LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS ON $k[x, y]$

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Abstract. We give a new proof of Miyanishi’s theorem on the classification of the additive group scheme actions on the affine plane.

1. Introduction. Let $k$ be a field of characteristic $p \geq 0$. For a positive integer $n$, we denote the polynomial ring in $n$ variables over $k$ by $k[n]$.

Let $A$ be a $k$-domain. A set $D = \{D_n\}_{n=0}^{\infty}$ of $k$-linear endomorphisms of $A$ is called a higher derivation on $A$ if:

(1) $D_0$ is the identity map of $A$.
(2) For all $a, b \in A$ and for all $n \geq 0$, $D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b)$.

For a higher derivation $D = \{D_n\}_{n=0}^{\infty}$ on $A$, we define the kernel $A_D$ of $D$ to be the $k$-subalgebra $\{a \in A \mid D_n(a) = 0 \text{ for all } n \geq 1\} = \bigcap_{n \geq 1} \text{Ker } D_n$.

A higher derivation $D = \{D_n\}_{n=0}^{\infty}$ on $A$ is said to be locally finite (resp. iterative) if the following additional condition (3) (resp. (4)) is satisfied:

(3) For all $a \in A$, there exists an integer $n \geq 0$ such that $D_m(a) = 0$ for all $m \geq n$.
(4) For all $i, j \geq 0$, $D_i \circ D_j = (i+j)D_{i+j}$.

A locally finite, iterative, higher derivation is abbreviated as lfihd. We note that, if $p = 0$ and $D = \{D_n\}_{n=0}^{\infty}$ is an lfihd on $A$, then the condition (4) implies that $D_n = \frac{1}{n!}D_1^n$ for all $n \geq 1$. So $D_1$ is a locally nilpotent derivation on $A$ and $A_D = \text{Ker } D_1$.

It is well-known that studying $G_{a}$-actions on an affine variety $X$ is equivalent to studying lfihds on the coordinate ring of $X$. When $p = 0$, locally nilpotent derivations and their kernels have been studied by many mathematicians (see [5], [1], [7], etc.). On the other hand, much less is known when $p > 0$. See [11, Chapter 1], [14], [3], [4] and the references therein for results on lfihds and their kernels in the case $p > 0$.

In [12], Rentschler classified the locally nilpotent derivations on $k[2]$ when $p = 0$. In particular, he classified the $G_{a}$-actions on the affine plane over $k$.

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an algebraically closed field of characteristic zero. Later on, Miyanishi [9] classified the $G_a$-actions on the affine plane over an algebraically closed field of positive characteristic (see Corollary 1.2 below). His proof in [9] uses some results from algebraic geometry (Zariski’s Main Theorem, etc.). In this short note, we classify the lfihds on $k^2$ by using purely algebraic methods.

In order to state our results, we give some definitions. Let $D = \{D_n\}_{n=0}^{\infty}$ be a higher derivation on a $k$-domain $A$. Then we can define a $k$-algebra homomorphism $\varphi_D : A \to A[[t]]$ by $\varphi_D(a) = \sum_{n \geq 0} D_n(a)t^n$. We call $\varphi_D$ the homomorphism associated to $D$. We note that, for an element $a$ of $A$, $a \in A^D$ if and only if $\varphi_D(a) = a$, and that $\operatorname{Im} \varphi_D \subset A[t]$ provided $D$ is locally finite. Suppose that $D$ is an lfihd. Using the homomorphism $\varphi_D$, we can define the $D$-degree of an element $a$ of $A$ by $\deg_D(a) = \deg(\varphi_D(a))$, where we set $\deg_D(0) = -\infty$. An element $s$ of $A$ is said to be a local slice of $D$ if it satisfies the following conditions:

(i) $s \not\in A^D$;
(ii) $\deg_D(s) = \min\{\deg_D(f) \mid f \in A \setminus A^D\}$.

The main result of this note is stated as follows.

**Theorem 1.1.** Let $k$ be a field of characteristic $p > 0$ and $D = \{D_n\}_{n=0}^{\infty}$ a higher derivation on $A = k^2$. Then $D$ is an lfihd if and only if there exists a system of variables $\{x, y\}$ for $A$ such that $\varphi_D(x) = x$ and $\varphi_D(y) = y + \sum_{i=0}^{\ell} f_i(x)t^\ell$, where $\varphi_D$ is the homomorphism associated to $D$, $\ell \geq 0$ and $f_0(x), \ldots, f_\ell(x) \in k[x]$.

As a direct consequence of Theorem 1.1, we obtain the following corollary.

**Corollary 1.2.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Then every $G_a$-action on the affine plane $\mathbb{A}_k^2 = \operatorname{Spec}(k^2)$ is equivalent to an action of the form

$$a \cdot (\xi, \eta) = (\xi, \eta + \sum_{i=0}^{\ell} f_i(\xi)a^{i^{\ell}})$$

for all $a \in G_a(k) = k$, $(\xi, \eta) \in \mathbb{A}_k^2(k) = k^2$, where $\ell \geq 0$ and $f_0, \ldots, f_\ell \in k[1]$.

