to A$ satisfying Leibniz's formula:

$$\delta(f \cdot g) = p^A(f) \cdot \delta(g) + p^A(g) \cdot \delta(f). \tag{1}$$

If D is a tangent vector to M^A at p^A then it defines a derivation given by

$$\delta(f) = \sum_k (Df_k) a_k,$$

and it is easy to prove that the spaces $\mathrm{Der}_{p^A}(C^\infty(M), A)$ and $T_{p^A}(M^A)$ are identified in this way [6]. Fron now on we will assume this identification. It applies not just to the vector spaces, but it is also compatible with Proposition 1. The following theorem summarizes some results of [6].

Theorem 2 (Muñoz, Rodríguez, Muriel [6]). *Consider a smooth map $f\colon M \to N$, a Weil algebra morphism $\phi\colon A \to B$, and the induced smooth map*

$$w(f,\phi)\colon M^A \to N^B, \qquad p^A \to q^B = \phi \circ p^A \circ f^*.$$

Then the linearized map $w(f,\phi)'\colon T_{p^A}(M^A) \to T_{q^B}(N^B)$ coincides (under the above identification) with the map

$$\mathrm{Der}_{p^A}(C^\infty(M), A) \to \mathrm{Der}_{q^B}(C^\infty(N), B), \qquad \delta \mapsto \phi \circ \delta \circ f^*.$$

1.2. *Affine structure.* In this section we will analyze the structure of the fibre bundle induced by a surjective morphism $A \to B$ which has been introduced in Remark 1. In some specific cases it has been proved that those bundles are endowed with a canonical structure of affine bundles. We will see that this structure has its foundation in the algebraic construction of the *spaces of near-points.* Indeed, it is easy to give an algebraic characterization of this fact. A morphism will induce an affine structure if and only if its kernel ideal has null square.

The key point is to consider both *near-points* and *tangent vectors* to M^A as $\mathbb{R}$-linear maps from $C^\infty(M)$ to a Weil algebra. Thus, they can be added as $\mathbb{R}$-linear maps. Under some assumptions we will obtain a new near-point when we add a derivation to a near-point.

Lemma 1. *Let $p^A \in M^A$ and $D \in T_{p^A}(M^A)$. The sum $p^A + D$ is an A-point of M if and only if* $(\mathrm{Im}(D))^2 = 0$.

Proof. Define $\tau = p^A + D$. Then, for all $f, g \in C^\infty(M)$,
$$\tau(f \cdot g) = \tau(f) \cdot \tau(g) - D(f) \cdot D(g);$$
since τ is an $\mathbb{R}$-linear map, it is an algebraic morphism if and only if for any pair f and g of smooth functions on M we have $D(f) \cdot D(g) = 0$. ∎

LEMMA 2. *Let $p^A, q^A \in M^A$. The difference $\delta = q^A - p^A$ is a derivation and belongs to $T_{p^A}(M^A)$ if and only if $(\mathrm{Im}(\delta))^2 = 0$.*

Proof. For all f and g in $C^\infty(M)$ we have
$$\delta(f \cdot g) = p^A(f) \cdot \delta(g) + p^A(g) \cdot \delta(f) + \delta(f) \cdot \delta(g).$$
Thus, δ satisfies Leibniz's formula if and only if for all f and g in $C^\infty(M)$ the product $\delta(f) \cdot \delta(g)$ vanishes. ∎

LEMMA 3. *Let I be an ideal of a k-algebra A with k a field of characteristic different from 2. The square I^2 of the ideal vanishes if and only if for all $x \in I$ its square x^2 also vanishes.*

Proof. If I^2 vanishes it is clear that $x^2 = 0$ for all $x \in I$. Conversely, assume that the squares of all elements of I vanish. Let x and y be in I. Then
$$0 = (x + y)^2 = x^2 + y^2 + 2xy = 2xy.$$
Hence, $xy = 0$ and I^2 vanishes. ∎

Let $\phi\colon A \to B$ be a surjective morphism of Weil algebras, and let I be its kernel ideal. Consider a smooth manifold M, and the induced fibre bundle $\hat{\phi}\colon M^A \to M^B$. The linearization $\hat{\phi}'$ gives rise to an exact sequence
$$0 \to TV^{\hat{\phi}}_{p^A}(M^A) \to T_{p^A}(M^A) \to T_{p^B}(M^B) \to 0$$
which defines the vertical tangent subbundle $TV^{\hat{\phi}}(M^A) \subset T(M^A)$. Taking into account that tangent vectors are derivations from $C^\infty(M)$ to A, we notice that $D \in T_{p^A}(M^A)$ belongs to $TV^{\hat{\phi}}_{p^A}(M^A)$ if and only if $\mathrm{Im}(D) \subseteq I$. Thus $TV^{\hat{\phi}}_{p^A}(M^A)$ is the space of derivations from $C^\infty(M)$ to I, where the structure of $C^\infty(M)$-module in I is given by the morphism $p^A\colon C^\infty(M) \to A$.

