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SEPARABLE AND FROBENIUS MONOIDAL HOM-ALGEBRAS

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Abstract. As generalizations of separable and Frobenius algebras, separable and Frobenius monoidal Hom-algebras are introduced. They are all related to the Hom-Frobenius-separability equation (HFS-equation). We characterize these two Hom-algebraic structures by the same central element and different normalizing conditions, and the structure of these two types of monoidal Hom-algebras is studied. The Nakayama automorphisms of Frobenius monoidal Hom-algebras are considered.

1. Introduction. Hom-type algebras appeared first in physical contexts, in connection with twisted, discretized or deformed derivatives and corresponding generalizations, discretizations and deformations of vector fields and differential calculus. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov [19, 20, 16] as part of a study of deformations of Witt algebras and Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map called the *Hom-Jacobi identity*,

$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0,$$

where α is a Lie algebra endomorphism. Because of their close relation to discrete and deformed vector fields and differential calculus, Hom-Lie algebras are widely studied recently: see [26, 2, 36, 38, 1, 17, 10, 30].

Hom-associative algebras play the role of associative algebras in the Hom-Lie setting. They were introduced by Makhlouf and Silvestrov [24]. Hom-associative algebras and their related structures have recently become rather popular, due to the prospect of having a general framework in which one can produce many types of natural deformations of algebras, including Hom-coassociative coalgebras, Hom-Hopf algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-Poisson algebras, Hom-Leibniz algebras, infinitesimal Hom-bialgebras, Hom-power associative algebras, quasi-triangular Hom-bialgebras (see [15, 23, 26, 37, 39, 6, 11]). Furthermore, some categories of Hom-modules on Hom-Hopf algebras are studied, such as the

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category of Hom-Hopf modules and the category of Yetter–Drinfel'd Hom-modules [13].

Makhlouf and Silvestrov [25, 26] further investigate Hom-associative algebras and Hom-coassociative coalgebras. Here the associativity of algebras and the coassociativity of coalgebras are twisted by two different endomorphisms. Hom-bialgebras are both Hom-associative algebras and Hom-coassociative coalgebras such that comultiplication and counit are morphisms of algebras. These objects are slightly different from the ones studied in this paper (see Section 2).

Frobenius extensions and separable extensions are two fundamental concepts in the theory of (non)commutative rings. The concept of Frobenius algebras is important because of its connections to such diverse areas as group representations, homology of a compact oriented manifold, topological quantum field theories, quantum cohomology, Gorenstein rings in commutative algebras, Hopf algebras, coding theory, Lie quasi-Frobenius algebras, the classical (quantum) Yang–Baxter equation (see [3, 4, 5]). In addition, there is a "quantum version" of the classical result that any finite-dimensional Hopf algebra is Frobenius. The main properties of Frobenius algebras were developed by Nakayama [28]. The Nakayama automorphism is a distinguished k-algebra automorphism of a Frobenius algebra A which measures how far A is from being a symmetric algebra, where k is a fixed field. The automorphism is the identity if and only if A is symmetric. To solve the problem of whether a symmetric algebra is independent of k, Murray [27] shows that the Nakayama automorphism of a Frobenius algebra over k is independent of k.

The existing versions of Maschke's Theorem provide the notion of separable functors (see [8]), which is applied to the category of representations. Caenepeel et al. [7] present a unified approach to the study of separable and Frobenius algebras. To do this, Frobenius-separability equation (FS-equation) is introduced, whose solution also satisfies the braided equation (which is in a sense equivalent to the quantum Yang–Baxter equation). As the main result, the structure of separable and Frobenius algebras is investigated in [9]. Naturally, we are interested in these two structures in Hom-setting. We have performed a preliminary study of monoidal Hom-Hopf algebras with Frobenius and separable property. In [11] we discuss the problem of when finite-dimensional monoidal Hom-Hopf algebras are Frobenius associated with integral spaces and describe the semisimplicity and separability of monoidal Hom-Hopf algebras by proving a Maschke type theorem for monoidal Hom-Hopf algebras. We now wish to study Frobenius monoidal Hom-algebras and separable monoidal Hom-algebras from a unifying point of view. Moreover, we want to see whether the Nakayama automorphism of a Frobenius monoidal Hom-algebras in the Hom-category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is independent of the field.

The main purpose of this paper is to study the structure of Frobenius and separable monoidal Hom-algebras related to the Hom-Frobenius-separability equation (HFS-equation). The paper is organized as follows. In Section 3, we introduce the HFS-equation and study its matrix form. In Section 4, we characterize Frobenius and separable monoidal Hom-algebras by the same central element and different normalizing conditions, and study the structure of these two types of monoidal Hom-algebras. In Section 5, we prove that the Nakayama automorphism of a Frobenius monoidal Hom-algebra in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is independent of the field k.

Throughout the paper, all vector spaces, tensor products and homomorphisms are over k. We use Sweedler's notation for coalgebras and comodules: For a coalgebra C, we write its comultiplication $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$; for a right C-comodule M, we denote its coaction by $\rho : m \mapsto m_{(0)} \otimes m_{(1)}$, for any $m \in M$, where we omit the summation symbols for convenience. The symbol τ denotes the transposition map.

