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INVARIANCE IDENTITY IN THE CLASS OF GENERALIZED QUASIARITHMETIC MEANS

BY

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Abstract. An invariance formula in the class of generalized p-variable quasiarithmetic means is provided. An effective form of the limit of the sequence of iterates of mean-type mappings of this type is given. An application to determining functions which are invariant with respect to generalized quasiarithmetic mean-type mappings is presented.

1. Introduction. Let X, Y be sets and T a selfmap of X. A function $\Phi : X \to Y$ is called *invariant with respect to* T (briefly, T-*invariant*) if $\Phi \circ T = \Phi$. The problem of determining such functions occurs in iteration theory and fixed point theory [7]. Assuming, for instance, that (X, d) is a metric space, $T : X \to X$ is a continuous mapping such that the sequence $(T^n)_{n\in\mathbb{N}}$ of iterates of T is pointwise convergent then, obviously, the function $\Phi : X \to X$ defined by $\Phi(x) := \lim_{n\to\infty} T^n(x)$ is T-invariant.

Mean-type mappings, i.e. mappings of the form $\mathbf{M} = (M_1, \ldots, M_p)$, where the coordinate functions $M_1, \ldots, M_p$ are p-variable means, form a broad class of maps for which this question is particularly interesting. This follows from the fact that, under some general weak conditions, the sequence $(\mathbf{M}^n)_{n\in\mathbb{N}}$ of iterates converges to a unique $\mathbf{M}$-invariant mean-type mapping $\mathbf{K}$ of the same invariant coordinate mean K (Theorem 1) ([8], cf. also [5], [4]). In general, given a mean-type mapping $\mathbf{M}$, it is either difficult or impossible to find the explicit form of $\mathbf{M}$-invariant means and $\mathbf{M}$-invariant functions.

In the case $p = 2$, assuming $X = (0,\infty)^2$, $\mathbf{M} = (A, H)$ and $\mathbf{K} = (G, G)$, where A, G, H are the arithmetic, geometric and harmonic means, respectively, we have the identity $\mathbf{K} \circ \mathbf{M} = \mathbf{K}$, that is, $G \circ (A, H) = G$, an example of invariance that is equivalent to the classical Pythagorean harmony proportion

$$\frac{A(x, y)}{G(x, y)} = \frac{G(x, y)}{H(x, y)}, \qquad x, y > 0.$$

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This invariance identity allows one to deduce the nontrivial fact that

$$\lim_{n\to\infty}(A,H)^n(x,y)=(G(x,y),G(x,y)),\qquad x,y>0,$$

which appeared to be useful in [2].

In this note we present an invariance formula for a broad family of generalized quasiarithmetic mean-type mappings (Theorem 2(i)) that, besides invariant means, gives the explicit form of the limit of the sequence of iterates of mean-type mappings (Theorem 2(ii)). Applying this result we determine the form of a large class of functions that are invariant with respect to mean-type mappings.

2. Invariant generalized quasiarithmetic means. Fix an interval $I\subset\mathbb{R}$ and $p\in\mathbb{N}$, $p\geq 2$. Recall that a function $M:I^p\to I$ is called a *p-variable mean* if

$$\min(x_1,\dots,x_p)\leq M(x_1,\dots,x_p)\leq\max(x_1,\dots,x_p)$$

for all $x_1,\dots,x_p\in I$. The mean M is called *strict* if these inequalities are sharp for all $(x_1,\dots,x_p)\in I^p\setminus\Delta_p$ where

$$\Delta_p:=\{(x_1,\dots,x_p)\in I^p:x_1=\dots=x_p\};$$

and *symmetric* if $M(x_{\sigma(1)},\dots,x_{\sigma(p)})=M(x_1,\dots,x_p)$ for all permutations σ of $\{1,\dots,p\}$.