Assume that I^2 vanishes. Let p^A and q^A be in the same fibre of the bundle, that is, $\hat{\phi}(p^A) = \hat{\phi}(q^A) = p^B$. Then, p^A and q^A induce the same structure of $C^\infty(M)$-module in I. Hence, the space of derivations $TV^{\hat{\phi}}_{p^A}(M^A)$ is canonically isomorphic to $TV^{\hat{\phi}}_{q^A}(M^A)$. *We denote this space by* $TV^{\hat{\phi}}_{p^B}(M^B)$.

Using Lemmas 1 and 2 we conclude that for any pair of A-points p^A and q^A in the fibre of p^B as above the difference $p^A - q^A$ is a derivation which belongs to $TV^{\hat{\phi}}_{p^B}(M^B)$. Moreover, for any derivation $D \in TV^{\hat{\phi}}_{p^B}(M^B)$, the sum $p^A + D$ is a near-point of type A in the fiber of p^B. Thus, the natural addition law of linear maps,

$$\hat{\phi}^{-1}(p^B) \times TV^{\hat{\phi}}_{p^B}(M^B) \to \hat{\phi}^{-1}(p^B), \quad (p^A, D) \mapsto p^A + D,$$

induces an affine structure on the fibre $\hat{\phi}^{-1}(p^B)$ associated with the vector space $TV^{\hat{\phi}}_{p^B}(M^B)$ of derivations from $C^\infty(M)$ to I.

We define the vector bundle

$$TV^{\hat{\phi}}(M^B) \to M^B,$$

whose fibre over a B-point p^B is the space $TV^{\hat{\phi}}_{p^B}(M^B)$. Hence, $TV^{\hat{\phi}}(M^B)$ is the vector bundle associated with the affine bundle $\hat{\phi}\colon M^A \to M^B$,

$$M^A \times_{M^B} TV^{\hat{\phi}}(M^B) \to M^A, \quad (p^A, D) \mapsto p^A + D.$$

On the other hand, if I^2 does not vanish, by applying Lemma 3 we find a derivation $D\colon C^\infty(M) \to I$ such that $(\mathrm{Im}(D))^2$ does not vanish. Hence, $p^A + D$ does not belong to M^A. We have proved the following:

Theorem 3. *Let $\phi\colon A \to B$ be a surjective Weil algebra morphism, and let I be its kernel ideal. For any manifold M, the natural addition law of linear maps induces a structure of affine bundle in the fibre bundle $\hat{\phi}\colon M^A \to M^B$ if and only if $I^2 = 0$.*

By elementary computations on the algebras A_k and $\mathbb{R}^l_m$ we deduce the following corollaries to Theorem 3.

Corollary 1. *Let A be a Weil algebra of height l. The natural projection $M^A \to M^{A_k}$ is endowed with a canonical structure of affine bundle if and only if $2k + 1 \geq l$.*

Corollary 2. *The natural projection of spaces of frames, $M^l_m \to M^k_m$, is endowed with a canonical structure of affine bundle if and only if $2k + 1 \geq l$.*

2. Jet spaces

Definition 2. A *jet* of M is an ideal $\mathfrak{p} \subset C^\infty(M)$ such that the quotient algebra $A_{\mathfrak{p}} = C^\infty(M)/\mathfrak{p}$ is a Weil algebra. A jet $\mathfrak{p}$ is said to be *of type A*, or an *A-jet*, if $A_{\mathfrak{p}}$ is isomorphic to A. The set $J^A M$ of A-jets of M is called the *A-jet space* of M.

An A-point p^A of M is said to be *regular* if it is a surjective morphism. The set of regular A-points of M is denoted by $\check{M}^A$. It is a dense open subset of M^A. It is obvious that an A-point is regular if and only if its kernel is an A-jet. Thus, we have a surjective map

$$\ker\colon \check{M}^A \to J^A M. \tag{2}$$

Let $\mathrm{Aut}(A)$ be the group of *automorphisms* of A. It is a linear algebraic group, as can be easily seen by representing it as a subgroup of $\mathrm{GL}(\mathfrak{m}_A)$ (see [3]). The group $\mathrm{Aut}(A)$ acts on $\check{M}^A$ by composition. Two A-points related by an automorphism have the same kernel ideal. Conversely, two A-points with the same kernel ideal are related by an automorphism. In this way J^AM is identified with the space of orbits $\check{M}^A/\mathrm{Aut}(A)$, and its manifold structure is determined in this way (see [1]).

EXAMPLE 6. The group G^l_m of automorphisms of $\mathbb{R}^l_m$ is called the *lth prolongation of the linear group of rank m* (see [7]), also called the *jet group*. In particular G^1_m is a linear group of rank m. The group G^l_m is the group of Taylor series of transformations of $\mathbb{R}^l$ around a fixed point up to order l.

THEOREM 4 (Alonso-Blanco [1]). *There is a unique structure of smooth manifold on J^AM such that the map* ker (*appearing in* (2)) *is a principal bundle with structure group* $\mathrm{Aut}(A)$.