Throughout this paper we freely use the Hopf algebras and coalgebras terminology introduced in [14, 29, 32, 35]. The reader is also referred to [34] for basic facts on Frobenius algebras, related Hopf algebras, and their applications.

The authors were informed by the Editor that the three papers [12, 21, 22], related to the subject of our paper, are accepted for publication.

2. Preliminaries. Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the category of k-modules. We define a new monoidal category $\mathcal{H}(\mathcal{M}_k)$. The objects of $\mathcal{H}(\mathcal{M}_k)$ are the couples (M, μ), where $M \in \mathcal{M}_k$ and $\mu \in \mathrm{Aut}_k(M)$. The morphisms of $\mathcal{H}(\mathcal{M}_k)$ are the morphisms $f : (M, \mu) \to (N, \nu)$ in $\mathcal{M}_k$ such that $\nu \circ f = f \circ \mu$. For any objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$, the monoidal structure is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu) \quad \text{and} \quad (k, \mathrm{id}).$$

Briefly, all Hom-structures are objects in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, \mathrm{id}), \widetilde{a}, \widetilde{l}, \widetilde{r})$ introduced in [6], where the associator $\widetilde{a}$ is given by the formula

$$\widetilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes \mathrm{id}) \otimes \varsigma^{-1}) = (\mu \otimes (\mathrm{id} \otimes \varsigma^{-1})) \circ a_{M,N,L}, \tag{2.1}$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$, and the unitors $\widetilde{l}$ and $\widetilde{r}$ are

$$\widetilde{l}_M = \mu \circ l_M = l_M \circ (\mathrm{id} \otimes \mu), \qquad \widetilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes \mathrm{id}).$$

The category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called the Hom-category associated to the monoidal category $\mathcal{M}_k$. A k-submodule $N \subseteq M$ is called a *subobject* of (M, μ) if μ restricts to an automorphism of N, that is, $(N, \mu|_N) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$. Since the category $\mathcal{M}_k$ has left duality, so does the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Now let $M^* = \mathrm{Hom}_k(M, k)$ be the left dual of $M \in \mathcal{M}_k$, and let $b_M : k \to M \otimes M^*$

and $d_M : M^* \otimes M \to k$ be the coevaluation and evaluation maps. Then the left dual of $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ is $(M^*, (\mu^*)^{-1})$, and the coevaluation and evaluation maps are given by the formulas

$$\widetilde{b}_M = (\mu \otimes \mu^*)^{-1} \circ b_M, \quad \widetilde{d}_M = d_M \circ (\mu^* \otimes \mu).$$

We now recall from [6] some information about Hom-structures.

Definition 2.1. A *unital monoidal Hom-associative algebra* (called a monoidal Hom-algebra in [6, Proposition 2.1]) is a vector space A together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \to A,\ a \otimes b \mapsto ab, \quad \alpha \in \mathrm{Aut}_k(A),$$

such that

$$\alpha(a)(bc) = (ab)\alpha(c), \tag{2.2}$$

$$\alpha(ab) = \alpha(a)\alpha(b), \tag{2.3}$$

$$a1_A = 1_A a = \alpha(a), \tag{2.4}$$

$$\alpha(1_A) = 1_A, \tag{2.5}$$

for all $a, b, c \in A$.

Throughout, we use the concepts of [6] for convenience. The definition of unital monoidal Hom-associative algebra is different from that of unital Hom-associative algebra in [25, 26] in the following sense. The same twisted associativity condition (2.2) holds in both cases. However, the unitality condition for unital Hom-associative algebras is the usual untwisted one: $a1_A = 1_A a = a$, for any $a \in A$, and the twisting map α does not need to be monoidal (that is, (2.3) and (2.5) are not required).

In the language of Hopf algebras, m is called *Hom-multiplication*, α is the *twisting automorphism* and 1_A is the *unit*. Let (A, α) and (A', α') be two monoidal Hom-associative algebras. A *monoidal Hom-algebra map* $f : (A, \alpha) \to (A', \alpha')$ is a linear map such that $f \circ \alpha = \alpha' \circ f$, $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

Definition 2.2. A *counital monoidal Hom-coassociative coalgebra* (called a monoidal Hom-coalgebra in [6, Proposition 2.4]) is a vector space C together with linear maps $\Delta : C \to C \otimes C, \Delta(c) = c_1 \otimes c_2$, and $\varepsilon : C \to k$ and $\gamma \in \mathrm{Aut}_k(C)$ such that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \tag{2.6}$$

$$\Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \tag{2.7}$$

$$c_1\varepsilon(c_2) = \gamma^{-1}(c) = \varepsilon(c_1)c_2, \tag{2.8}$$

$$\varepsilon(\gamma(c)) = \varepsilon(c), \tag{2.9}$$

for all $c \in C$.

Note that (2.6) is equivalent to $c_1 \otimes (c_{21} \otimes \gamma(c_{22})) = (\gamma(c_{11}) \otimes c_{12}) \otimes c_2$, which is often used in the rest of the paper. Let (C, γ) and (C', γ') be two monoidal Hom-coassociative coalgebras. A *monoidal Hom-coalgebra map* $f : (C, \gamma) \to (C', \gamma')$ is a linear map such that $f \circ \gamma = \gamma' \circ f, \Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$.

