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AN EXPONENTIAL DIOPHANTINE EQUATION RELATED TO THE SUM OF POWERS OF TWO CONSECUTIVE k-GENERALIZED FIBONACCI NUMBERS

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Abstract. A generalization of the well-known Fibonacci sequence $\{F_n\}_{n\ge 0}$ given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 0$ is the k-generalized Fibonacci sequence $\{F_n^{(k)}\}_{n\ge -(k-2)}$ whose first k terms are $0, \dots, 0, 1$ and each term afterwards is the sum of the preceding k terms. For the Fibonacci sequence the formula $F_n^2 + F_{n+1}^2 = F_{2n+1}$ holds for all $n \ge 0$. In this paper, we show that there is no integer $x \ge 2$ such that the sum of the xth powers of two consecutive k-generalized Fibonacci numbers is again a k-generalized Fibonacci number. This generalizes a recent result of Chaves and Marques.

1. Introduction. Let $\{F_n\}_{n\ge 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 0$. The formula

$$F_n^2 + F_{n+1}^2 = F_{2n+1} \tag{1}$$

holds for all $n \ge 0$. Marques and Togbé [9] investigated analogues of (1) in higher powers, obtaining the following partial result.

THEOREM 1. *If $x \ge 1$ is an integer such that $F_n^x + F_{n+1}^x$ is a Fibonacci number for all sufficiently large n, then $x \in \{1, 2\}$.*

Later, Luca and Oyono [8] extended the above result on the nonexistence of positive integer solutions (n, m, x) to the Diophantine equation

$$F_n^x + F_{n+1}^x = F_m \tag{2}$$

by proving the following result.

THEOREM 2. *Equation (2) has no positive integer solutions (n, m, x) with $n \ge 2$ and $x \ge 3$.*

In this paper, we prove an analogue of Theorem 2 when the sequence of Fibonacci numbers is replaced by the sequence of k-generalized Fibonacci numbers. In what follows, we adopt some definitions and notation from Bravo and Luca [1], [2].

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Let $k \geq 2$ be an integer. One of numerous generalizations of the Fibonacci sequence, which is sometimes called the *k-generalized Fibonacci sequence* $\{F_n^{(k)}\}_{n \geq -(k-2)}$, is given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all } n \geq 2,$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We refer to $F_n^{(k)}$ as the nth *k-generalized Fibonacci number*. Note that for $k = 2$, we have $F_n^{(2)} = F_n$, the familiar nth Fibonacci number. For $k = 3$ such numbers are called *Tribonacci* numbers. They are followed by the *Tetranacci* numbers for $k = 4$, and so on.

Recently, Chaves and Marques [3] proved that the analogue of the Diophantine equation (1) in k-generalized Fibonacci numbers has no positive integer solution (k, n, m) with $k \geq 3$ and $n \geq 1$.

In this paper, we look at the Diophantine equation (2), in k-generalized Fibonacci numbers, in this way generalizing both the results from [8] and from [3]. More precisely, we prove:

Main Theorem. *The Diophantine equation*

$$(F_n^{(k)})^x + (F_{n+1}^{(k)})^x = F_m^{(k)} \tag{3}$$

has no positive integer solutions (k, n, m, x) *with* $k \geq 3$, $n \geq 2$ *and* $x \geq 2$.

Before getting into details, we give a brief description of our method. We first use lower bounds for linear forms in logarithms of algebraic numbers to bound n, m and x polynomially in terms of k. When k is small, we use the theory of continued fractions by means of a variation of a result of Dujella and Pethő to lower such bounds to cases that allow us to treat our problem computationally. When k is large, we use the fact that the dominant root of the k-generalized Fibonacci sequence is exponentially close to 2, to replace this root by 2 in our calculations with linear forms in logarithms, obtaining in this way a simpler linear form in logarithms which allows us to bound k and then complete the calculations.

2. Preliminary results. Note that the characteristic polynomial of the k-generalized Fibonacci sequence is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

The above polynomial has just one root $\alpha(k)$ outside the unit circle. It is real and positive, so it satisfies $\alpha(k) > 1$. The other roots are strictly inside the unit circle. In particular, $\Psi_k(x)$ is irreducible over $\mathbb{Q}$. Lemma 2.3 in [7] shows that

$$2(1 - 2^{-k}) < \alpha(k) < 2 \quad \text{for all } k \geq 2. \tag{4}$$

This inequality was rediscovered by Wolfram [11]. In particular, we have $\alpha(k) > 7/4 = 1.75$ for all $k \geq 3$. This fact will be used in our work.

We write $\alpha := \alpha(k)$. This is called the *dominant root* of $\Psi_k(x)$ for reasons that we present below. Dresden [4] gave the following Binet-like formula for $F_n^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha^{(i)} - 1}{2 + (k+1)(\alpha^{(i)} - 2)} (\alpha^{(i)})^{n-1}, \tag{5}$$

where $\alpha = \alpha^{(1)}, \ldots, \alpha^{(k)}$ are the roots of $\Psi_k(x)$. Dresden also showed that the contribution of the roots which are inside the unit circle to the right-hand side of (5) is very small. More precisely, he proved that

$$\left| F_n^{(k)} - \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \right| < \frac{1}{2} \quad \text{for all } n \geq 1. \tag{6}$$

We will also use the following results.

Lemma 1. *We have* $F_n^{(k)} = 2^{n-2}$ *for all* $n = 2, \ldots, k+1$.

Bravo and Luca [2] showed that $F_n^{(k)} < 2^{n-2}$ for all $n \geq k+2$.

Lemma 2. *The inequality*

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}$$

holds for all $n \geq 1$.