EXAMPLE 7. Let J^l_mM denote the space of jets of type $\mathbb{R}^l_m$ of M. Thus, J^l_mM is the space of germs of m-submanifolds of M up to order l.

The space J^AM is a bundle over M. We will say that $\mathfrak{p}$ is a *jet over the point p* if $\mathfrak{p} \subset \mathfrak{m}_p$, where $\mathfrak{m}_p$ is the ideal of smooth functions vanishing at p. If p^A is an A-point *near p* then $\ker(p^A)$ is a jet *over p*.

2.1. *Functorial behaviour.* In contrast with Weil bundles, jet spaces do not show a functorial behaviour. A smooth map $f\colon M \to N$ induces a smooth map on jet spaces, but in the general case it is defined only on an open dense subset of J^AM, which depends on f. There is no natural object associated to a Weil algebra morphism $A \to B$. There is a natural, highly interesting, object associated to a pair (A, B) of Weil algebras: the *Lie correspondence*. This is a submanifold $\Lambda_{A,B}M$ of the fibred product $J^AM \times_M J^BM$,

$$\Lambda_{A,B}M = \{(\mathfrak{p}, \overline{\mathfrak{p}}) \in J^AM \times_M J^BM \colon \mathfrak{p} \subset \overline{\mathfrak{p}}\}.$$

The Lie correspondence is empty if and only if there does not exist any surjective morphism from A to B. There is a special case to be analyzed in which it is the graph of a bundle $J^AM \to J^BM$.

Let I be an ideal of A. Then, for each automorphism σ of A, the space $\sigma(I)$ is another ideal of A; the group $\mathrm{Aut}(A)$ acts on the set of ideals of A. We say that I is an *invariant ideal* of A if $I = \sigma(I)$ for all $\sigma \in \mathrm{Aut}(A)$. Each positive power of the maximal ideal $\mathfrak{m}^k_A$ is an invariant ideal, and any other ideals obtained from these by general processes of division and derivation are also invariant; some examples are shown in [2]. Let $I \subset A$ be an invariant ideal and $\phi\colon A \to A/I = B$ the canonical projection. Let p^A be an A-point and $\mathfrak{p}$ be its kernel. It is obvious that the kernel ideal $\overline{\mathfrak{p}}$ of the composition

$\phi \circ p^A$ is the unique B-jet containing $\mathfrak{p}$. Let $\check{\phi}$ be the restriction of $\hat{\phi}$ to the space of regular points $\check{M}^A$. We have a commutative diagram

$$\begin{tikzcd} \check{M}^A \arrow[r, "\check{\phi}"] \arrow[d] & \check{M}^B \arrow[d] & p^A \arrow[r] \arrow[d] & \check{\phi}(p^A) = p^B \arrow[d] \\ J^A M \arrow[r, "\phi^J"] & J^B M & \mathfrak{p} \arrow[r] & \overline{\mathfrak{p}} \end{tikzcd} \tag{3}$$

The Lie correspondence is precisely the set

$$\Lambda_{A,B}M = \{(\mathfrak{p}, \phi^J(\mathfrak{p}))\colon \mathfrak{p} \in J^A M\}.$$

Summarizing, the following result holds:

Theorem 5. *If $I \subset A$ is an invariant ideal and B is the quotient algebra A/I then there is a canonical bundle structure $J^A M \to J^B M$.*

2.2. *Tangent structure.* In order to study the linearization of ϕ^J in diagram (3) we need some characterization of the tangent space to $J^A M$ at a jet $\mathfrak{p}$.

Theorem 6 ([1, 6]). *The space $T_{\mathfrak{p}}(J^A M)$ is canonically realized as a quotient of the space of derivations $C^\infty(M) \to A_{\mathfrak{p}}$. A derivation δ defines the null vector if and only if $\delta(\mathfrak{p}) = 0$. Thus,*

$$T_{\mathfrak{p}}(J^A M) \simeq \mathrm{Der}(C^\infty(M), A_{\mathfrak{p}})/\mathrm{Der}(A_{\mathfrak{p}}, A_{\mathfrak{p}}).$$

For a better understanding let us sketch the proof. Recall that the Lie algebra of $\mathrm{Aut}(A)$ is the space of derivations $\mathrm{Der}(A, A)$, as can be shown in a matrix representation of the group (see [6, 3]). Taking $p^A \in \check{M}^A$ such that $\ker(p^A) = \mathfrak{p}$, the representation of $\mathrm{Der}(A, A)$ as fundamental vector fields of the action of $\mathrm{Aut}(A)$ on $\check{M}^A$ gives rise to an exact sequence

$$0 \to \mathrm{Der}(A, A) \xrightarrow{\text{fun. vec. fields}} T_{p^A}(M^A) \xrightarrow{\ker'} T_{\mathfrak{p}} J^A(M) \to 0.$$

Note that $T_{p^A}(M^A)$ is the space of derivations $\mathrm{Der}_{p^A}(C^\infty(M), A)$ and that p^A induces an isomorphism of C^∞-algebras between A and $A_{\mathfrak{p}}$. This yields the isomorphism of the theorem. This isomorphism does not depend on the A-point p^A representing the A-jet $\mathfrak{p}$. This can be seen by means of the principal bundle structure stated in Theorem 4.