The definition of monoidal Hom-coassociative coalgebra here is somewhat different from the counital Hom-coassociative coalgebra in [25, 26]. Their coassociativity condition is twisted by some endomorphism, not necessarily by the inverse of the automorphism γ. The counitality condition is the usual untwisted one. Counital Hom-coassociative coalgebras are not monoidal, that is, (2.7) and (2.9) are not required.

With the same compatibility conditions as for Hom-bialgebras in [26], we introduce the concept of monoidal Hom-bialgebras.

DEFINITION 2.3. A *monoidal Hom-bialgebra* $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that (H, α, m, η) is a monoidal Hom-associative algebra and $(H, \alpha, \Delta, \epsilon)$ is a monoidal Hom-coassociative coalgebra such that Δ and ε are algebra maps, that is, for any $h, g \in H$,

$$\begin{aligned} \Delta(hg) &= \Delta(h)\Delta(g), & \Delta(1_H) &= 1_H \otimes 1_H, \\ \varepsilon(hg) &= \varepsilon(h)\varepsilon(g), & \varepsilon(1_H) &= 1. \end{aligned}$$

REMARK 2.4. The definition of Hom-bialgebra proposed in [25] contains two different endomorphisms governing the Hom-associativity and Hom-coassociativity. For any bialgebra $(H, m, \eta, \Delta, \varepsilon)$, and any bialgebra endomorphism α of H, the authors in [25] show that $(H, \alpha, \alpha \circ m, \eta, \Delta \circ \alpha, \varepsilon)$ is a Hom-bialgebra in their sense. In our case, there is a monoidal Hom-bialgebra $(H, \alpha, \alpha \circ m, \eta, \Delta \circ \alpha^{-1}, \varepsilon)$, provided that $\alpha : H \to H$ is a bialgebra automorphism. The superiority of our definition is that these objects in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ are self-dual.

DEFINITION 2.5. A monoidal Hom-bialgebra (H, α) is a *monoidal Hom-Hopf algebra* if there exists a morphism (called *antipode*) $S : H \to H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e. $S \circ \alpha = \alpha \circ S$) such that

$$S * \mathrm{id} = \eta \circ \varepsilon = \mathrm{id} * S.$$

Note that a monoidal Hom-Hopf algebra is by definition a Hopf algebra in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Further, the antipode of monoidal Hom-Hopf algebras has almost all the properties of the antipode of Hopf algebras such as

$$\begin{aligned} S(hg) &= S(g)S(h), & S(1_H) &= 1_H, \\ \Delta(S(h)) &= S(h_2) \otimes S(h_1), & \varepsilon \circ S &= \varepsilon. \end{aligned}$$

That is, S is a monoidal Hom-anti-(co)algebra homomorphism. Since α is bijective and commutes with the antipode S, we can also see that the inverse α^{-1} commutes with S, that is, $S \circ \alpha^{-1} = \alpha^{-1} \circ S$.

For a finite-dimensional monoidal Hom-Hopf algebra $(H, \alpha, m, \eta, \Delta, \varepsilon, S)$, the dual $(H^*, (\alpha^*)^{-1})$ is also a monoidal Hom-Hopf algebra with the following structure: for all $g, h \in H$ and $\phi, \varphi \in H^*$,

$$\begin{aligned} &\langle \phi\varphi, h\rangle = \langle \phi, h_1\rangle\langle \varphi, h_2\rangle, && 1_{H^*} = \varepsilon, \\ &\langle \Delta(\phi), g \otimes h\rangle = \langle \phi, gh\rangle, && \varepsilon_{H^*} = \eta, \\ &(\alpha^*)^{-1}(\phi) = \phi \circ \alpha^{-1}, && S^*(\phi) = \phi \circ S^{-1}. \end{aligned}$$

In the following, we recall the actions of monoidal Hom-associative algebras and coactions of monoidal Hom-coassociative coalgebras.

Definition 2.6. Let (A, α) be a unital monoidal Hom-associative algebra. A *left (A, α)-Hom-module* consists of (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \to M$, $\psi(a \otimes m) = a \cdot m$, such that

$$\begin{gathered} \alpha(a) \cdot (b \cdot m) = (ab) \cdot \mu(m), \\ \mu(a \cdot m) = \alpha(a) \cdot \mu(m), \quad 1_A \cdot m = \mu(m), \end{gathered}$$

for all $a, b \in A$ and $m \in M$.

A monoidal Hom-associative algebra (A, α) can be considered as a Hom-module over itself via Hom-multiplication. Let $(M, \mu), (N, \nu)$ be two left (A, α)-Hom-modules. A map $f : (M, \mu) \to (N, \nu)$ is called a *morphism of left (A, α)-Hom-modules* if $f(a \cdot m) = a \cdot f(m)$ and $f \circ \mu = \nu \circ f$. We denote by $\widetilde{\mathcal{H}}({}_A\mathcal{M})$ the category of left (A, α)-Hom-modules and left (A, α)-linear morphisms between them.