Let $M_i:I^p\to I$, $i=1,\dots,p$, be some means. A mean $K:I^p\to I$ is called *invariant with respect to the mean-type mapping* $\mathbf{M}:I^p\to I^p$, $\mathbf{M}:=(M_1,\dots,M_p)$, (briefly $\mathbf{M}$-*invariant*) if

$$K\circ(M_1,\dots,M_p)=K.$$

From [5, Theorem 1] and [8, Theorem 3], we have

Theorem 1. *If $M_i:I^p\to I$ for $i=1,\dots,p$ are continuous means and*

$$\max(M_1(\mathbf{x}),\dots,M_p(\mathbf{x}))-\min(M_1(\mathbf{x}),\dots,M_p(\mathbf{x}))<\max(\mathbf{x})-\min(\mathbf{x})$$

for all $\mathbf{x}=(x_1,\dots,x_p)\in I^p\setminus\Delta_p$, then the sequence of iterates of the mean-type mapping $\mathbf{M}=(M_1,\dots,M_p)$ converges to a mean-type mapping $\mathbf{K}=(K,\dots,K)$, where $K:I^p\to I$ is a continuous and $\mathbf{M}$-invariant mean, i.e. $K\circ\mathbf{M}=K$; moreover, the $\mathbf{M}$-invariant mean is unique.

Set

$$A(x,y)=\frac{x+y}{2},\qquad H(x,y)=\frac{2xy}{x+y},\qquad G(x,y)=\sqrt{xy},\qquad x,y>0.$$

Remark 1. Since $G\circ(A,H)=G$, the geometric mean G is (A,H)-invariant. From Theorem 1 we conclude that the sequence $((A,H)^n)_{n\in\mathbb{N}}$ of iterates of the mean-type mapping (A,H) converges and

$$\lim_{n\to\infty}(A,H)^n=(G,G).$$

This remark shows that the knowledge of the **M**-invariant mean and the invariance formula

$$G \circ (A, H) = G$$

can be very useful. However, it is known that in the class of quasiarithmetic means, this invariance formula is rather exceptional (cf. [3], [1]).

It turns out that, in this respect, a natural extension of the notion of quasiarithmetic means significantly improves the situation.

To see this consider the following fact easy to verify:

Remark 2 ([6]). Let $p \in \mathbb{N}$, $p \geq 2$. If $f_1, \ldots, f_p : I \to \mathbb{R}$ are continuous increasing functions such that $f_1 + \cdots + f_p$ is strictly increasing, then the function $A^{[f_1,\ldots,f_p]} : I^p \to I$ defined by

$$A^{[f_1,\ldots,f_p]}(x_1, \ldots, x_p) := (f_1 + \cdots + f_p)^{-1}(f_1(x_1) + \cdots + f_p(x_p)), \qquad x_1, \ldots, x_p \in I,$$

is a p-variable strict mean. (This remark remains true on replacing "increasing" by "decreasing".)

Taking $f_i = w_i f$, where $f : I \to \mathbb{R}$ is continuous and strictly monotonic, and $w_i \in (0, 1)$, $i = 1, \ldots, p$, are such that $w_1 + \cdots + w_p = 1$, we obtain

$$A^{[f_1,\ldots,f_p]}(x_1, \ldots, x_p) = f^{-1}(w_1 f(x_1) + \cdots + w_p f(x_p)), \qquad x_1, \ldots, x_p \in I.$$

Therefore $A^{[f_1,\ldots,f_p]}$ is called a *generalized weighted quasiarithmetic mean* with generators $f_1, \ldots, f_p$ (cf. [6]). If $A^{[f_1,\ldots,f_p]}$ is symmetric then it is quasiarithmetic.

For $p = 2$, setting $f = f_1$ and $g = f_2$, we get

$$A^{[f,g]}(x, y) := (f + g)^{-1}(f(x) + g(y)), \qquad x, y \in I.$$

Example 1. Since $f, g : (-\pi/2, \pi/2) \to \mathbb{R}$, $f(x) = \sin x$, $g(x) = x - \sin x$, are continuous and strictly increasing, the function

$$A^{[f,g]}(x, y) = \sin x + y - \sin y, \qquad x, y \in (-\pi/2, \pi/2),$$

is a generalized *weighted quasiarithmetic mean.*

The following result provides another invariance formula in the class of generalized quasiarithmetic means.