3. Affine structure on jet spaces

3.1. *Space of regular points.* Let I be an ideal of the Weil algebra A, and $\phi\colon A \to B$ the canonical projection onto the quotient algebra $B = A/I$.

Lemma 4 ([6]). *A finite set $\{a_1, \ldots, a_m\} \subset \mathfrak{m}_A$ is a system of generators of A if and only if the set $\{\overline{a}_1, \ldots, \overline{a}_m\}$ of their classes in $\mathfrak{m}_A/\mathfrak{m}_A^2$ is a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$.*

LEMMA 5. *If $I \not\subseteq \mathfrak{m}_A^2$ then there exists a non-trivial subalgebra $S \subset A$ such that $S/(S \cap I) \simeq B$.*

Proof. If $I \not\subseteq \mathfrak{m}_A^2$ then the canonical projection $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ has non-trivial kernel. There exists a finite set $\{a_1, \ldots, a_r\}$ such that $\{\overline{\phi(a_1)}, \ldots, \overline{\phi(a_r)}\}$ is a basis of $\mathfrak{m}_B/\mathfrak{m}_B^2$ but $\{\overline{a}_1, \ldots, \overline{a}_r\}$ is not a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$. Then $S = \mathbb{R}[a_1, \ldots, a_r]$ is a proper subalgebra and satisfies $S/(S \cap I) = B$. ∎

Note that each subalgebra of a Weil algebra A is a Weil algebra. For each subset $X \subset \mathfrak{m}_A$, $\mathbb{R}[X]$ is a Weil algebra and its maximal ideal is spanned by X.

LEMMA 6. *The following conditions are equivalent*:

(1) $I \subseteq \mathfrak{m}_A^2$.
(2) $\hat{\phi}^{-1}(\check{M}^B) \subseteq \check{M}^A$.

Proof. Assume $I \subseteq \mathfrak{m}_A^2$, and consider $p^A \in M^A$ such that $\hat{\phi}(p^A)$ is a regular B-point. There are differentiable functions $f_1, \ldots, f_m$ in M such that $\{\phi(p^A(f_1)), \ldots, \phi(p^A(f_m))\}$ is a system of generators of B. Then their classes modulo $\mathfrak{m}_B^2$ form a basis $\{\overline{\phi(p^A(f_1))}, \ldots, \overline{\phi(p^A(f_m))}\}$ of $\mathfrak{m}_B/\mathfrak{m}_B^2$. Since I is contained in $\mathfrak{m}_A^2$ and $\mathfrak{m}_B = \mathfrak{m}_A/I$ we see that $\mathfrak{m}_B/\mathfrak{m}_B^2 \simeq \mathfrak{m}_A/\mathfrak{m}_A^2$. Then $\{\overline{p^A(f_1)}, \ldots, \overline{p^A(f_m)}\}$ is a basis of $\mathfrak{m}_A/\mathfrak{m}_A^2$ and $\{p^A(f_1), \ldots, p^A(f_m)\}$ is a system of generators of A. Thus, A is regular.

Conversely, assume $I \not\subseteq \mathfrak{m}_A^2$. Consider a subalgebra $S \subset A$ as in Lemma 5. Then $M^S \to M^B$ is a bundle. Let $p^B \in \check{M}^B$ be a regular B-point, and p^S any preimage of p^B. Hence, p^S is an S-point, and thus a non-regular A-point, but $\hat{\phi}(p^S) = p^B$. ∎

The *annihilator ideal* of I is defined by

$$\operatorname{Ann}(I) = \{a \in A \colon \forall b \in I \ ab = 0\}.$$

Notice that $I \subseteq \operatorname{Ann}(I)$ if and only if $I^2 = 0$.

THEOREM 7. *The bundle $\check{\phi} \colon \check{M}^A \to \check{M}^B$ is endowed with a canonical structure of affine bundle* (*given by addition of morphisms and derivations*) *if and only if $I \subseteq \mathfrak{m}_A^2 \cap \operatorname{Ann}(I)$.*

Proof. Suppose that $I \subseteq \mathfrak{m}_A^2 \cap \operatorname{Ann}(I)$. Then addition of A-points and derivations induces an affine structure on $\hat{\phi}$. Let $p^B \in \check{M}^B$ be a regular B-point. In view of Lemma 6 the fibre $\hat{\phi}^{-1}(p^B)$ consists of regular points. Thus, $\check{\phi}^{-1}(p^B) = \hat{\phi}^{-1}(p^B)$ so that the bundle $\check{M}^A \to \check{M}^B$ is the restriction of $M^A \to M^B$ to the open submanifold $\check{M}^B$, which is an affine bundle.