If (M, μ) is both a left (A, α)-Hom-module and a right (A, α)-Hom-module such the compatibility condition

$$\alpha(a) \cdot (m \cdot b) = (a \cdot m) \cdot \alpha(b) \tag{2.10}$$

holds, then (M, μ) is called an *(A, α)-Hom-bimodule.*

Dually, we can define Hom-comodules. Now let (C, γ) be a monoidal Hom-coassociative coalgebra.

Definition 2.7. A *right (C, γ)-Hom-comodule* is an object (M, μ) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that

$$\begin{aligned} &\mu^{-1}(m_{(0)}) \otimes \triangle_C(m_{(1)}) = (m_{(0)(0)} \otimes m_{(0)(1)}) \otimes \gamma^{-1}(m_{(1)}), \\ &\rho_M(\mu(m)) = \mu(m_{(0)}) \otimes \gamma(m_{(1)}), \quad m_{(0)}\varepsilon(m_{(1)}) = \mu^{-1}(m), \end{aligned} \tag{2.11}$$

for all $m \in M$.

Note that (2.11) can be equivalently restated as $m_{(0)} \otimes (m_{(1)1} \otimes \gamma(m_{(1)2})) = (\mu(m_{(0)(0)}) \otimes m_{(0)(1)}) \otimes m_{(1)}$. (C, γ) is a Hom-comodule over itself via Hom-comultiplication. Let (M, μ) and (N, ν) be two right (C, γ)-Hom-comodules. A map $g : (M, \nu) \to (N, \nu)$ is called a *right* (C, γ)*-Hom-comodule morphism* if $g \circ \mu = \nu \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$. We denote by $\widetilde{\mathcal{H}}(\mathcal{M}^C)$ the category of right (C, γ)-Hom-comodules and right (C, γ)-colinear morphisms between them.

3. Hom-Frobenius-separability equation. In this section, we introduce the HFS-equation and show that it implies the Hom-braided equation. Also we study the matrix form of the HFS-equation.

For any monoidal Hom-algebra (A, α), $(A \otimes A, \alpha \otimes \alpha)$ can be considered as an (A, α)-Hom-bimodule with the actions

$$a \rhd (b \otimes c) = ab \otimes \alpha(c), \qquad (b \otimes c) \lhd a = \alpha(b) \otimes ca,$$

for all $a, b, c \in A$.

Indeed, under the action $a \rhd (b \otimes c) = ab \otimes \alpha(c)$, $(A \otimes A, \alpha \otimes \alpha)$ is a left (A, α)-Hom-module. Explicitly,

$$\begin{aligned}
\alpha(a) \rhd (b \rhd (c \otimes d)) &= \alpha(a) \rhd (bc \otimes \alpha(d)) = \alpha(a)(bc) \otimes \alpha^2(d) \\
&= (ab)\alpha(c) \otimes \alpha^2(d) = (ab) \rhd (\alpha(c) \otimes \alpha(d)) \\
&= (ab) \rhd (\alpha^{\otimes 2}(c \otimes d)), \\
1_A \rhd (c \otimes d) = 1_A &\rhd (c \otimes d) = 1_A c \otimes \alpha(d) = \alpha^{\otimes 2}(c \otimes d),
\end{aligned}$$

for all $a, b, c, d \in A$. Similarly, $(A \otimes A, \alpha \otimes \alpha)$ is a right (A, α)-Hom-module with the action $(b \otimes c) \lhd a = \alpha(b) \otimes ca$. Moreover, the compatibility condition (2.10) holds, that is,

$$(a \rhd (b \otimes c)) \lhd \alpha(d) = \alpha(ab) \otimes \alpha(cd) = \alpha(a) \rhd ((b \otimes c) \lhd d).$$

So $(A \otimes A, \alpha \otimes \alpha)$ is an (A, α)-Hom-bimodule.

An $\alpha^{\otimes 2}$-invariant element $e = \sum e^1 \otimes e^2 \in A \otimes A$ will be called (A, α)-*central* if for any $a \in A$,

$$a \rhd e = e \lhd a.$$

Explicitly, e satisfies the following two conditions:

$$(\alpha \otimes \alpha)(e) = e$$

and

$$\sum ae^1 \otimes \alpha(e^2) = \sum \alpha(e^1) \otimes e^2 a. \tag{3.1}$$

For an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$, let $\mathcal{R} : M \otimes M \to M \otimes M$ be a linear map and consider the maps $\mathcal{R}^{12}, \mathcal{R}^{13}, \mathcal{R}^{23} : M \otimes M \otimes M \to M \otimes M \otimes M$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ given by the formulas

$$\mathcal{R}^{12} = \mathcal{R} \otimes \mu, \qquad \mathcal{R}^{23} = \mu \otimes \mathcal{R}, \qquad \mathcal{R}^{13} = (\mathrm{id} \otimes \tau) \circ (\mathcal{R} \otimes \mu) \circ (\mathrm{id} \otimes \tau).$$

A similar notation will be used for elements $e \in A \otimes A$, where (A, α) is a monoidal Hom-algebra with unit 1_A. If $e = \sum e^1 \otimes e^2$, then

$$e^{12} = \sum e^1 \otimes e^2 \otimes 1_A, \quad e^{13} = \sum e^1 \otimes 1_A \otimes e^2, \quad e^{23} = \sum 1_A \otimes e^1 \otimes e^2.$$

Definition 3.1. Let $(A, m_A, 1_A, \alpha)$ be a Hom-algebra and let $\mathcal{R} = \sum \mathcal{R}^1 \otimes \mathcal{R}^2 \in A \otimes A$.