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$. Suppose that $f_1, \ldots, f_{2p-1} : I \to \mathbb{R}$ are continuous increasing and such that*

$$F_i := \sum_{j=i}^{p+i-1} f_j \quad \text{is strictly increasing for } i \in \{1, \ldots, p\}.$$

Then

(i) *the mean* $A^{[F_1,\dots,F_p]}$ *is invariant with respect to the mean-type mapping* $(A^{[f_1,\dots,f_p]}, A^{[f_2,\dots,f_{p+1}]}, \dots, A^{[f_p,\dots,f_{2p-1}]})$*, that is,*

$$A^{[F_1,\dots,F_p]} \circ (A^{[f_1,\dots,f_p]}, A^{[f_2,\dots,f_{p+1}]}, \dots, A^{[f_p,\dots,f_{2p-1}]}) = A^{[F_1,\dots,F_p]};$$

(ii) *the sequence* $((A^{[f_1,\dots,f_p]}, A^{[f_2,\dots,f_{p+1}]}, \dots, A^{[f_p,\dots,f_{2p-1}]})^n)_{n\in\mathbb{N}}$ *of iterates converges in* I^p*, and*

$$\lim_{n\to\infty}(A^{[f_1,\dots,f_p]}, A^{[f_2,\dots,f_{p+1}]}, \dots, A^{[f_p,\dots,f_{2p-1}]})^n = (A^{[F_1,\dots,F_p]}, \dots, A^{[F_1,\dots,F_p]}).$$

Proof. (i) Applying in turn the definition of the generalized quasiarithmetic mean $A^{[F_1,\dots,F_p]}$, the definition of $F_i := \sum_{j=i}^{p+i-1} f_j$, the commutativity of addition, and again the definition of F_i, we have

$$\begin{aligned}
\Big(\sum_{j=1}^{p} F_j\Big) &\circ A^{[F_1,\dots,F_p]} \circ (A^{[f_1,\dots,f_p]}, \dots, A^{[f_p,\dots,f_{2p-1}]})(x_1,\dots,x_p) \\
&= F_1(A^{[f_1,\dots,f_p]}(x_1,\dots,x_p)) + \dots + F_p(A^{[f_p,\dots,f_{2p-1}]}(x_1,\dots,x_p)) \\
&= [f_1(x_1) + \dots + f_p(x_p)] + [f_2(x_1) + \dots + f_{p+1}(x_p)] \\
&\quad + \dots + [f_p(x_1) + \dots + f_{2p-1}(x_p)] \\
&= [f_1(x_1) + \dots + f_p(x_1)] + [f_2(x_2) + \dots + f_{p+1}(x_2)] \\
&\quad + \dots + [f_p(x_p) + \dots + f_{2p-1}(x_p)] \\
&= F_1(x_1) + F_2(x_2) + \dots + F_p(x_p) = \sum_{j=1}^{p} F_j(x_j)
\end{aligned}$$

for all $x_1,\dots,x_p \in I$. Hence, again by the definition of $A^{[F_1,\dots,F_p]}$, we obtain

$$A^{[F_1,\dots,F_p]} \circ (A^{[f_1,\dots,f_p]}, \dots, A^{[f_p,\dots,f_{2p-1}]})(x_1,\dots,x_p) = A^{[F_1,\dots,F_p]}(x_1,\dots,x_p)$$

for all $x_1,\dots,x_p \in I$.

Result (ii) follows from (i) and Theorem 1. ■

Taking $f_i = f_{p+i}$ for $i = 1,\dots,p-1$, we get

Corollary 3. *Let* $I \subset \mathbb{R}$ *be an interval and* $p \in \mathbb{N}$*,* $p \ge 2$*. Suppose that* $f_1,\dots,f_p : I \to \mathbb{R}$ *are continuous increasing and such that*

$$F := \sum_{j=1}^{p} f_j \quad \textit{is strictly increasing.}$$

Then

(i) *the quasiarithmetic mean* $A^{[F]}$ *is invariant with respect to the cyclic mean-type mapping*

$$(A^{[f_1,\dots,f_p]}, A^{[f_2,\dots,f_p,f_1]}, A^{[f_3,\dots,f_p,f_1,f_2]}, \dots, A^{[f_p,f_1,\dots,f_{p-1}]}),$$

that is,

$$A^{[F]} \circ (A^{[f_1,\ldots,f_p]}, A^{[f_2,\ldots,f_p,f_1]}, \ldots, A^{[f_p,f_1,\ldots,f_{p-1}]}) = A^{[F]};$$