We note that, for an arbitrary non-negative integer $\ell$ and an arbitrary choice of the $f_0, \ldots, f_\ell \in k[1]$ via the displayed formula in Corollary 1.2, a $G_a$-action on the affine plane and hence an lfihd on $k^2$ is defined.

Moreover, we obtain the following result.

**Corollary 1.3** (cf. [10] Corollary 5). Let $k$ be an algebraically closed field of characteristic $p > 0$ and $D = \{D_n\}_{n=0}^{\infty}$ an lfihd on $A = k^2$ with $A^D \neq A$ (i.e., $D$ is a non-trivial lfihd on $A$). Then $A^D = k[1] = k[x]$ and $D_p^i(s) \in A^D = k[x]$ for every local slice $s$ of $D$ and for every non-negative integer $i$. Moreover, the following conditions are equivalent to each other:
(1) The $\mathbb{G}_a$-action on $\mathbb{A}_k^2 = \text{Spec}(A)$ corresponding to $D$ is fixed point free.
(2) There exists a local slice $s \in A$ of $D$ such that the ideal generated by
$\{D_{p^i}(s)\}_{i=0}^\infty$ in $A^D$ equals $A^D$.

2. Proofs. Let $A = k[n]$ be the polynomial ring in $n$ variables over a field $k$ of characteristic $p > 0$, $D = \{D_n\}_{n=0}^\infty$ an lfihd on $A$ and $\varphi_D$ the homomorphism associated to $D$.

Lemma 2.1. Let $s \in A$ be a local slice of $D$. Then:

(1) Let $c$ be the leading coefficient of $\varphi_D(s)$. Then $A[c^{-1}] = A^D[c^{-1}][s]$ and $s$ is indeterminate over $A^D$.
(2) $\varphi_D(s)$ can be expressed as
$$\varphi_D(s) = s + \sum_{i=0}^{\ell} c_i t^{p^i},$$
where $\ell \geq 0$ and $c_0, \ldots, c_\ell \in A^D$.

Proof. The assertion (1) is well-known (see, e.g., [11, 1.5 (p. 20)]). The assertion (2) follows from [11, 1.5.3 (pp. 22–23)].

We define the rank of $D$ which is a word-for-word translation of that of a derivation.

Definition 2.2. With the same notations as above, we define the rank of $D$ to be the minimal integer $r$ such that a part of a system of variables $\{x_1, \ldots, x_{n-r}\}$ for $A$ is contained in $A^D$. We denote the rank of $D$ by $\text{rank}_D$.

It is clear that $\text{rank}_D = 0$ if and only if $D$ is trivial, i.e., $A^D = A$. We prove the following result, which is given in [6] when $p = 0$ (see [6] Corollary to Proposition 1]).

Lemma 2.3. Assume that $\text{rank}_D = 1$. Then there exists a system of variables $\{x_1, \ldots, x_n\}$ for $A$ such that $A^D = k[x_1, \ldots, x_{n-1}]$ and
$$\varphi_D(x_n) = x_n + \sum_{i=0}^{\ell} f_i t^{p^i},$$
where $\ell \geq 0$ and $f_0, \ldots, f_\ell \in k[x_1, \ldots, x_{n-1}]$. Moreover, the set of all local slices of $D$ equals $\{a x_n + b \mid a \in A^D \setminus \{0\}, b \in A^D\}$.

Proof. Since $\text{rank}_D = 1$, there exists a system of variables $\{x_1, \ldots, x_n\}$ for $A$ such that $x_1, \ldots, x_{n-1} \in A^D$. Since $D$ is non-trivial, $x_n \not\in A^D$.

Let $f = f(x_1, \ldots, x_n) \in A \setminus \{0\}$ be a non-zero polynomial. We put $m = \deg_{x_n} f(x_1, \ldots, x_n)$ and write
$$f(x_1, \ldots, x_n) = a_0 x_n^m + a_1 x_n^{m-1} + \cdots + a_{m-1} x_n + a_m$$
with $a_0, \ldots, a_m \in k[x_1, \ldots, x_{n-1}]$. Since $\varphi_D$ is a $k$-algebra homomorphism and $a_0, \ldots, a_m \in A^D$, we have

$$\varphi_D(f) = a_0\varphi_D(x_n)^m + a_1\varphi_D(x_n)^{m-1} + \cdots + a_{m-1}\varphi_D(x_n) + a_m.$$ 

It is then clear that $f \in A^D$ if and only if $m = 0$, i.e., $f \in k[x_1, \ldots, x_{n-1}]$. Moreover, $\deg_D(f) = m \deg_D(x_n) \geq \deg_D(x_n)$ provided $f \notin A^D$. So $x_n$ is a local slice of $D$. We see that $f$ is a local slice of $D$ if and only if $m = 1$, i.e., $f = ax_n + b$ for some $a \in A^D \setminus \{0\}$ and $b \in A^D$, which proves the last assertion. It follows from (2) of Lemma 2.1 that $f_0, \ldots, f_\ell \in A^D = k[x_1, \ldots, x_{n-1}]$. □