For a proof of Lemma 2, see [1]. We consider the function

$$f_k(z) := \frac{z-1}{2 + (k+1)(z-2)} \quad \text{for } k \geq 2.$$

If $z \in (2(1 - 2^{-k}), 2)$, a straightforward verification shows that $\partial_z f_k(z) < 0$. Indeed,

$$\partial_z f_k(z) = \frac{1-k}{(2 + (k+2)(z-2))^2} < 0 \quad \text{for all } k \geq 2.$$

Thus, from inequality (4), we conclude that

$$1/2 = f_k(2) \leq f_k(\alpha) \leq f_k(2(1 - 2^{-k})) = \frac{2^{k-1} - 1}{2^k - k - 1} \leq 3/4$$

for all $k \geq 3$. Even more, since $f_2((1+\sqrt{5})/2) = 0.72360\ldots < 3/4$, we deduce that $f_k(\alpha) \leq 3/4$ for all $k \geq 2$. On the other hand, if $z = \alpha^{(i)}$ with $i = 2, \ldots, k$, then $|f_k(\alpha^{(i)})| < 1$ for all $k \geq 2$. Indeed, as $|\alpha^{(i)}| < 1$, then $|\alpha^{(i)} - 1| < 2$ and $|2 + (k+1)(\alpha^{(i)} - 2)| > k - 1$. Further, $f_2((1-\sqrt{5})/2) = 0.2763\ldots$.

The following lemma is due to Bravo and Luca [2].

Lemma 3. *If* $1 \le r < 2^{k/2}$, *then*

$$\alpha^r = 2^r + \delta \qquad \text{with} \quad |\delta| < \frac{2^{r+1}}{2^{k/2}}, \tag{7}$$

$$f_k(\alpha) = f_k(2) + \eta \quad \text{with} \quad |\eta| < \frac{2k}{2^k}. \tag{8}$$

The idea of the proof of Lemma 3 is as follows. We estimate the error of approximating α^r with 2^r. Let $\lambda > 0$ be such that $\lambda + \alpha = 2$. Since α is located between $2(1 - 2^k)$ and 2, we get $\lambda \in (0, 1/2^{k-1})$. Therefore,

$$\alpha^r = (2 - \lambda)^r = 2^r e^{r \log(1-\lambda/2)} \ge 2^r e^{-\lambda r} \ge 2^r(1 - \lambda r),$$

where we have used the fact that $\log(1 - x) \ge -2x$ for all $x < 1/2$ and that $e^{-x} \ge 1 - x$ for all $x \in \mathbb{R}$. Moreover, $\lambda r < r/2^{k-1} < 2/2^{k/2}$. It then follows that

$$|\alpha^r - 2^r| < \frac{2^{r+1}}{2^{k/2}}.$$

Writing $\delta = \alpha^r - 2^r$, we get (7).

We now estimate the error of approximating $f_k(\alpha)$ with $f_k(2) = 1/2$. By the Mean-Value Theorem, there exists $\theta \in (\alpha, 2)$ such that

$$|f_k(\alpha) - f_k(2)| = |2 - \alpha|\, |\partial_z f_k(\theta)| < \frac{2k}{2^k},$$

where we have used the fact that $|\partial_z f_k(\theta)| < k$. Writing $\eta = f_k(\alpha) - f_k(2)$, we obtain (8).

In particular,

$$|f_k(\alpha)\alpha^r - 2^{r-1}| < \frac{2^r}{2^{k/2}} + \frac{2^{r+1}k}{2^k} + \frac{2^{r+2}k}{2^{3k/2}}. \tag{9}$$

Lemma 4. *The sequences* $\{F_n^{(k)}\}_{n\ge1}$, $\{F_n^{(k)}\}_{k\ge3}$ *and* $\{\alpha(k)\}_{k\ge3}$ *are non-decreasing.*

The following lemma is crucial in our applications of linear forms in logarithms.

Lemma 5. *The number* $f_k(\alpha)$ *is an algebraic integer for no* $k \ge 2$.

Proof. Assume that $f_k(\alpha)$ is an algebraic integer. Then its norm (from $\mathbb{K}$ to $\mathbb{Q}$) is an integer. Applying the norm and taking absolute values, we obtain

$$1 \le |\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(f_k(\alpha))| = f_k(\alpha) \prod_{i=2}^{k} |f_k(\alpha^{(i)})|.$$

However, $f_k(\alpha) \le 0.75$ and $|f_k(\alpha^{(i)})| < 2/(k-1) \le 1$ for $i = 2, \dots, k$ and all $k \ge 3$, contradicting the above inequality. The case $k = 2$ is clear. ∎

We need two more ingredients from Diophantine approximation, which are Matveev's lower bound for nonzero linear forms in logarithms of algebraic numbers and a generalization of the Baker and Davenport Lemma on continued fractions due essentially to Dujella and Pethő.

Let γ be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) := a_0 \prod_{i=1}^{d} (X - \gamma^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The logarithmic height of γ is given by

$$h(\gamma) := \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d} \log \max\{|\gamma^{(i)}|, 1\}\Big).$$

One of the most cited results today when it comes to the effective solution of exponential Diophantine equations is the following theorem of Matveev [10].

Theorem 3. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, let $\gamma_1, \dots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and let $b_1, \dots, b_t$ be rational integers. Suppose*

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and set

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \dots, A_t$ be real numbers such that

$$A_i \geq \max\{D h(\gamma_i), |\log \gamma_i|, 0.16\}, \qquad i = 1, \dots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp\big(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\big).$$

We make repeated use of the following result, which is a slight variation of a result due to Dujella and Pethő which itself is a generalization of a result of Baker and Davenport (see [5] and [1]). For a real number x, we write $\|x\| = \min\{|x - n| \,:\, n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 6. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\gamma q\|$. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-k}$$

in positive integers m, n *and* k *with*

$$m \le M \quad \textit{and} \quad k \ge \frac{\log(Aq/\epsilon)}{\log B}.$$

3. An inequality for x **in terms of** k **and** n**.** From now on, $k \ge 2$, $n \ge 1$, m, $x \ge 2$ are integers satisfying (3).