(1) $\mathcal{R}$ is called a solution of *the Hom-Frobenius-separability equation* (or *HFS-equation* for brevity) if
$$\mathcal{R}^{12}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13} = \mathcal{R}^{13}\mathcal{R}^{12}$$
in $A \otimes A \otimes A$.

(2) $\mathcal{R}$ is called a solution of *the Hom-separability equation* (or *HS-equation*) if $\mathcal{R}$ is a solution of the HFS-equation and satisfies the normalizing separability condition
$$\sum \mathcal{R}^1\mathcal{R}^2 = 1_A.$$

(3) $(\mathcal{R}, \varphi)$ is called a solution of *the Hom-Frobenius equation* (or *HF-equation*) if $\mathcal{R}$ is a solution of the HFS-equation and there exists an element $\varphi \in A^*$ such that the normalizing Frobenius condition holds:
$$\sum \varphi(\mathcal{R}^1)\mathcal{R}^2 = \sum \mathcal{R}^1\varphi(\mathcal{R}^2) = 1_A.$$

(4) $\mathcal{R}$ is called a solution of *the Hom-braided equation* if
$$(\mathcal{R}^{12}\mathcal{R}^{23})\alpha^{\otimes 3}(\mathcal{R}^{12}) = \alpha^{\otimes 3}(\mathcal{R}^{23})(\mathcal{R}^{12}\mathcal{R}^{23}).$$

Note that if $\mathcal{R}$ is a solution of the HFS-equation then
$$\begin{aligned}(\mathcal{R}^{12}\mathcal{R}^{23})\alpha^{\otimes 3}(\mathcal{R}^{12}) &= (\mathcal{R}^{23}\mathcal{R}^{13})\alpha^{\otimes 3}(\mathcal{R}^{12}) = \alpha^{\otimes 3}(\mathcal{R}^{23})(\mathcal{R}^{13}\mathcal{R}^{12}) \\ &= \alpha^{\otimes 3}(\mathcal{R}^{23})(\mathcal{R}^{12}\mathcal{R}^{23}),\end{aligned}$$
so $\mathcal{R}$ is a solution of the Hom-braided equation.

In addition, if e is (A, α)-central, then writing $E = \sum E^1 \otimes E^2$ as another copy of e, we have
$$\begin{aligned}e^{12}e^{23} &= \sum (e^1 \otimes e^2 \otimes 1_A)(1_A \otimes E^1 \otimes E^2) \\ &= \sum \alpha(e^1) \otimes e^2E^1 \otimes \alpha(E^2) \\ &\overset{(3.1)}{=} \sum E^1e^1 \otimes \alpha(e^2) \otimes \alpha(E^2) = e^{13}e^{12},\end{aligned}$$
and
$$\begin{aligned}e^{12}e^{23} &= \sum \alpha(e^1) \otimes e^2E^1 \otimes \alpha(E^2) \\ &\overset{(3.1)}{=} \sum \alpha(e^1) \otimes \alpha(E^1) \otimes E^2e^2 = e^{23}e^{13},\end{aligned}$$
so e is a solution of the HFS-equation.

For any monoidal Hom-algebra (A,α), there is another (A,α)-Hom-bimodule structure on $(A\otimes A,\alpha\otimes\alpha)$ given by

$$a\to(b\otimes c)=\alpha^{-1}(a)b\otimes\alpha(c),$$
$$(a\otimes b)\leftarrow c=\alpha(a)\otimes b\alpha^{-1}(c),$$

for any $a,b,c\in A$.

PROPOSITION 3.2. *Let $(A,m_A,1_A,\alpha)$ be a monoidal Hom-algebra and let $\Delta: A\to A\otimes A$ be an (A,α)-Hom-bimodule map with the left action $\to$ and right action $\leftarrow$. Set $e=\Delta(1_A)$. Then*

(1) *e is $\alpha^{\otimes 2}$-invariant and a solution of the HFS-equation.*
(2) *Δ is Hom-coassociative.*
(3) *If $(A,\Delta,\varepsilon,\alpha)$ is a monoidal Hom-coassociative coalgebra structure on A, then A is finitely generated and projective in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$.*

Proof. Firstly,

$$e=\Delta(1_A)=\Delta(\alpha(1_A))=(\alpha\otimes\alpha)(\Delta(1_A))=(\alpha\otimes\alpha)(e),$$

so e is $\alpha^{\otimes 2}$-invariant. Secondly, since Δ is an (A,α)-Hom-bimodule map, we obtain

$$\Delta(\alpha^2(a))=\Delta(1_A\alpha(a))=\Delta(1_A)\leftarrow\alpha(a)=e\lhd a,$$
$$\Delta(\alpha^2(a))=\Delta(\alpha(a)1_A)=\alpha(a)\to\Delta(1_A)=a\rhd e.$$

So $e\lhd a=a\rhd e$, that is, e is (A,α)-central. Thus e is a solution of the HFS-equation.