On the other hand, assume that $I \not\subseteq \mathfrak{m}_A^2 \cap \operatorname{Ann}(I)$. If $I \not\subseteq \operatorname{Ann}(I)$, then the sum of an A-point and a derivation is not in general an A-point and there is no affine structure. Finally, assume that $I \subseteq \operatorname{Ann}(I)$ but $I \not\subseteq \mathfrak{m}_A^2$. Then there is an affine structure on $\hat{\phi}$. However, by Lemma 6 there is a non-regular A-point $p^A M$ such that its projection p^B is regular. Consider $q^A \in \check{\phi}^{-1}(p^B)$, and $D = p^A - q^A \in TV^{\check{\phi}}_{q^A}\check{M}^A$. Thus $q^A + D \in \check{M}^A$, and there is no affine structure. ∎

Corollary 3. *Let A be of height l. Then for each $l > k > 0$ the natural projection $\check{M}^A \to \check{M}^{A_k}$ is an affine bundle if and only if $2k+1 \geq l$.*

Corollary 4. *For any $l > k > 0$, the natural projection $\check{M}^l_m \to \check{M}^k_m$ is an affine bundle if and only if $2k+1 \geq l$.*

3.2. *Affine structure on the group of automorphisms.* Let $I \subset A$ be an invariant ideal of the Weil algebra A, and $\phi\colon A \to B = A/I$ the canonical projection. Each automorphism $\sigma \in \mathrm{Aut}(A)$ satisfies $\sigma(I) = I$, so it induces an automorphism $\phi_*(\sigma) \in \mathrm{Aut}(B)$.

Definition 3. The *affine sequence* associated to I is the following sequence of algebraic groups:

$$K(I) \to \mathrm{Aut}(A) \xrightarrow{\phi_*} \mathrm{Aut}(B),$$

where

$$K(I) = \{\sigma \in \mathrm{Aut}(A) : \sigma(a) - a \in I,\ \sigma(b) = b,\ \forall a \in A\ \forall b \in I\}$$

is the subgroup of automorphisms of A inducing the identity both in B and I.

We will say that the affine sequence is *exact on the left* if $K(I) = \ker \phi_*$. Analogously, we will say that it is *exact on the right* if ϕ_* is surjective. Note that if a sequence is exact both on the right and on the left then it is an exact sequence.

Notice that if $I \subseteq \mathrm{Ann}(I)$ then the A-module I is also a B-module. By composition we have a canonical injection $\mathrm{Der}(B, I) \subseteq \mathrm{Der}(A, I)$ identifiying derivations from B to I with derivations from A to I which vanish on $I \subset A$.

Proposition 2. *Asume $I \subseteq \mathrm{Ann}(I)$. The affine sequence associated to I is exact on the left if and only if* $\mathrm{Der}(B, I) = \mathrm{Der}(A, I)$.

Proof. Assuming that the affine sequence is exact on the left, consider the sequence of Lie algebras induced by the sequence of algebraic groups associated to I. The Lie algebra of $K(I)$ is, by the definition of $K(I)$, the space of derivations from A to I which vanish on I. Thus, it is identified with $\mathrm{Der}(B, I)$. On the other hand, the kernel of the Lie algebra morphism induced by ϕ_* is $\mathrm{Der}(A, I)$. If the affine sequence is exact on the left, then the Lie algebra of $K(I)$ coincides with this last space, and $\mathrm{Der}(B, I) = \mathrm{Der}(B, A)$.

Conversely, assume that $\mathrm{Der}(A, I) = \mathrm{Der}(B, I)$, i.e. all derivations from A to I vanish on I. Let σ be an automorphism of A. The difference $\mathrm{Id}_A - \sigma$

is a derivation from A to I. It vanishes on I, and thus for any $a \in I$ we have $\sigma(a) = a$, and then σ induces the identity in I, i.e. $\sigma \in K(I)$. ∎

THEOREM 8. *If* $I \subseteq \mathrm{Ann}(A) \cap \mathfrak{m}_A^2$ *and the affine sequence is exact*, *then* ϕ_* *is endowed with a natural structure of affine bundle associated with the space* $\mathrm{Der}(A, I)$ *with the following addition law*:

$$\sigma \oplus D = \sigma + \sigma \circ D.$$

Proof. Let D be a derivation from A to I. Then $\mathrm{Id}_A + D$ is an automorphism of A. Conversely, let σ be an automorphism of A such that $\phi_*(\sigma) = \mathrm{Id}_B$. Then $\sigma - \mathrm{Id}_A$ is a derivation with values in I. We have

$$\mathrm{Der}(A, I) = \ker(\phi_*).$$

By definition of the addition law we have $\sigma \oplus D = \sigma(\mathrm{Id} + D)$, so that $\sigma \oplus \mathrm{Der}(A, I) = \sigma \circ \ker(\phi_*)$. Finally, let us see that the addition law of the bundle is compatible with the vector space structure of $\mathrm{Der}(A, I)$, i.e.