Observe that when $n = 1$ we get $F_m^{(k)} = 2$. This has the solution $m = 3$, for all $k \ge 2$ and $x \ge 2$. Furthermore, if $k = 2$ and $x = 2$, then (3) holds with $m = 2n + 1$ for all $n \ge 1$, as shown by identity (1). If $k = 2$ and $x \ge 3$, then Theorem 2 shows that equation (3) has no positive solutions (n, m). Thus, from now on, we assume that $n \ge 2$ and $k \ge 3$. Moreover, since $x \ge 2$, by Lemma 4 we get $F_m^{(k)} \ge (F_2^{(k)})^2 + (F_3^{(k)})^2 = 5$, so $m \ge 5$.

Hence, our equation reduces to

$$(10) \qquad (F_n^{(k)})^x + (F_{n+1}^{(k)})^x = F_m^{(k)}$$

in integers subject to the inequalities $n \ge 2$, $m \ge 5$, $k \ge 3$ and $x \ge 2$. By Lemma 2,

$$\alpha^{m-2} \le F_m^{(k)} = (F_n^{(k)})^x + (F_{n+1}^{(k)})^x \le \alpha^{(n-1)x} + \alpha^{nx} = \alpha^{nx}(1 + \alpha^{-x}) < \alpha^{nx+1},$$

and

$$\alpha^{(n-1)x} \le (F_{n+1}^{(k)})^x < (F_n^{(k)})^x + (F_{n+1}^{(k)})^x = F_m^{(k)} \le \alpha^{m-1}.$$

Thus,

$$(11) \qquad (n-1)x + 1 < m < nx + 3.$$

Estimate (11) is essential for our purpose.

From formula (5) and estimate (6), we can write

$$(12) \qquad F_m^{(k)} = f_k(\alpha)\alpha^{m-1} + e_k(m), \quad \text{where} \quad |e_k(m)| < 1/2.$$

Hence, equation (10) can be rewritten as

$$(13) \qquad f_k(\alpha)\alpha^{m-1} - (F_{n+1}^{(k)})^x = (F_n^{(k)})^x - e_k(m).$$

Dividing (13) by $(F_{n+1}^{(k)})^x$ and taking absolute values, we get

$$(14) \qquad |f_k(\alpha)\alpha^{m-1}(F_{n+1}^{(k)})^{-x} - 1| < 2\left(\frac{F_n^{(k)}}{F_{n+1}^{(k)}}\right)^x < \frac{2}{1.75^x},$$

where we have used the fact that $F_n^{(k)}/F_{n+1}^{(k)} \le 4/7$ for all $n \ge 2$ and $k \ge 3$. Indeed,

$$\begin{aligned} 7F_n^{(k)} \le 4F_{n+1}^{(k)} &\Leftrightarrow 7F_n^{(k)} \le 4(F_n^{(k)} + \cdots + F_{n-(k-1)}^{(k)}) \\ &\Leftrightarrow 3F_n^{(k)} \le 4(F_{n-1}^{(k)} + \cdots + F_{n-(k-1)}^{(k)}) \\ &\Leftrightarrow 3(F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)}) \le 4(F_{n-1}^{(k)} + \cdots + F_{n-(k-1)}^{(k)}) \\ &\Leftrightarrow 3F_{n-k}^{(k)} \le F_{n-1}^{(k)} + \cdots + F_{n-(k-1)}^{(k)}, \end{aligned}$$

and the last statement is true since $F_{n-k}^{(k)}$ is less than or equal to each of $F_{n-1}^{(k)}, F_{n-2}^{(k)}, \ldots, F_{n-(k-1)}^{(k)}$ for $n \ge 2$.

We apply Theorem 3 with $t := 3$, $\gamma_1 := f_k(\alpha)$, $\gamma_2 := \alpha$, $\gamma_3 := F_{n+1}^{(k)}$, $b_1 := 1$, $b_2 := m-1$, $b_3 := -x$. Hence,

$$\Lambda_1 := f_k(\alpha)\alpha^{m-1}(F_{n+1}^{(k)})^{-x} - 1$$

and from (14) we have

$$|\Lambda_1| < \frac{2}{1.75^x}. \tag{15}$$

Furthermore, $\mathbb{K} := \mathbb{Q}(\alpha)$ contains $\gamma_1, \gamma_2, \gamma_3$ and has $D = [\mathbb{K} : \mathbb{Q}] = k$. To see that $\Lambda_1 \ne 0$, we note that otherwise we would get the relation

$$f_k(\alpha)\alpha^{m-1} = (F_{n+1}^{(k)})^x.$$

The above inequality implies that $f_k(\alpha)$ is an algebraic integer, which is false by Lemma 5. Thus, $\Lambda_1 \ne 0$.

Bravo and Luca [2] showed that $h(\gamma_1) < 4\log k$. Furthermore, by the properties of the roots of $\Psi_k(x)$ we obtain

$$\begin{aligned} h(\gamma_2) &= (\log\alpha)/k < (\log 2)/k < 0.7/k, \\ h(\gamma_3) &= \log(F_{n+1}^{(k)}) \le n\log\alpha < 0.7n, \end{aligned}$$

by Lemma 2. Thus, we can take $A_1 := 4k\log k$, $A_2 := 0.7$ and $A_3 := 0.7nk$. Finally, from (11), we have $m > (n-1)x + 1 > x$, so we can take $B := m$.

Theorem 3 gives the following lower bound for $|\Lambda_1|$:

$$\exp\big(-1.4 \cdot 30^6 \cdot 3^{4.5} k^2 (1+\log k)(1+\log m)(4k\log k)(0.7)(0.7nk)\big),$$

which is smaller than $2/1.75^x$ by (15). Taking logarithms and performing the calculations, we get

$$\begin{aligned} x &< \frac{\log 2}{\log 1.75} + \frac{1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 0.7^2 \cdot 4}{\log 1.75} nk^4 (\log k)(1+\log k)(1+\log m) \\ &< \frac{\log 2}{\log 1.75} + \left(\frac{1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 0.7^2 \cdot 4^2}{\log 1.75}\right) nk^4 (\log k)^2 \log m \\ &< 3 \cdot 10^{12} nk^4 (\log k)^2 \log(nx), \end{aligned} \tag{16}$$

where we have used the fact that $1 + \log k < 2\log k$ for all $k \ge 3$, the similar inequality with k replaced by m, and inequality (11).