(ii) *the sequence* $((A^{[f_1,\ldots,f_p]}, A^{[f_2,\ldots,f_p,f_1]}, \ldots, A^{[f_p,f_1,\ldots,f_{p-1}]})^n)_{n\in\mathbb{N}}$ *of iterates converges in* I^p, *and*

$$\lim_{n\to\infty} (A^{[f_1,\ldots,f_p]}, A^{[f_2,\ldots,f_p,f_1]}, \ldots, A^{[f_p,f_1,\ldots,f_{p-1}]})^n = (A^{[F]}, \ldots, A^{[F]}).$$

Here the invariant mean $A^{[F]}$, being quasiarithmetic, is symmetric, though the coordinates of the mean-type mapping are nonsymmetric.

It turns out that the converse holds true. Namely, we have the following obvious

Remark 3. Let the assumptions of Theorem 2 be satisfied. If the invariant mean $A^{[F_1,\ldots,F_p]}$ is quasiarithmetic, then $f_i = f_{p+i}$ for $i = 1, \ldots, p-1$.

Taking $p = 2$ in Theorem 2, and setting $f_1 = f$, $f_2 = g$, $f_3 = h$, we obtain

Corollary 4. *Let* $I \subset \mathbb{R}$ *be an interval. Suppose that the functions* $f, g, h : I \to \mathbb{R}$ *are continuous increasing and such that* $f + g$ *and* $g + h$ *are strictly increasing. Then*

(i) *the mean* $A^{[f+g,g+h]}$ *is invariant with respect to the mean-type mapping* $(A^{[f,g]}, A^{[g,h]})$, *that is,*

$$A^{[f+g,g+h]} \circ (A^{[f,g]}, A^{[g,h]}) = A^{[f+g,g+h]};$$

(ii) *the sequence* $((A^{[f,g]}, A^{[g,h]})^n)_{n\in\mathbb{N}}$ *of iterates converges in* I^2, *and*

$$\lim_{n\to\infty} (A^{[f,g]}, A^{[g,h]})^n = (A^{[f+g,g+h]}, A^{[f+g,g+h]}).$$

Example 2. The functions $f, g, h : (0, \infty) \to \mathbb{R}$,

$$f(x) = \sqrt{x} - \log(1 + x), \qquad g(x) = \log(1 + x), \qquad h(x) = \sqrt{x} - \log(1 + x),$$

are continuous and strictly increasing in $I = (0, \infty)$. Thus, for all $x, y \in I$,

$$A^{[f,g]}(x, y) = \left(\sqrt{x} + \log\frac{y+1}{x+1}\right)^2, \qquad A^{[g,h]}(x, y) = \left(\sqrt{y} + \log\frac{x+1}{y+1}\right)^2,$$

$$A^{[f+g,g+h]}(x, y) = \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2.$$

By Corollary 4, the mean $A^{[f+g,g+h]}$ is $(A^{[f,g]}, A^{[g,h]})$-invariant, the sequence $((A^{[f,g]}, A^{[g,h]})^n)_{n\in\mathbb{N}}$ of iterates converges, and

$$\lim_{n\to\infty} (A^{[f,g]}, A^{[g,h]})^n = (A^{[f+g,g+h]}, A^{[f+g,g+h]}).$$

For $p = 3$, setting $f_1 = c$, $f_2 = d$, $f_3 = f$, $f_4 = g$, $f_5 = h$, we obtain

COROLLARY 5. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $c, d, f, g, h : I \to \mathbb{R}$ are continuous increasing and such that $c + d + f$, $d + f + g$ and $f + g + h$ are strictly increasing. Then*

(i) *the mean $A^{[c+d+f,d+f+g,f+g+h]}$ is invariant with respect to the mean-type mapping $(A^{[c,d,f]}, A^{[d,f,g]}, A^{[f,g,h]})$, that is,*

$$A^{[c+d+f,d+f+g,f+g+h]} \circ (A^{[c,d,f]}, A^{[d,f,g]}, A^{[f,g,h]}) = A^{[c+d+f,d+f+g,f+g+h]};$$