(2) From [11, proof of Proposition 5.2],

$$\Delta(a)=\alpha(e^1)\otimes e^2\alpha^{-2}(a)=\alpha^{-2}(a)e^1\otimes\alpha(e^2)$$

is Hom-coassociative.

(3) For all $a\in A$, applying $\varepsilon\otimes\mathrm{id}$ and $\mathrm{id}\otimes\varepsilon$ to $\Delta(\alpha(a))$,

$$a=(\varepsilon\otimes\mathrm{id})\circ\Delta(\alpha(a))=\sum(\varepsilon\otimes\mathrm{id})(\alpha^{-1}(a)e^1\otimes\alpha(e^2))=\sum\varepsilon(ae^1)e^2,$$
$$a=(\mathrm{id}\otimes\varepsilon)\circ\Delta(\alpha(a))=\sum(\mathrm{id}\otimes\varepsilon)(\alpha(e^1)\otimes e^2\alpha^{-1}(a))=\sum\varepsilon(e^2a)e^1,$$

which implies that $\{e^1,\varepsilon(e^2\cdot)\}$ or $\{e^2,\varepsilon(\cdot\, e^1)\}$ are dual bases of (A,α) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. ∎

Let (H,α) be a monoidal Hom-Hopf algebra with antipode S. It follows from [11, Section 4] that $\mathcal{R}=\sum t_1\otimes S(t_2)\in H\otimes H$ is (H,α)-central, where $t\in H$ is a left integral for (H,α). Therefore it is a solution of the HFS-equation and Hom-braided equation.

Observe that if t is a right integral for a monoidal Hom-Hopf algebra (H,α), then a similar argument can show that $S(t_1)\otimes t_2$ is also a solution of the HFS-equation.

Proposition 3.3. *Let (A,α) be a monoidal Hom-algebra, and $e\in A\otimes A$ an (A,α)-central element. Then for any left (A,α)-Hom-module (M,μ), the map $\mathcal{R}=\mathcal{R}_e : M\otimes M\to M\otimes M$ given by*

$$\mathcal{R}(m\otimes n)=\sum e^1\cdot m\otimes e^2\cdot n$$

is a solution of the HFS-equation in $\mathrm{End}_k(M\otimes M\otimes M)$.

Proof. For all $m,l,n\in M$, we have

$$\begin{aligned}\mathcal{R}^{12}\mathcal{R}^{23}(m\otimes l\otimes n)&=\mathcal{R}^{12}(\mu(m)\otimes e^1\cdot l\otimes e^2\cdot n)\\&=E^1\cdot\mu(m)\otimes E^2\cdot(e^1\cdot l)\otimes\mu(e^2\cdot n)\\&=\alpha(E^1)\cdot\mu(m)\otimes\alpha(E^2)\cdot(e^1\cdot l)\otimes\alpha(e^2)\cdot\mu(n)\\&=\alpha(E^1)\cdot\mu(m)\otimes(E^2e^1)\cdot\mu(l)\otimes\alpha(e^2)\cdot\mu(n).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{R}^{13}\mathcal{R}^{12}(m\otimes l\otimes n)&=(E^1e^1)\cdot\mu(m)\otimes\alpha(e^2)\cdot\mu(l)\otimes\alpha(E^2)\cdot\mu(n),\\\mathcal{R}^{23}\mathcal{R}^{13}(m\otimes l\otimes n)&=\alpha(e^1)\cdot\mu(m)\otimes\alpha(E^1)\cdot\mu(l)\otimes(E^2e^2)\cdot\mu(n).\end{aligned}$$

Since e is (A,α)-central, we know it is a solution to the HFS-equation, that is, $\alpha(E^1)\otimes(E^2e^1)\otimes\alpha(e^2)=(E^1e^1)\otimes\alpha(e^2)\otimes\alpha(E^2)=\alpha(e^1)\otimes\alpha(E^1)\otimes(E^2e^2)$, which implies that $\mathcal{R}^{12}\mathcal{R}^{23}=\mathcal{R}^{13}\mathcal{R}^{12}=\mathcal{R}^{23}\mathcal{R}^{13}$. ∎

In particular, we have the following corollary.

Corollary 3.4. *If (A,α) is a monoidal Hom-algebra and $e\in A\otimes A$ is an (A,α)-central element, then $\mathcal{R}_e : A\otimes A\to A\otimes A, \mathcal{R}_e(a\otimes b)=\sum e^1a\otimes e^2b$, is a solution of the HFS-equation. Moreover, if e is a separable idempotent element (resp. (e,ε) is a Frobenius pair), then $\mathcal{R}_e$ is a solution of the HS-equation (resp. HF-equation).*

In the following, we introduce the notation of matrix Hom-algebras and comatrix Hom-coalgebras. Then we can study the HFS-equation in the form of matrix equations.