We next extract from (16) an upper bound for x depending on n and k. Multiplying both sides of (16) by n we obtain

$$nx < 3 \cdot 10^{12} n^2 k^4 (\log k)^2 \log(nx),$$

or equivalently

$$\frac{nx}{\log(nx)} < 3 \cdot 10^{12} n^2 k^4 (\log k)^2. \tag{17}$$

Now we use the fact that

$$\text{if} \quad A > 3 \text{ and } \frac{y}{\log y} < A \quad \text{then} \quad y < 2A \log A \tag{18}$$

(see [8]). Taking $y := nx$ and $A := 3 \cdot 10^{12} n^2 k^4 (\log k)^2$, we see from (17) and (18) that

$$\begin{aligned} nx &< 2(3 \cdot 10^{12} n^2 k^4 (\log k)^2) \log(3 \cdot 10^{12} n^2 k^4 (\log k)^2) \\ &< 6 \cdot 10^{12} n^2 k^4 (\log k)^2 (29 + 2 \log n + 4 \log k + 2 \log\log k) \\ &< 3 \cdot 10^{14} n^2 k^4 (\log k)^2 \max\{\log n, \log k\}. \end{aligned}$$

In the last inequality, we have used the fact that

$$29 + 2 \log n + 4 \log k + 2 \log\log k < 42 \max\{\log n, \log k\}$$

for all $n \geq 2$ and $k \geq 3$.

We record what we have just proved.

LEMMA 7. *If* (n, m, k, x) *is a solution of* (10) *with* $n \geq 2$, $k \geq 3$ *and* $x \geq 2$, *then*

$$x < 3 \cdot 10^{14} n k^4 (\log k)^2 \max\{\log n, \log k\}. \tag{19}$$

4. Inequalities on x, n and m in terms of k. We assume first that $n > 1750$. We suppose that $k < n$ and we find an upper bound for n, m and x in terms of k only.

From (19), we have

$$x < 3 \cdot 10^{14} n^5 (\log n)^3. \tag{20}$$

For equation (12) (with m replaced by n), we can write

$$(F_n^{(k)})^x = f_k(\alpha)^x \alpha^{(n-1)x} \left(1 + \frac{e_k(n)}{f_k(\alpha)\alpha^{n-1}}\right)^x.$$

We look at the elements

$$z := xr \quad \text{and} \quad (1+r)^x, \quad \text{where} \quad r := \frac{e_k(n)}{f_k(\alpha)\alpha^{n-1}}.$$

We have $k \geq 3$, $\alpha > 1.75$ and $f_k(\alpha) > 1/2$. So, $|r| < 1/1.75^{n-1}$ and

$$|z| = x|r| < \frac{3 \cdot 10^{14} n^5 (\log n)^3}{1.75^{n-1}} < \frac{1}{1.75^{0.921n}},$$

where the last inequality holds for all $n > 1750$. In particular, we have $|z| < 10^{-391}$.

Now, if $r < 0$ then

$$1 > (1+r)^x = \exp(x \log(1-|r|)) \geq \exp(-2|z|) > 1 - 2|z|,$$

while if $r > 0$, then

$$1 < (1+r)^x = \left(1 + \frac{|z|}{x}\right)^x < \exp|z| < 1 + 2|z|,$$

because $|r| < 1/2$ and $|z| < 10^{-391}$ is very small.

Thus, in either case we have

$$|(F_n^{(k)})^x - f_k(\alpha)^x \alpha^{(n-1)x}| < 2|z| f_k(\alpha)^x \alpha^{(n-1)x}. \tag{21}$$

The same inequality is true if we replace n by $n+1$:

$$|(F_{n+1}^{(k)})^x - f_k(\alpha)^x \alpha^{nx}| < 2|z| f_k(\alpha)^x \alpha^{nx}. \tag{22}$$

We rewrite (10) using (21) and (22) as

$$\begin{aligned} F_m^{(k)} = (F_n^{(k)})^x + (F_{n+1}^{(k)})^x &= f_k(\alpha)^x \alpha^{(n-1)x} + f_k(\alpha)^x \alpha^{nx} \\ &+ [(F_n^{(k)})^x - f_k(\alpha)^x \alpha^{(n-1)x}] + [(F_{n+1}^{(k)})^x - f_k(\alpha)^x \alpha^{nx}], \end{aligned}$$

or

$$\begin{aligned} &|f_k(\alpha)\alpha^{m-1} - f_k(\alpha)^x \alpha^{(n-1)x}(1+\alpha^x)| \\ &\quad < |(F_n^{(k)})^x - f_k(\alpha)^x \alpha^{(n-1)x}| + |(F_{n+1}^{(k)})^x - f_k(\alpha)^x \alpha^{nx}| + \frac{1}{2} \\ &\quad < 2|z| f_k(\alpha)^x \alpha^{(n-1)x}(1+\alpha^x) + \frac{1}{2}. \end{aligned} \tag{23}$$

Dividing by $f_k(\alpha)^x \alpha^{nx}$, we conclude that

$$\begin{aligned} |f_k(\alpha)^{1-x} \alpha^{m-1-nx} - (1+\alpha^{-x})| &< 2|z|(1+\alpha^{-x}) + \frac{1}{2 f_k(\alpha)^x \alpha^{nx}} \\ &< 3|z| + \frac{1}{2}\left(\frac{1}{1.75^{n-2}}\right)^x \\ &< \frac{4}{1.75^{0.921n}}, \end{aligned} \tag{24}$$

where we have used the following facts: $\alpha^x > 1.75^2 > 2$, $f_k(\alpha)\alpha^n > 1.75^{n-2}$ and $(n-2)x + 1 \geq 0.921n$ for all $n > 1750$, $x \geq 2$. Hence,

$$|f_k(\alpha)^{1-x} \alpha^{m-1-nx} - 1| < \frac{4}{1.75^{0.921n}} + \frac{1}{1.75^x} < \frac{5}{1.75^\ell}, \tag{25}$$

where we have set $\ell := \min\{0.921n, x\}$.