$$(\sigma \oplus D) \oplus D' = \sigma \oplus (D + D').$$

Indeed, from Proposition 2 we have

$$\sigma \oplus D \oplus D' = \sigma + \sigma \circ D(\sigma + \sigma \circ D) \circ D' = \sigma \oplus (D + D') + \sigma \circ D \circ D',$$

and because each derivation vanishes on I we see that $D \circ D'$ vanishes. ∎

LEMMA 7. *If* $I \subseteq (\mathrm{Ann}(I))^2$ *then the affine sequence associated to* I *is exact on the left.*

Proof. Consider a derivation $D\colon A \to I$, and a in I; thus a is also in $(\mathrm{Ann}(I))^2$ and so we can write $a = \sum b_k c_k$ for suitable b_k and c_k in $\mathrm{Ann}(I)$. We have

$$D(a) = \sum_k b_k D(c_k) + c_k D(b_k) = 0.$$

Thus, D annihilates I. We conclude that $\mathrm{Der}(A, I) = \mathrm{Der}(B, I)$. Our assertion now follows directly from Proposition 2. ∎

COROLLARY 5. *If the natural numbers* $l > r > 0$ *satisfy* $3r + 1 \geq 2l$ *then the natural projection* $G_m^l \to G_m^r$ *is an affine bundle.*

Proof. In general $G_m^l \to G_m^k$ is a surjective morphism. We apply Lemma 7 to the case $A = \mathbb{R}_m^l$, $I = \mathfrak{m}_A^{k+1}$. Then $\mathrm{Ann}(I) = \mathfrak{m}_A^{l-k}$, and we have $\mathfrak{m}_A^{k+1} \subseteq (\mathrm{Ann}(\mathfrak{m}_A^{k+1}))^2$ if and only if $k + 1 \geq 2(l - k)$. ∎

3.3. *Affine structure on jet bundles.* Let $I \subset A$ be an invariant ideal with $I \subseteq \mathrm{Ann}(I) \cap \mathfrak{m}_A^2$ and denote by B the quotient algebra A/I as above. For each $\mathfrak{p} \in J^A M$ denote by $\pi_{\mathfrak{p}}\colon C^\infty(M) \to A_{\mathfrak{p}}$ the canonical projection, and set $\overline{\mathfrak{p}} = \phi^J(\mathfrak{p}) \in J^B M$. Then $A_{\mathfrak{p}} \simeq \mathfrak{p}$ and $\overline{\mathfrak{p}}/\mathfrak{p} \simeq I$. For each $D \in \mathrm{Der}(C^\infty(M), \overline{\mathfrak{p}}/\mathfrak{p})$ define

$$\mathfrak{p} + D = \ker(\pi_{\mathfrak{p}} + D). \tag{4}$$

Because $I \subseteq \mathrm{Ann}(I)$ we see that $\pi_{\mathfrak{p}} + D$ is an $A_{\mathfrak{p}}$-point. It is regular because $I \subseteq \mathfrak{m}_A^2$. Hence, $\mathfrak{p} + D$ is an A-jet. We also have $\phi^J(\mathfrak{p} + D) = \overline{\mathfrak{p}}$, because D takes values in $\overline{\mathfrak{p}}/\mathfrak{p}$.

LEMMA 8. *Each derivation $D\colon C^\infty(M) \to \overline{\mathfrak{p}}/\mathfrak{p}$ which vanishes on $\mathfrak{p}$ also vanishes on $\overline{\mathfrak{p}}$ if and only if the affine sequence associated to I is exact on the left.*

Proof. A derivation $C^\infty(M) \to \overline{\mathfrak{p}}/\mathfrak{p}$ which annihilates $\mathfrak{p}$ factorizes through a derivation $A_{\mathfrak{p}} \to \overline{\mathfrak{p}}/\mathfrak{p}$. Then the claim is equivalent to Lemma 2. ∎

THEOREM 9. *The addition law* (4) *defines an affine structure on the bundle $\phi^J\colon J^AM \to J^BM$ for any smooth manifold M if and only if the affine squence associated to I is exact.*

Proof. A derivation $D\colon C^\infty(M) \to \overline{\mathfrak{p}}/\mathfrak{p}$ defines a tangent vector $[D] \in T_{\mathfrak{p}}(J^AM)$ as shown in Theorem 6. Moreover, $[D] \in TV_{\mathfrak{p}}^{\phi^J} J^AM$, because D takes values in $\overline{\mathfrak{p}}/\mathfrak{p}$. Let us prove that the following conditions, which are equivalent to the assertion of the theorem, hold if and only if the affine sequence associated to I is exact.