**Lemma 2.4.** Assume that rank $D = 1$ and $k$ is algebraically closed. Then the following conditions are equivalent to each other:

1. The $G_a$-action $\sigma$ on $A^n_k = \text{Spec}(A)$ corresponding to $D$ is fixed point free.
2. There exists a local slice $s \in A$ of $D$ such that the ideal generated by $\{D_{p_i}(s) \mid i \in \mathbb{Z}_{\geq 0}\}$ in $A^D$ equals $A^D$.

**Proof.** Since rank $D = 1$, there exists a system of variables $\{x_1, \ldots, x_n\}$ for $A$ such that $x_1, \ldots, x_{n-1} \in A^D$. By Lemma 2.3, $A^D = k[x_1, \ldots, x_{n-1}]$ and $x_n$ is a local slice of $D$.

(1)⇒(2). With the same notations as in the previous paragraph, the $G_a$-action $\sigma$ is equivalent to an action of the form

$$a \cdot (\xi_1, \ldots, \xi_n) = (\xi_1, \ldots, \xi_{n-1}, \xi_n + \sum_{i=0}^\ell f_i(\xi_1, \ldots, \xi_{n-1})a^{p^i})$$

for all $a \in k$, $(\xi_1, \ldots, \xi_n) \in k^n$, where $\ell \geq 0$ and $f_0, \ldots, f_\ell \in k[x_1, \ldots, x_{n-1}]$. We can easily see that the $G_a$-action $\sigma$ is fixed point free if and only if $f_0, \ldots, f_\ell$ have no common zeros in $A^{n-1}_k(k) = k^{n-1}$. This implies (2).

(2)⇒(1). Let $s$ be the local slice of $D$ in (2). By Lemma 2.3, $s$ can be expressed as $s = ax_n + b$ with $a, b \in k[x_1, \ldots, x_{n-1}]$ and $a \neq 0$. Then $D_{p_i}(s) = aD_{p_i}(x_n)$ for every non-negative integer $i$. Since the ideal generated by $\{D_{p_i}(s) \mid i \in \mathbb{Z}_{\geq 0}\}$ in $A^D$ contains 1, we have $a \in k[x_1, \ldots, x_{n-1}]$ and $x_n$ is fixed point free if and only if $a \neq 0$. Hence the ideal generated by $\{D_{p_i}(x_n) \mid i \in \mathbb{Z}_{\geq 0}\}$ in $A^D$, which can be expressed as $(f_0, f_1, \ldots, f_\ell)$, equals $A^D$. This implies (1). □

We now prove the results stated in the previous section. The outline of the proof of Theorem 1.1 below is almost the same as that of Rentschler’s theorem [12] in Daigle–Frenicle [2] Section 2.

**Proof of Theorem 1.1** The “if” part is clear. We prove the “only if” part. The assertion is clear if $A^D = A$. So we assume that $A^D \neq A$. Put $S = A^D \setminus \{0\}$. It follows from [11, p. 39] (or [8, Corollary 1.2]) that $A^D = k^{[1]} = k[x]$ for some $x \in A^D \setminus k$. Let $s$ be a local slice of $D$. Lemma 2.1
then implies that $S^{-1}A = k(x)[1] = k(x)[s]$. By [8, Lemma 2.3], $k(x) \cap A = Q(A^D) \cap A = A^D = k[x]$. Since $A$ is a polynomial ring over $k$, it is finitely generated over $k$ and is geometrically factorial over $k$ (for the definition, see Lemma 2.5 below). So all the hypotheses (i)–(iii) in Lemma 2.5 are satisfied. Hence, it follows from that lemma that $A = (A^D)[1] = A^D[y] = k[x,y]$. Then $\{x,y\}$ is a system of variables for $A$. Since $D$ is non-trivial and $A^D = k[x]$, we infer from Lemma 2.3 that $y$ is a local slice of $D$. Therefore, by using Lemma 2.1, we obtain Theorem 1.1.

**Lemma 2.5** (Russell–Sathaye [13, Theorem 2.4.2]). Let $k$ be a field, $A$ a finitely generated $k$-algebra and $x \in A$. Put $S = k[x] \setminus \{0\}$. Suppose that:

(i) $S^{-1}A = k(x)[1]$;
(ii) $k(x) \cap A = k[x]$;
(iii) $A$ is geometrically factorial over $k$, i.e., $E \otimes_k A$ is a UFD for any algebraic extension field $E$ of $k$.

Then $A = k[x][1]$.

**Proof.** See [13, 2.4], where we consider $L$ and $F$ as $k$ and $x$, respectively. ■

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