We apply again Theorem 3 with $t := 2$, $\gamma_1 := f_k(\alpha)$, $\gamma_2 := \alpha$, $b_1 := 1-x$, $b_2 := m-1-nx$. So, $\Lambda_2 := f_k(\alpha)^{1-x}\alpha^{m-1-nx} - 1$, and from (25),

$$|\Lambda_2| < \frac{5}{1.75^{\ell}}. \tag{26}$$

As in the previous application of Theorem 3, we have $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := k$, $A_1 := 4k\log k$, $A_2 := 0.7$. Moreover, we can take $B := x$, since $|m-1-nx| \leq x$ by inequality (11).

Let us see that $\Lambda_2 \neq 0$. Indeed, if $\Lambda_2 = 0$, then

$$f_k(\alpha)^{x-1} = \alpha^{m-1-nx}.$$

This implies that $f_k(\alpha)$ is an algebraic integer, which is not possible by Lemma 5. Thus, $\Lambda_2 \neq 0$.

The conclusion of Theorem 3 and inequality (26) yield, after taking logarithms, the following upper bound for ℓ:

$$\begin{aligned} \ell &< \frac{\log 5}{\log 1.75} + \frac{1.4\cdot 30^5\cdot 2^{4.5}\cdot 4\cdot 0.7}{\log 1.75} k^3(\log k)(1+\log k)(1+\log x) \\ &< \frac{\log 5}{\log 1.75} + \frac{1.4\cdot 30^5\cdot 2^{4.5}\cdot 4\cdot 0.7\cdot 2^2}{\log 1.75} k^3(\log k)^2\log x, \end{aligned}$$

which leads to

$$\ell < 1.6\cdot 10^{10}\, k^3(\log k)^2\log x. \tag{27}$$

If $\ell = 0.921n$, then from (27),

$$n < 1.8\cdot 10^{10}k^3(\log k)^2\log x$$

and using inequality (20), we obtain

$$\begin{aligned} n &< 1.8\cdot 10^{10}k^3(\log k)^2(\log(3\cdot 10^{14}) + 5\log n + 3\log\log n) \\ &< 1.8\cdot 10^{10}k^3(\log k)^2(57\log n) \\ &< 1.1\cdot 10^{12}k^3(\log k)^2\log n, \end{aligned}$$

where we have used the fact that $\log(3\cdot 10^{14}) < 49\log n$ for all $n \geq 2$. Hence,

$$\frac{n}{\log n} < 1.1\cdot 10^{12}k^3(\log k)^2.$$

Applying the argument (18) with $y := n$ and $A := 1.1\cdot 10^{12}k^3(\log k)^2$, we obtain an upper bound on n depending only on k. Inserting this bound in (20) and using inequality (11), we obtain

$$\begin{aligned} n &< 7\cdot 10^{13}k^3(\log k)^3, \\ x &< 5.1\cdot 10^{83}k^{15}(\log k)^{18}, \\ m &< 3.5\cdot 10^{97}k^{18}(\log k)^{21}, \end{aligned} \tag{28}$$

where we have used the fact that $\log(1.1\cdot 10^{12}) < 26\log k$ for all $k \geq 3$.

If $\ell = x$, then from (27) we get

$$\frac{x}{\log x} < 1.6 \cdot 10^{10} k^3 (\log k)^2,$$

which implies, via (18) again, that

$$x < 2(1.6 \cdot 10^{10} k^3 (\log k)^2) \log(1.6 \cdot 10^{10} k^3 (\log k)^2).$$

Since $\log(1.6 \cdot 10^{10} k^3 (\log k)^2) < 27 \log k$ for $k \geq 3$, we conclude that

$$x < 10^{12} k^3 (\log k)^3. \tag{29}$$

In order to estimate n in terms of k only, we recall inequality (23):

$$|f_k(\alpha)\alpha^{m-1} - f_k(\alpha)^x \alpha^{(n-1)x}(1+\alpha^x)| < 2|z| f_k(\alpha)^x \alpha^{(n-1)x}(1+\alpha^x) + \frac{1}{2}.$$

Dividing both sides by $f_k(\alpha)\alpha^{m-1}$, we obtain

$$\begin{aligned} &|f_k(\alpha)^{x-1}\alpha^{(n-1)x-(m-1)}(1+\alpha^x) - 1| \\ &\qquad < 2|z| f_k(\alpha)^{x-1}\alpha^{nx-(m-1)}(1+\alpha^{-x}) + \frac{1}{2f_k(\alpha)\alpha^{m-1}} \\ &\qquad < \frac{2n(f_k(\alpha)\alpha)^{x-1}}{1.75^{n-1}}(1+\alpha^{-x}) + \frac{1}{\alpha^{m-1}} \\ &\qquad < 6\left(\frac{n(3/2)^{0.921n}}{1.75^n}\right) + \frac{1}{1.75^{0.32n}} < \frac{2}{1.75^{0.32n}}, \end{aligned}$$

where we have used the following facts:

- (i) $\ell = x \leq 0.921n$, so $|z| = x|r| < n/1.75^{n-1}$;
- (ii) by (11), we have $(n-1)x - (m-1) + x \leq x - 1$ and $m - 1 > 0.32n$;
- (iii) since $k \geq 3$ and $1/2 < f_k(\alpha) \leq 3/4$, we have $f_k(\alpha)\alpha < 3/2$;
- (iv) $1 + \alpha^{-x} < 3/2$;
- (v) the very last inequality holds for all $n > 1750$.