(ii) *the sequence $((A^{[c,d,f]}, A^{[d,f,g]}, A^{[f,g,h]})^n)_{n\in\mathbb{N}}$ of iterates converges in I^3, and*

$$\lim_{n\to\infty} (A^{[c,d,f]}, A^{[d,f,g]}, A^{[f,g,h]})^n = (A^{[c+d+f,d+f+g,f+g+h]}, A^{[c+d+f,d+f+g,f+g+h]}, A^{[c+d+f,d+f+g,f+g+h]}).$$

3. Invariant functions. In this section we prove

THEOREM 6. *Let $I \subset \mathbb{R}$ be an interval and $p \in \mathbb{N}$, $p \geq 2$. Suppose that $f_1, \ldots, f_{2p-1} : I \to \mathbb{R}$ are continuous increasing and such that*

$$F_i := \sum_{j=i}^{p+i-1} f_j \quad \textit{is strictly increasing for } i \in \{1, \ldots, p\}.$$

Assume that a function $\Phi : I^p \to \mathbb{R}$ is continuous on the diagonal Δ_p. Then Φ is $(A^{[f_1,\ldots,f_p]}, A^{[f_2,\ldots,f_{p+1}]}, \ldots, A^{[f_p,\ldots,f_{2p-1}]})$-invariant, i.e.

$$\Phi(A^{[f_1,\ldots,f_p]}, A^{[f_2,\ldots,f_{p+1}]}, \ldots, A^{[f_p,\ldots,f_{2p-1}]}) = \Phi$$

if and only if there is a continuous single variable function $\varphi : I \to \mathbb{R}$ such that

$$\Phi = \varphi \circ A^{[F_1,\ldots,F_p]}.$$

Proof. If a function Φ is $(A^{[f_1,\ldots,f_p]}, A^{[f_2,\ldots,f_{p+1}]}, \ldots, A^{[f_p,\ldots,f_{2p-1}]})$-invariant, then, by induction,

$$\Phi(x_1, \ldots, x_p) = \Phi((A^{[f_1,\ldots,f_p]}, A^{[f_2,\ldots,f_{p+1}]}, \ldots, A^{[f_p,\ldots,f_{2p-1}]})^n(x_1, \ldots, x_p))$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_p \in I$. From Theorem 2(ii), letting $n \to \infty$, and making use of the continuity of Φ on Δ_p, we obtain

$$\Phi(x_1, \ldots, x_p) = \Phi((A^{[F_1,\ldots,F_p]}, \ldots, A^{[F_1,\ldots,F_p]})(x_1, \ldots, x_p)), \quad x_1, \ldots, x_p \in I.$$

Hence, setting $\varphi(x) := \Phi(x, \ldots, x)$, $x \in I$, we get

$$\Phi(x_1, \ldots, x_p) = \varphi(A^{[F_1,\ldots,F_p]}(x_1, \ldots, x_p)), \quad x_1, \ldots, x_p \in I.$$

Since the converse implication is easy to verify, the proof is complete. ∎

For $p = 2$ setting $f_1 = f$, $f_2 = g$, $f_3 = h$, we hence get

COROLLARY 7. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $f, g, h : I \to \mathbb{R}$ are continuous, increasing and such that $f + g$ and $g + h$ are strictly increasing. Assume that $\Phi : I^p \to \mathbb{R}$ is continuous on the diagonal Δ_2. Then Φ is $(A^{[f,g]}, A^{[g,h]})$-invariant, i.e.*

$$\Phi \circ (A^{[f,g]}, A^{[g,h]}) = \Phi,$$

if and only if there is a continuous single variable function $\varphi : I \to \mathbb{R}$ such that

$$\Phi = \varphi \circ A^{[f+g,g+h]}.$$

Take $f, g, h : I \to \mathbb{R}$ as in Example 2. Applying Corollary 7 we obtain

EXAMPLE 3. Assume that a two-variable function $\Phi : I^2 \to \mathbb{R}$ is continuous at every point of the diagonal Δ_2. Then Φ satisfies the functional equation

$$\Phi\left(\left(\sqrt{x} + \log\frac{y+1}{x+1}\right)^2, \left(\sqrt{y} + \log\frac{x+1}{y+1}\right)^2\right) = \Phi(x, y), \qquad x, y \in I,$$

if and only if

$$\Phi(x, y) = \varphi\left(\left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2\right), \qquad x, y \in I,$$

where $\varphi : I \to \mathbb{R}$ is a continuous function.

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