(i) If two derivations D and D' from $C^\infty(M)$ to $A_{\mathfrak{p}}$ define the same tangent vector at $\mathfrak{p}$ then $\mathfrak{p} + D = \mathfrak{p} + D'$.
(ii) The natural projection $\mathrm{Der}(C^\infty(M), \overline{\mathfrak{p}}/\mathfrak{p}) \to TV_{\mathfrak{p}}^{\phi^J}(J^AM)$ is surjective.
(iii) For each $\mathfrak{q} \subset \overline{\mathfrak{p}}$ there is a unique $[D] \in TV_{\mathfrak{p}}^{\phi^J}(J^AM)$ such that $\mathfrak{p} + [D] = \mathfrak{q}$.
(iv) For each A-jet $\mathfrak{q}$ contained in the B-jet $\overline{\mathfrak{p}}$ there is a canonical isomorphism $TV_{\mathfrak{p}}^{\phi^J}(J^AM) \simeq TV_{\mathfrak{q}}^{\phi^J}(J^AM)$.

- *Condition* (i) *holds if and only if the affine sequence is exact on the left.*

Let D and D' define the same tangent vector $[D] \in TV_{\mathfrak{p}}^{\phi^J}(J^AM)$. Then the difference $\delta = D - D'$ vanishes on $\mathfrak{p}$. By Lemma 8, each derivation vanishing on $\mathfrak{p}$ also vanishes on $\overline{\mathfrak{p}}$ if and only if the affine sequence associated to I is exact on the left. In the case of exact affine sequence we have

$$\ker(\pi_{\mathfrak{p}} + D) = \ker(\pi_{\mathfrak{p}} + D').$$

- *If the affine sequence is exact on the right then condition* (ii) *holds.*

This is an application of the classical *Snake Lemma*. We have a natural diagram of exact columns and arrows:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Der}(A_{\mathfrak{p}}, \bar{\mathfrak{p}}/\mathfrak{p}) \arrow[r] \arrow[d] & \mathrm{Der}(A_{\mathfrak{p}}, A_{\mathfrak{p}}) \arrow[r, "\psi"] \arrow[d] & \mathrm{Der}(A_{\bar{\mathfrak{p}}}, A_{\bar{\mathfrak{p}}}) \arrow[d] & \\
0 \arrow[r] & \mathrm{Der}(C^\infty(M), \bar{\mathfrak{p}}/\mathfrak{p}) \arrow[r] \arrow[d, "\overline{\psi}"] & \mathrm{Der}(C^\infty(M), A_{\mathfrak{p}}) \arrow[r] \arrow[d] & \mathrm{Der}(C^\infty(M), A_{\bar{\mathfrak{p}}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & TV_{\mathfrak{p}}^{\phi^J}(J^A M) \arrow[r] & T_{\mathfrak{p}}(J^A M) \arrow[r] \arrow[d] & T_{\bar{\mathfrak{p}}}(J^B M) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

According to the Snake Lemma, if $\operatorname{coker}(\psi)$ vanishes then we have an exact sequence

$$\cdots \to 0 \to \operatorname{coker}(\overline{\psi}) \to 0 \to \cdots$$

and vice versa. Hence, $\operatorname{coker}(\psi)$ vanishes if and only if $\operatorname{coker}(\overline{\psi})$ vanishes. Note that the natural mapping ψ is the linearization of the algebraic group morphism $\mathrm{Aut}(A_{\mathfrak{p}}) \to \mathrm{Aut}(A_{\overline{\mathfrak{p}}})$. Since $A_{\mathfrak{p}} \simeq A$ and $A_{\overline{\mathfrak{p}}} \simeq B$ we conclude that if the affine sequence associated to I is exact on the right then (ii) holds.

- *Condition* (iii) *holds if and only if the affine sequence associated to I is exact on the right.*

Let us consider any other A-jet $\mathfrak{q} \subset \overline{\mathfrak{p}}$ and an isomorphism $\tau\colon A_{\mathfrak{q}} \to A_{\mathfrak{p}}$. Thus, we have a diagram (not commutative):

$$
\begin{tikzcd}
C^\infty(M) \arrow[r, "\pi_{\mathfrak{q}}"] \arrow[dr, "\pi_{\mathfrak{q}}"'] & A_{\mathfrak{p}} \arrow[dr, "\overline{\pi}_{\mathfrak{p}}"] & \\
 & A_{\mathfrak{q}} \arrow[u, "\tau"] \arrow[r, "\overline{\pi}_{\mathfrak{q}}"'] & A_{\overline{\mathfrak{p}}}
\end{tikzcd}
$$

Let us prove the following assertion: for each $\mathfrak{q}$ as above we can find an isomorphism τ such that $\overline{\pi}_{\mathfrak{p}} \circ \tau = \overline{\pi}_{\mathfrak{q}}$ if and only if the affine sequence associated to I is exact on the right. First, assume that the affine sequence is exact on the right. Let p^A and q^A be A-jets representing $\mathfrak{p}$ and $\mathfrak{q}$ respectively. The B-points $\check{\phi}(p^A)$ and $\check{\phi}(q^A)$ represent the same B-jet $\overline{\mathfrak{p}}$. Hence, $\check{\phi}(p^A)$ and $\check{\phi}(q^A)$ are related by an automorphism τ_1 of B. If the affine sequence associated to I is exact on the right, then $\mathrm{Aut}(B)$ is a quotient of $\mathrm{Aut}(A)$. Hence, τ_1 lifts to an automorphism τ_2 of A. This automorphism induces the isomorphism τ when we substitute A for $A_{\mathfrak{p}}$ and $A_{\mathfrak{q}}$. Conversely, if the affine sequence is not exact on the right, we can choose $\mathfrak{q}$ and A-points p^A and q^A

such that $\check{\phi}(p^A)$ and $\check{\phi}(p^B)$ are related by an automorphism which cannot be lifted to A. In that case, we cannot find such an isomorphism τ.