In conclusion, we have shown that

$$|f_k(\alpha)^{x-1}\alpha^{(n-1)x-(m-1)}(1+\alpha^x) - 1| < \frac{2}{1.75^{0.32n}}. \tag{30}$$

We apply again Theorem 3 with $t := 3$, $\gamma_1 := f_k(\alpha)$, $\gamma_2 := \alpha$, $\gamma_3 := 1 + \alpha^x$, $b_1 := x - 1$, $b_2 := (n-1)x - (m-1)$, $b_3 := 1$. Hence, from (30),

$$\Lambda_3 := f_k(\alpha)^{x-1}\alpha^{(n-1)x-(m-1)}(1+\alpha^x) - 1$$

satisfies

$$|\Lambda_3| < \frac{2}{1.75^{0.32n}}. \tag{31}$$

We can take again $\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$, $A_1 := 4k \log k$, $A_2 := 0.7$. For A_3, we note that $1 + \alpha^x \in \mathcal{O}_{\mathbb{K}}$, $1 + \alpha^x < 2^{x+1}$ for all $x \geq 2$ and $|1 + (\alpha^{(i)})^x| < 2$

for all $i = 2, \ldots, k$. Therefore, if $1 \le d \le k$ is the degree of the minimal polynomial of $1 + \alpha^x$ over $\mathbb{Z}$, then

$$h(1+\alpha^x) = \frac{1}{d}\Big(\log(1+\alpha^x) + \sum_{i=2}^{d} \log\max\{|1 + (\alpha^{(i)})^x|, 1\}\Big)$$
$$< \log 2(x+1) + \log 2(d-1) < 0.7(x+k).$$

Thus, we can take $A_3 := 0.7(x+k)k$. For B, we observe that, by (11), $|(n-1)x - (m-1)| < x+2$, so we take $B := x+2$.

Before applying Theorem 3, it remains to prove that $\Lambda_3 \neq 0$. Assuming the contrary, we get

$$f_k(\alpha)^{1-x}\alpha^{m-1-(n-1)x} = 1 + \alpha^x.$$

This again implies (as in the argument used to show that $\Lambda_1 \neq 0$ and $\Lambda_2 \neq 0$) that $f_k(\alpha)$ is an algebraic integer, which is false by Lemma 5. Hence, $\Lambda_3 \neq 0$.

Combining the conclusion of Theorem 3 with inequality (31), we get, after taking logarithms, the following upper bound for n:

$$(0.32n)\log 1.75 < \log 2 + (1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2 \cdot 4 \cdot 4 \cdot (0.7)^2)k^4(\log k)^2(\log x)(x+k), \tag{32}$$

where we have used the inequality $1 + \log(x+2) < 4\log x$ for all $x \ge 2$.

By (29), we have $x < 10^{12}k^3(\log k)^3$ so $x + k < 1.1 \cdot 10^{12}k^3(\log k)^3$ and therefore

$$\log x < \log(10^{12}) + 3\log k + 3\log\log k < 28 + 6\log k < 32\log k.$$

Here, we have used the fact that $28 < 26\log k$ for all $k \ge 3$.

Hence, returning to inequality (32), we get

$$n < 4.5 \cdot 10^{26}k^7(\log k)^6.$$

Using also the inequality $m < nx + 3$, we have in summary

$$\begin{aligned} n &< 4.5 \cdot 10^{26}k^7(\log k)^6,\\ x &< 10^{12}k^3(\log k)^3,\\ m &< 4.6 \cdot 10^{38}k^{10}(\log k)^9. \end{aligned} \tag{33}$$

Combining (28) and (33), we get

$$\begin{aligned} n &< 4.5 \cdot 10^{26}k^7(\log k)^6,\\ x &< 5.1 \cdot 10^{83}k^{15}(\log k)^{18},\\ m &< 3.5 \cdot 10^{97}k^{18}(\log k)^{21}. \end{aligned}$$

We note that the above inequalities have been obtained under the assumptions that $n > 1750$ and $k < n$. However, we can see that when $n \le k$, the upper bounds for n, x and m in terms of k, arising from (19), are smaller than the above upper bounds. Moreover, the case $n \le 1750$ together with inequalities (19) yields upper bounds for x and m in terms of k which are also smaller than the ones above. Thus, we can state the following result.

LEMMA 8. *Let* (n,m,k,x) *be a solution of* (10). *Then*

$$
\begin{aligned}
&n < 4.5 \cdot 10^{26} k^7 (\log k)^6,\\
(34)\qquad &x < 5.1 \cdot 10^{83} k^{15} (\log k)^{18},\\
&m < 3.5 \cdot 10^{97} k^{18} (\log k)^{21}.
\end{aligned}
$$

5. The case of small k. We first treat the case $n > 1750$ and $k \in [3, 1131]$ (so $n > k$); we show that in this range equation (10) has no solution.

Returning to inequality (25), we take

$$\Gamma_2 := (x-1)\log(f_k(\alpha)^{-1}) + (m-1-nx)\log\alpha,$$

and conclude that

$$(35)\qquad |\Lambda_2| = |e^{\Gamma_2} - 1| < \frac{4}{1.75^{0.921n}} + \frac{1}{1.75^x} < \frac{1}{3},$$

because $n > 1750$ and $x \ge 2$. Thus, $e^{|\Gamma_2|} < 3/2$, and from (26),

$$|\Gamma_2| \le e^{|\Gamma_2|}|e^{\Gamma_2} - 1| < \frac{7.5}{1.75^{\ell}}$$

with $\ell = \min\{0.921n, x\}$.

