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LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS ON $k[x,y]$

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Abstract. We give a new proof of Miyanishi's theorem on the classification of the additive group scheme actions on the affine plane.

1. Introduction. Let k be a field of characteristic $p \geq 0$. For a positive integer n, we denote the polynomial ring in n variables over k by $k^{[n]}$.

Let A be a k-domain. A set $D = \{D_n\}_{n=0}^{\infty}$ of k-linear endomorphisms of A is called a *higher derivation* on A if:

(1) D_0 is the identity map of A.
(2) For all $a, b \in A$ and for all $n \geq 0$, $D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b)$.

For a higher derivation $D = \{D_n\}_{n=0}^{\infty}$ on A, we define the *kernel* A^D of D to be the k-subalgebra $\{a \in A \mid D_n(a) = 0 \text{ for all } n \geq 1\} = \bigcap_{n\geq 1} \operatorname{Ker} D_n$. A higher derivation $D = \{D_n\}_{n=0}^{\infty}$ on A is said to be *locally finite* (resp. *iterative*) if the following additional condition (3) (resp. (4)) is satisfied:

(3) For all $a \in A$, there exists an integer $n \geq 0$ such that $D_m(a) = 0$ for all $m \geq n$.
(4) For all $i, j \geq 0$, $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$.

A locally finite, iterative, higher derivation is abbreviated as *lfihd*. We note that, if $p = 0$ and $D = \{D_n\}_{n=0}^{\infty}$ is an lfihd on A, then the condition (4) implies that $D_n = \frac{1}{n!}D_1^n$ for all $n \geq 1$. So D_1 is a locally nilpotent derivation on A and $A^D = \operatorname{Ker} D_1$.

It is well-known that studying $\mathbb{G}_{\mathrm{a}}$-actions on an affine variety X is equivalent to studying lfihds on the coordinate ring of X. When $p = 0$, locally nilpotent derivations and their kernels have been studied by many mathematicians (see [5], [1], [7], etc.). On the other hand, much less is known when $p > 0$. See [11, Chapter 1], [14], [3], [4] and the references therein for results on lfihds and their kernels in the case $p > 0$.

In [12], Rentschler classified the locally nilpotent derivations on $k^{[2]}$ when $p = 0$. In particular, he classified the $\mathbb{G}_{\mathrm{a}}$-actions on the affine plane over

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an algebraically closed field of characteristic zero. Later on, Miyanishi [9] classified the $\mathbb{G}_a$-actions on the affine plane over an algebraically closed field of positive characteristic (see Corollary 1.2 below). His proof in [9] uses some results from algebraic geometry (Zariski's Main Theorem, etc.). In this short note, we classify the lfihds on $k^{[2]}$ by using purely algebraic methods.

In order to state our results, we give some definitions. Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher derivation on a k-domain A. Then we can define a k-algebra homomorphism $\varphi_D : A \to A[[t]]$ by $\varphi_D(a) = \sum_{n\geq 0} D_n(a)t^n$. We call φ_D the *homomorphism associated to* D. We note that, for an element a of A, $a \in A^D$ if and only if $\varphi_D(a) = a$, and that $\operatorname{Im}\varphi_D \subset A[t]$ provided D is locally finite. Suppose that D is an lfihd. Using the homomorphism φ_D, we can define the *D-degree* of an element a of A by $\deg_D(a) = \deg_t(\varphi_D(a))$, where we set $\deg_t(0) = -\infty$. An element s of A is said to be a *local slice* of D if it satisfies the following conditions:

(i) $s \notin A^D$;
(ii) $\deg_D(s) = \min\{\deg_D(f) \mid f \in A \setminus A^D\}$.

The main result of this note is stated as follows.

THEOREM 1.1. *Let k be a field of characteristic $p > 0$ and $D = \{D_n\}_{n=0}^{\infty}$ a higher derivation on $A = k^{[2]}$. Then D is an lfihd if and only if there exists a system of variables $\{x, y\}$ for A such that $\varphi_D(x) = x$ and $\varphi_D(y) = y + \sum_{i=0}^{\ell} f_i(x)t^{p^i}$, where φ_D is the homomorphism associated to D, $\ell \geq 0$ and $f_0(x), \dots, f_\ell(x) \in k[x]$.*

As a direct consequence of Theorem 1.1, we obtain the following corollary.

COROLLARY 1.2. *Let k be an algebraically closed field of characteristic $p > 0$. Then every $\mathbb{G}_a$-action on the affine plane $\mathbb{A}_k^2 = \operatorname{Spec}(k^{[2]})$ is equivalent to an action of the form*

$$a \cdot (\xi, \eta) = \Big(\xi, \eta + \sum_{i=0}^{\ell} f_i(\xi)a^{p^i}\Big)$$

for all $a \in \mathbb{G}_a(k) = k$, $(\xi, \eta) \in \mathbb{A}_k^2(k) = k^2$, where $\ell \geq 0$ and $f_0, \dots, f_\ell \in k^{[1]}$.