Now, suppose that the affine sequence is exact on the right and let τ be as above. Then $\pi_{\mathfrak{p}}$ and $\tau \circ \pi_{\mathfrak{p}}$ are regular $A_{\mathfrak{p}}$-points that project onto the same $A_{\overline{\mathfrak{p}}}$-point. Then $D = \pi_{\mathfrak{p}} - \tau \circ \pi_{\tau_q}$ is a derivation of $C^\infty(M)$ and it takes values in $\overline{\mathfrak{p}}/\mathfrak{p}$. It defines a vertical vector $[D] \in TV_{\mathfrak{p}}^{\phi^J}(J^A M)$ and it follows that

$$\mathfrak{p} + [D] = \mathfrak{q}.$$

- *If the affine sequence is exact then condition* (iv) *holds.*

If the affine sequence is exact, we can find τ and $\overline{\tau}\colon A_{\mathfrak{q}} \to A_{\mathfrak{p}}$ as above. Then $\sigma = \tau \circ \overline{\tau}$ is an automorphism of $A_{\mathfrak{p}}$ which induces the identity on $A_{\overline{\mathfrak{p}}}$. Since the affine sequence is exact on the right, σ induces the identity map on $\overline{\mathfrak{p}}/\mathfrak{p}$. It follows that the restriction of τ to $\overline{\mathfrak{p}}/\mathfrak{q}$ is canonical and does not depend on τ. This canonical identification $\tau\colon \overline{\mathfrak{p}}/\mathfrak{q} \to \overline{\mathfrak{p}}/\mathfrak{p}$ induces canonical isomorphisms

$$\begin{tikzcd}
\mathrm{Der}(A_{\overline{\mathfrak{p}}}, \overline{\mathfrak{p}}/\mathfrak{q}) \arrow[r] \arrow[d] & \mathrm{Der}(C^\infty(M), \overline{\mathfrak{p}}/\mathfrak{q}) \arrow[r] \arrow[d] & TV_{\mathfrak{q}}^{\phi^J}(J^A M) \arrow[d, "\tau^*"] \\
\mathrm{Der}(A_{\overline{\mathfrak{p}}}, \overline{\mathfrak{p}}/\mathfrak{p}) \arrow[r] & \mathrm{Der}(C^\infty(M), \overline{\mathfrak{p}}/\mathfrak{p}) \arrow[r] & TV_{\mathfrak{p}}^{\phi^J}(J^A M)
\end{tikzcd}$$

Thus, condition (iv) is satisfied.

If the affine sequence is exact then the vector space $TV_{\mathfrak{p}}^{\phi^J}(J^A M)$ depends only on the base B-jet $\overline{p}$. Those spaces define a vector bundle $TV^{\phi^J}(J^B M) \to J^B M$, and the composition law

$$J^A M \times_{J^B M} TV^{\phi^J}(J^B M) \to J^A M, \quad (\mathfrak{p} + [D]) \mapsto \mathfrak{p} + [D],$$

is an affine structure on the bundle ϕ^J. ∎

Corollary 6. *Let A_l be of height l, and $l > k > 0$. The natural projection $J^{A_l} M \to J^{A_k} M$ is endowed with a canonical structure of affine bundle if and only if $3k + 1 \geq 2l$ and $\mathrm{Aut}(A_l) \to \mathrm{Aut}(A_k)$ is surjective.*

Corollary 7. *The natural projection $J^l_m M \to J^r_m M$ for $l > r > 0$ is endowed with a canonical structure of affine bundle if and only if $3r + 1 \geq 2l$.*

Remark 2. Those results extend the well known affine structure of the spaces of jets of sections. First, they show that this structure arises not only for the projection $J^l_m \to J^{l-1}_m$, but also for projections between jet spaces of orders satisfying an unexpected condition, different from that of *duplicating orders*. Second, this affine structure is inherent to the spaces $J^l_m M$ as spaces of ideals, it does not depend on their realization as spaces of sections of fibre bundles.

Acknowledgements. I thank Prof. J. Muñoz and R. J. Alonso-Blanco for their help with this work. I am grateful to J. Rodríguez and S. Jiménez for their suggestions and helpful discussions on the topic of Weil bundles. I would also like to thank Professor C. Simó who hosted me at Barcelona University, and J. J. Morales-Ruiz, for their support.

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Received 26 October 2006;
revised 11 August 2008 (4810)