Dividing the above inequality by $(x-1)\log\alpha$, we obtain

$$
(36)\qquad \left|\frac{\log(f_k(\alpha)^{-1})}{\log\alpha} - \frac{nx-(m-1)}{x-1}\right| < \frac{7.5}{1.75^{\ell}(x-1)\log\alpha} < \frac{14}{1.75^{\ell}(x-1)}.
$$

Now for $3 \le k \le 1131$, we set $\gamma_k := \log(f_k(\alpha)^{-1})/\log\alpha$, compute its continued fraction $[a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, \dots]$ and its convergents $p_1^{(k)}/q_1^{(k)}, p_2^{(k)}/q_2^{(k)}, \dots$. In each case we find an integer t_k such that

$$q_{t_k}^{(k)} > 5.1 \cdot 10^{83} k^{15} (\log k)^{18} > x - 1$$

(by (34)), and take

$$a_M := \max_{3 \le k \le 1131} \{a_i^{(k)} : 0 \le i \le t_k\}.$$

Then, from the known properties of continued fractions, we have

$$(37)\qquad \left|\gamma_k - \frac{nx-(m-1)}{x-1}\right| > \frac{1}{(a_M+2)(x-1)^2}.$$

Hence, combining (36) and (37), and taking into account that $a_M + 2 < 3.6 \cdot 10^{337}$ (confirmed by Mathematica), we obtain

$$1.75^{\ell} < 5.1 \cdot 10^{337} x.$$

If $\ell = 0.921n$, then

$$1.75^{0.921n} < 1.6 \cdot 10^{352} n^5 (\log n)^3,$$

which is a consequence of (20), since $n > k$. The last inequality above leads to $n \leq 1657$, contradicting the assumption on n.

If $\ell = x$, then we get

$$1.75^x < 5.1 \cdot 10^{337} x,$$

so $x \leq 1402$. Below we show that (10) has no solution for $x \in [2, 1402]$ with k in our range.

We go back to inequality (24) and rewrite it as

$$|f_k(\alpha)^{1-x}(\alpha^{-1})^{nx-(m-1)}(1+\alpha^{-x})^{-1} - 1| < \frac{4}{1.75^{0.921n}(1+\alpha^{-x})} < \frac{4}{1.75^{0.921n}}. \tag{38}$$

Before continuing, we note that $|\Lambda_2| < 1/3$ by (35), therefore

$$f_k(\alpha)^{1-x}\alpha^{m-1-nx} \in [2/3, 4/3]$$

and, in particular, $0.4x - 1 < nx - (m-1) < 1.3x$.

Set $d := nx - (m-1)$. With the help of Mathematica, we calculated the numbers $|f_k(\alpha)^{1-x}(\alpha^{-1})^d(1+\alpha^{-x})^{-1} - 1|$ for all $k \in [3, 1131]$, all $x \in [2, 1402]$ and all $d \in [\lfloor 0.4x - 1 \rfloor, \lfloor 1.3x \rfloor]$. It turns out that the smallest of these numbers is $> 10^{-340}$. Hence, by (38), $10^{-340} < 4/1.75^{0.921n}$, so $n < 1521$, which is false.

We now continue with the case $n \in [2, 1750]$ and $k \in [3, 1131]$. In order to apply Lemma 6, we let

$$\Gamma_1 := \log f_k(\alpha) + (m-1)\log\alpha - x \log F_{n+1}^{(k)}.$$

Returning to Λ_1 given by (15), we have $e^{\Gamma_1} - 1 = \Lambda_1$. We note that Γ_1 is positive since Λ_1 is positive, which can be deduced by looking at the right-hand side of (13) and using

$$(F_n^{(k)})^x - e_k(m) > (F_2^{(3)})^2 - \frac{1}{2} > \frac{1}{2}.$$

Moreover,

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 < \frac{2}{1.75^x}. \tag{39}$$

Replacing Γ_1 and dividing by $\log F_{n+1}^{(k)}$, we get

$$0 < m\left(\frac{\log\alpha}{\log F_{n+1}^{(k)}}\right) - x + \frac{\log f_k(\alpha) - \log\alpha}{\log F_{n+1}^{(k)}} < \frac{2}{1.75^x \log F_{n+1}^{(k)}} < \frac{3}{(1.75^{\frac{1}{1750}})^m}, \tag{40}$$

where we have used $(m-3)/1750 < (m-3)/n < x$ as well as the inequalities $\log F_{n+1}^{(k)} > \log F_3^{(3)} = \log 2 > 2/3$.

We set

$$\gamma := \frac{\log \alpha}{\log F_{n+1}^{(k)}}, \qquad \mu := \frac{\log f_k(\alpha) - \log \alpha}{\log F_{n+1}^{(k)}},$$

and

$$A := 3, \qquad B := 1.00032 \leq 1.75^{\frac{1}{1750}}.$$

The fact that α is a unit in $\mathcal{O}_{\mathbb{K}}$ ensures that γ is irrational. Inequality (40) can be rewriten as

$$0 < m\gamma - x + \mu < AB^{-m}. \tag{41}$$

Now, we take $M := \lfloor 3 \cdot 10^{14} n^2 k^4 (\log k)^2 \max\{\log n, \log k\} + 3 \rfloor$ (using (11) and (19)) and apply Lemma 6 for each $k \in [3, 1131]$ and $n \in [2, 1750]$ to inequality (41). A computer search with Mathematica showed that the maximum of $\log(Aq/\epsilon)/\log B$ is 5030930, which according to Lemma 6 is an upper bound on m.

Next, since $(n-1)x + 1 < m$, we have

$$x \leq m/(n-1) \leq 5030930/(n-1).$$

Thus, our problem is reduced to searching for solutions to equation (10) in the following range:

$$\begin{gathered} k \in [3, 1131], \quad n \in [2, 1750], \\ m \in [5, 5030930], \quad x \in [2, 5030930/(n-1)]. \end{gathered} \tag{42}$$

A computer search with Mathematica revealed that there are no solutions to (10) in the ranges given in (42). This completes the analysis of the case when k is small.