We note that, for an arbitrary non-negative integer ℓ and an arbitrary choice of the $f_0, \dots, f_\ell \in k^{[1]}$ via the displayed formula in Corollary 1.2, a $\mathbb{G}_a$-action on the affine plane and hence an lfihd on $k^{[2]}$ is defined.

Moreover, we obtain the following result.

COROLLARY 1.3 (cf. [10, Corollary 5]). *Let k be an algebraically closed field of characteristic $p > 0$ and $D = \{D_n\}_{n=0}^{\infty}$ an lfihd on $A = k^{[2]}$ with $A^D \neq A$ (i.e., D is a non-trivial lfihd on A). Then $A^D = k^{[1]} = k[x]$ and $D_{p^i}(s) \in A^D = k[x]$ for every local slice s of D and for every non-negative integer i. Moreover, the following conditions are equivalent to each other:*

(1) *The* $\mathbb{G}_{\mathrm{a}}$*-action on* $\mathbb{A}^2_k = \mathrm{Spec}(A)$ *corresponding to* D *is fixed point free.*
(2) *There exists a local slice* $s \in A$ *of* D *such that the ideal generated by* $\{D_{p^i}(s)\}_{i=0}^{\infty}$ *in* A^D *equals* A^D*.*

2. Proofs. Let $A = k^{[n]}$ be the polynomial ring in n variables over a field k of characteristic $p > 0$, $D = \{D_n\}_{n=0}^{\infty}$ an lfihd on A and φ_D the homomorphism associated to D.

LEMMA 2.1. *Let* $s \in A$ *be a local slice of* D*. Then:*

(1) *Let* c *be the leading coefficient of* $\varphi_D(s)$*. Then* $A[c^{-1}] = A^D[c^{-1}][s]$ *and* s *is indeterminate over* A^D*.*
(2) $\varphi_D(s)$ *can be expressed as*

$$\varphi_D(s) = s + \sum_{i=0}^{\ell} c_i t^{p^i},$$

where $\ell \geq 0$ *and* $c_0, \ldots, c_\ell \in A^D$*.*

Proof. The assertion (1) is well-known (see, e.g., [11, 1.5 (p. 20)]). The assertion (2) follows from [11, 1.5.3 (pp. 22–23)]. ∎

We define the rank of D which is a word-for-word translation of that of a derivation.

DEFINITION 2.2. With the same notations as above, we define the *rank* of D to be the minimal integer r such that a part of a system of variables $\{x_1, \ldots, x_{n-r}\}$ for A is contained in A^D. We denote the rank of D by $\mathrm{rank}\, D$.

It is clear that $\mathrm{rank}\, D = 0$ if and only if D is trivial, i.e., $A^D = A$. We prove the following result, which is given in [6] when $p = 0$ (see [6, Corollary to Proposition 1]).

LEMMA 2.3. *Assume that* $\mathrm{rank}\, D = 1$*. Then there exists a system of variables* $\{x_1, \ldots, x_n\}$ *for* A *such that* $A^D = k[x_1, \ldots, x_{n-1}]$ *and*

$$\varphi_D(x_n) = x_n + \sum_{i=0}^{\ell} f_i t^{p^i},$$

where $\ell \geq 0$ *and* $f_0, \ldots, f_\ell \in k[x_1, \ldots, x_{n-1}]$*. Moreover, the set of all local slices of* D *equals* $\{a x_n + b \mid a \in A^D \setminus \{0\}, b \in A^D\}$*.*

Proof. Since $\mathrm{rank}\, D = 1$, there exists a system of variables $\{x_1, \ldots, x_n\}$ for A such that $x_1, \ldots, x_{n-1} \in A^D$. Since D is non-trivial, $x_n \notin A^D$.

Let $f = f(x_1, \ldots, x_n) \in A \setminus \{0\}$ be a non-zero polynomial. We put $m = \deg_{x_n} f(x_1, \ldots, x_n)$ and write

$$f(x_1, \ldots, x_n) = a_0 x_n^m + a_1 x_n^{m-1} + \cdots + a_{m-1} x_n + a_m$$

with $a_0, \dots, a_m \in k[x_1, \dots, x_{n-1}]$. Since φ_D is a k-algebra homomorphism and $a_0, \dots, a_m \in A^D$, we have

$$\varphi_D(f) = a_0 \varphi_D(x_n)^m + a_1 \varphi_D(x_n)^{m-1} + \cdots + a_{m-1}\varphi_D(x_n) + a_m.$$