Let $(M,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ and M be a finite-dimensional vector space with a basis $\{e_1,\dots,e_n\}$, and $\{f^1,\dots,f^n\}$ the corresponding basis of the dual space M^* with $\langle f^i,e_j\rangle=\delta^i_j$ for all $i,j\in\{1,\dots,n\}$. Then $\{a^i_j=f^i\otimes e_j\mid i,j=1,\dots,n\}$ and $\{c^i_j=e_j\otimes f^i\mid i,j=1,\dots,n\}$ are bases for respectively $\mathrm{End}_k(M)\cong M^*\otimes M$ and $\mathrm{End}_k(M^*)\cong M\otimes M^*$. The isomorphisms are given by the formula

$$a^i_j(e_k)=\delta^i_ke_j,\qquad c^i_j(f^k)=\delta^k_jf^i.$$

Then there is a monoidal Hom-associative algebra structure on $(M^*\otimes M,\alpha=$

$\mu^* \otimes \mu)$ given by

$$a^i_j a^k_l = \delta^i_l \alpha(a^k_j), \qquad \sum_{i=1}^n a^i_i = 1, \tag{3.2}$$

which is isomorphic to the $n \times n$-matrix algebra $\mathcal{M}_n(k)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, i.e., *the $n \times n$-matrix Hom-algebra* $\widetilde{\mathcal{H}}(\mathcal{M}_n(k))$, where α is an algebra automorphism. Moreover there is a monoidal Hom-coassociative coalgebra structure on $(M \otimes M^*, \gamma = \mu \otimes \mu^*)$ given by

$$\Delta(c^i_j) = \sum_k \gamma^{-1}(c^i_k) \otimes \gamma^{-1}(c^k_j), \qquad \varepsilon(c^i_j) = \delta^i_j,$$

which is isomorphic to *the $n \times n$-comatrix Hom-coalgebra* $\widetilde{\mathcal{H}}(\mathcal{M}^n(k))$, where γ is a coalgebra automorphism.

A linear map $\mathcal{R} : M \otimes M \to M \otimes M$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ can be described by a matrix, with n^4 entries $x^{ij}_{kl} \in k$, where $i, j, k, l \in \{1, \dots, n\}$. This means that

$$\mathcal{R}(m_k \otimes m_l) = \sum_{i,j} x^{ij}_{kl} m_i \otimes m_j, \tag{3.3}$$

or

$$\mathcal{R} = \sum_{i,j,k,l} x^{ij}_{kl} e^k_i \otimes e^l_j. \tag{3.4}$$

In the following, we use the Einstein summation convention as in [9, Section 5.1]. In summations, all indices run from 1 to n. If an index occurs twice or more in an expression, then it is understood implicitly that we take the sum where this index runs from 1 to n, so we often omit the summation symbol. Indices that occur only once are not summation indices, and an index is not allowed to occur more than twice in one expression.

Furthermore, we can represent the automorphism μ by a matrix, with n^2 entries $z^j_i \in k$. Explicitly,

$$\mu(m_i) = \sum_j z^j_i m_j, \tag{3.5}$$

where $i, j \in \{1, \dots, n\}$. Then it is straightforward to compute

$$\mathcal{R}^{12}\mathcal{R}^{23}(m_u \otimes m_v \otimes m_w) = z^r_u z^k_t x^{st}_{vw} x^{ij}_{rs} m_i \otimes m_j \otimes m_k,$$
$$\mathcal{R}^{23}\mathcal{R}^{13}(m_u \otimes m_v \otimes m_w) = z^s_v z^i_r x^{rt}_{uw} x^{jk}_{st} m_i \otimes m_j \otimes m_k,$$
$$\mathcal{R}^{13}\mathcal{R}^{12}(m_u \otimes m_v \otimes m_w) = z^t_w z^j_s x^{rs}_{uv} x^{ik}_{rt} m_i \otimes m_j \otimes m_k.$$

Now, the HFS-equation can be rewritten as the following matrix equation.

Proposition 3.5. *Let $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ with a basis $\{m_1, \dots, m_n\}$, and let $\mathcal{R} \in \mathrm{End}_k(M \otimes M)$ and μ be given by* (3.3) *and* (3.5) *respectively.*

(1) $\mathcal{R}$ *is a solution of the HFS-equation if and only if*

$$z_u^r z_t^k x_{vw}^{st} x_{rs}^{ij} = z_v^s z_r^i x_{uw}^{rt} x_{st}^{jk} = z_w^t z_s^j x_{uv}^{rs} x_{rt}^{ik}$$

for all $u, v, w, i, j, k \in \{1, \dots, n\}$.

(2) $\mathcal{R}$ *satisfies the separability condition if and only if* $x_{kv}^{ij} z_i^k = \delta_k^j$.

(3) $\mathcal{R}$ *satisfies the Frobenius condition if and only if* $x_{ki}^{kj} = x_{ik}^{jk} = \delta_i^j$.

Proof. (1) This is obvious from the above analysis.