6. The case of large k. From now on, we assume that $k > 1131$. From (34), we have

$$n < 4.5 \cdot 10^{26} k^7 (\log k)^6 < 2^{k/2}, \qquad m < 3.5 \cdot 10^{97} k^{18} (\log k)^{21} < 2^{k/2}.$$

If $n \leq k$, then from (13) and Lemma 1, we obtain

$$|f_k(\alpha)\alpha^{m-1} - 2^{(n-1)x}| < 2^{(n-2)x} + \frac{1}{2}. \tag{43}$$

Taking $r := m - 1$ in (9) and using (43), we conclude that

$$\begin{aligned} |2^{m-2} - 2^{(n-1)x}| &< |2^{m-2} - f_k(\alpha)\alpha^{m-1}| + |f_k(\alpha)\alpha^{m-1} - 2^{(n-1)x}| \\ &< 2^{m-2}\left(\frac{2}{2^{k/2}} + \frac{4k}{2^k} + \frac{8k}{2^{3k/2}}\right) + 2^{(n-2)x} + \frac{1}{2}. \end{aligned} \tag{44}$$

Now, dividing by 2^{m-2} and using the inequalities $4k/2^k < 1/2^{k/2}$ and $8k/2^{3k/2} < 1/2^{k/2}$, which are valid for $k > 1131$, we get

$$(45)\qquad |1-2^{(n-1)x-(m-2)}| < \frac{1}{2^{(m-2)-(n-2)x}} + \frac{1}{2^{m-1}} + \frac{4}{2^{k/2}} < \frac{1}{2^x} + \frac{1}{8}.$$

The last inequality follows because $(m-2)-(n-2)x \geq x$ (by (11)), $m \geq 5$ and $k > 1131$.

The left side in (45) is greater than or equal to $1/2$ unless $(n-1)x = m-2$, in which case it is zero. However, $m-2 = (n-1)x$ is not possible: otherwise, from (10), we would get

$$2^{(n-2)x} + 2^{(n-1)x} = F^{(k)}_{(n-1)x+2} \leq 2^{(n-1)x},$$

which is a contradiction. This shows that the case $n \leq k$ does not yield any convenient solutions to our problem.

Assume now that $n > k$. From (13) again, we conclude that

$$(46)\qquad |f_k(\alpha)\alpha^{m-1} - (F^{(k)}_{n+1})^x| < (F^{(k)}_n)^x + \frac{1}{2} \leq 2^{(n-2)x} + \frac{1}{2}.$$

Performing an analysis similar to the one used to deduce (22), we get

$$(47)\qquad |(F^{(k)}_{n+1})^x - f_k(\alpha)^x\alpha^{nx}| < 2|z|(f_k(\alpha)\alpha^n)^x < \frac{2^{(n-1)x+1}}{1.75^{0.88n}},$$

where we have used the facts that $|z| < 1/1.75^{0.88n}$ for $n \geq k > 1131$ and $f_k(\alpha)\alpha^n < 2^{n-1}$.

Finally, we conclude that

$$\begin{aligned}(48)\qquad & |f_k(\alpha)^x\alpha^{nx} - 2^{(n-1)x}| \\ &= |f_k(\alpha)\alpha^n - 2^{n-1}|((f_k(\alpha)\alpha^n)^{x-1} + \cdots + 2^{(n-1)(x-1)}) \\ &< |f_k(\alpha)\alpha^n - 2^{n-1}|\, x(\max\{f_k(\alpha)\alpha^n, 2^{n-1}\})^{x-1} \\ &< x2^{(n-1)(x-1)}\left(\frac{2^n}{2^{k/2}} + \frac{2^{n+1}k}{2^k} + \frac{2^{n+2}k}{2^{3k/2}}\right) \\ &= x2^{(n-1)x+1}\left(\frac{1}{2^{k/2}} + \frac{2k}{2^k} + \frac{4k}{2^{3k/2}}\right) \\ &< x2^{(n-1)x+1}\left(\frac{3}{2^{k/2}}\right) < \frac{2^{(n-1)x+1}}{2^{k/14}}.\end{aligned}$$

In the above inequality, we have used (9) with $r := n$, and the inequalities

$$2k/2^k < 1/2^{k/2}, \qquad 4k/2^{3k/2} < 1/2^{k/2},$$
$$\frac{3x}{2^{k/2}} < \frac{3(5.1\cdot 10^{83}k^{15}(\log k)^{18})}{2^{k/2}} < \frac{1}{2^{k/14}},$$

which hold since $n > k > 1131$.

Hence, combining the estimate for $|2^{m-2} - f_k(\alpha)\alpha^{m-1}|$ used in (44) and the estimates (46)–(48), we obtain

$$\begin{aligned}|2^{m-2} - 2^{(n-1)x}| &< |2^{m-2} - f_k(\alpha)\alpha^{m-1}| + |f_k(\alpha)\alpha^{m-1} - (F_{n+1}^{(k)})^x| \\ &\quad + |(F_{n+1}^{(k)})^x - f_k(\alpha)^x\alpha^{nx}| + |f_k(\alpha)^x\alpha^{nx} - 2^{(n-1)x}| \\ &< \frac{2^m}{2^{k/2}} + \left(2^{(n-2)x} + \frac{1}{2}\right) + \frac{2^{(n-1)x+1}}{1.75^{0.88n}} + \frac{2^{(n-1)x+1}}{2^{k/14}}.\end{aligned}$$

Dividing by 2^{m-2}, we get

$$|1 - 2^{(n-1)x-(m-2)}| < \frac{4}{2^{k/2}} + \frac{1}{2^x} + \frac{1}{2^{m-1}} + \frac{2}{1.75^{0.88n}} + \frac{2}{2^{k/14}} < \frac{1}{2^x} + \frac{1}{8},$$

where we have used $m - 2 - (n - 2)x \geq x$ (by (11)), as well as the facts that $n > k > 1131$ and $m \geq 5$. But the last displayed inequality leads us again to

$$\frac{1}{2} < \frac{1}{2^x} + \frac{1}{8},$$

which is impossible for any $x \geq 2$.

Thus, we have in fact shown that there are no solutions (n, m, k, x) to (10) with $k > 1131$, which completes the proof of our Main Theorem.

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