It is then clear that $f \in A^D$ if and only if $m = 0$, i.e., $f \in k[x_1, \dots, x_{n-1}]$. Moreover, $\deg_D(f) = m \deg_D(x_n) \geq \deg_D(x_n)$ provided $f \notin A^D$. So x_n is a local slice of D. We see that f is a local slice of D if and only if $m = 1$, i.e., $f = ax_n + b$ for some $a \in A^D \setminus \{0\}$ and $b \in A^D$, which proves the last assertion. It follows from (2) of Lemma 2.1 that $f_0, \dots, f_\ell \in A^D = k[x_1, \dots, x_{n-1}]$. ∎

Lemma 2.4. *Assume that* $\operatorname{rank} D = 1$ *and k is algebraically closed. Then the following conditions are equivalent to each other:*

(1) *The $\mathbb{G}_{\mathrm{a}}$-action σ on $\mathbb{A}^n_k = \operatorname{Spec}(A)$ corresponding to D is fixed point free.*
(2) *There exists a local slice $s \in A$ of D such that the ideal generated by $\{D_{p^i}(s) \mid i \in \mathbb{Z}_{\geq 0}\}$ in A^D equals A^D.*

Proof. Since $\operatorname{rank} D = 1$, there exists a system of variables $\{x_1, \dots, x_n\}$ for A such that $x_1, \dots, x_{n-1} \in A^D$. By Lemma 2.3, $A^D = k[x_1, \dots, x_{n-1}]$ and x_n is a local slice of D.

(1)⇒(2). With the same notations as in the previous paragraph, the $\mathbb{G}_{\mathrm{a}}$-action σ is equivalent to an action of the form

$$a \cdot (\xi_1, \dots, \xi_n) = \Bigl(\xi_1, \dots, \xi_{n-1}, \xi_n + \sum_{i=0}^{\ell} f_i(\xi_1, \dots, \xi_{n-1}) a^{p^i}\Bigr)$$

for all $a \in k$, $(\xi_1, \dots, \xi_n) \in k^n$, where $\ell \geq 0$ and $f_0, \dots, f_\ell \in k[x_1, \dots, x_{n-1}]$. We can easily see that the $\mathbb{G}_{\mathrm{a}}$-action σ is fixed point free if and only if $f_0, \dots, f_\ell$ have no common zeros in $\mathbb{A}^{n-1}_k(k) = k^{n-1}$. This implies (2).

(2)⇒(1). Let s be the local slice of D in (2). By Lemma 2.3, s can be expressed as $s = ax_n + b$ with $a, b \in k[x_1, \dots, x_{n-1}]$ and $a \neq 0$. Then $D_{p^i}(s) = aD_{p^i}(x_n)$ for every non-negative integer i. Since the ideal generated by $\{D_{p^i}(s) \mid i \in \mathbb{Z}_{\geq 0}\}$ in A^D contains 1, we have $a \in k[x_1, \dots, x_{n-1}]^\times = k^\times$. Hence the ideal generated by $\{D_{p^i}(x_n) \mid i \in \mathbb{Z}_{\geq 0}\}$ in A^D, which can be expressed as $(f_0, f_1, \dots, f_\ell)$, equals A^D. This implies (1). ∎

We now prove the results stated in the previous section. The outline of the proof of Theorem 1.1 below is almost the same as that of Rentschler's theorem [12] in Daigle–Freudenburg [2, Section 2].

Proof of Theorem 1.1. The "if" part is clear. We prove the "only if" part. The assertion is clear if $A^D = A$. So we assume that $A^D \neq A$. Put $S = A^D \setminus \{0\}$. It follows from [11, p. 39] (or [8, Corollary 1.2]) that $A^D = k^{[1]} = k[x]$ for some $x \in A^D \setminus k$. Let s be a local slice of D. Lemma 2.1

then implies that $S^{-1}A = k(x)^{[1]} = k(x)[s]$. By [8, Lemma 2.3], $k(x) \cap A = Q(A^D) \cap A = A^D = k[x]$. Since A is a polynomial ring over k, it is finitely generated over k and is geometrically factorial over k (for the definition, see Lemma 2.5 below). So all the hypotheses (i)–(iii) in Lemma 2.5 are satisfied. Hence, it follows from that lemma that $A = (A^D)^{[1]} = A^D[y] = k[x, y]$. Then $\{x, y\}$ is a system of variables for A. Since D is non-trivial and $A^D = k[x]$, we infer from Lemma 2.3 that y is a local slice of D. Therefore, by using Lemma 2.1, we obtain Theorem 1.1. ∎

LEMMA 2.5 (Russell–Sathaye [13, Theorem 2.4.2]). *Let k be a field, A a finitely generated k-algebra and $x \in A$. Put $S = k[x] \setminus \{0\}$. Suppose that:*

(i) $S^{-1}A = k(x)^{[1]}$;
(ii) $k(x) \cap A = k[x]$;
(iii) *A is geometrically factorial over k, i.e., $E \otimes_k A$ is a UFD for any algebraic extension field E of k.*

Then $A = k[x]^{[1]}$.

Proof. See [13, 2.4], where we consider L and F as k and x, respectively. ∎

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