(2) and (3) follow from (3.4) by using the multiplication rule (3.2) and the formula $\varepsilon(c_j^i) = \delta_j^i$. ∎

EXAMPLE 3.6. (1) Let (A, α) be a monoidal Hom-algebra. If an element $a \in A$ satisfies the condition $a^2 = \alpha(a)$, then $a \otimes a$ is a solution of the HFS-equation.

(2) Let $(A, \alpha) = \widetilde{\mathcal{H}}(\mathcal{M}_n(k))$, and let $(e_i^j)_{1 \le i,j \le n}$ be the basis, where $\alpha(e_i^j) = -e_i^j$. Then

$$\mathcal{R} = \sum_{i=1}^{n} e_i^j \otimes e_j^i$$

is a solution of the HS-equation, but not a solution of the HF-equation.

4. The structure of separable and Frobenius monoidal Hom-algebras. In this section, we introduce separable monoidal Hom-algebras and Frobenius monoidal Hom-algebras, and describe them by the same (A, α)-central element and different normalizing conditions. Any (A, α)-central element is a solution of the HFS-equation. As the main result of this section, we show that any solution of the HFS-equation arises in this way. In the following, we assume that (A, α) is a finite-dimensional monoidal Hom-algebra.

DEFINITION 4.1. (1) A monoidal Hom-algebra $(A, m, 1_A, \alpha)$ is called *separable* if there exists a separability idempotent element, that is, an (A, α)-central element $e = \sum e^1 \otimes e^2 \in A \otimes A$ satisfying the *normalizing separability condition*

$$\sum e^1 e^2 = 1_A.$$

(2) A monoial Hom-algebra $(A, m, 1_A, \alpha)$ is called *Frobenius* if there exists a coalgebra structure (A, Δ, ε) such that the comultiplication $\Delta : A \to A \otimes A$ is an (A, α)-Hom-bimodule map.

We note that a monoidal Hom-algebra (A, m, α) is separable if and only if the Hom-multiplication m splits in the category of (A, α)-Hom-bimodules (see [11, Section 4]). A finite-dimensional monoidal Hom-algebra (A, α) is Frobenius if and only if $(A, \alpha) \cong (A^*, (\alpha^*)^{-1})$ as left (or right) (A, α)-Hom-modules if and only if there exists a Frobenius structure of (A, α) if and only if there exists a Hom-associative, nondegenerate bilinear form $B : (A, A) \to k$ for (A, α) (see [11, Proposition 5.2]).

By defining $\lambda : (A, \alpha) \to k, \lambda(a) := B(a, 1) = B(1, a)$ and $B(a, b) := \lambda(ab)$, it is not difficult to show that the above conditions are all equivalent to the assertion that there exists a morphism λ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ whose kernel contains no nonzero Hom-ideals.

If the bilinear form B is symmetric, that is, $B(a, b) = B(b, a)$ for any $a, b \in A$, then (A, α) is called *symmetric*.

Furthermore, we will study the relation between Frobenius monoidal Hom-algebras and the HF-equation.

PROPOSITION 4.2. *Let (A, α) be a monoidal Hom-algebra. Then (A, α) is Frobenius if and only if there exist $e = \sum e^1 \otimes e^2 \in A \otimes A$ and $\varphi \in A^*$ such that (e, φ) is a solution of the HF-equation.*

Proof. Firstly, the property of counit is satisfied if and only if the normalizing Frobenius condition holds.

Next, from the fact that $\Delta : A \to A \otimes A$ is an (A, α)-Hom-bimodule map, set $e = \Delta(1_A)$. Then e is (A, α)-central by the proof of the first part of Proposition 3.2. So e is a solution of the HFS-equation. Conversely, the (A, α)-central element determines the (A, α)-Hom-bimodule map Δ and it is Hom-coassociative by Proposition 3.2(2). ■

Recall that any finite-dimensional monoidal Hom-Hopf algebra is semisimple if and only if it is separable (see [11, Theorem 4.6]). We will see below that a Frobenius monoidal Hom-algebra is separable under some assumption.

PROPOSITION 4.3. *Let (A, α) be a Frobenius monoidal Hom-algebra such that $w = m_A \circ \Delta(1_A)$ is invertible. Then (A, α) is a separable monoidal Hom-algebra.*

Proof. Let $e = \Delta(1_A) = e^1 \otimes e^2$. Then e is (A, α)-central of (A, α). So,

$$(e^1 e^2)a = \alpha(e^1)(e^2 \alpha^{-1}(a)) = (\alpha^{-1}(a) e^1)\alpha(e^2) = a(e^1 e^2),$$

that is, $w \in Z(A)$, the center of (A, α). And the $\alpha^{\otimes 2}$-invariance of e implies that w is α-invariant too. It follows that its inverse w^{-1} is also α-invariant and also in $Z(A)$. Now set

$$\mathcal{R} = w^{-1} e^1 \otimes \alpha(e^2),$$

which is a separability idempotent. Indeed, for any $a \in A$,

$$\begin{aligned} a(w^{-1}e^1) \otimes \alpha^2(e^2) &= (\alpha^{-1}(a) w^{-1})\alpha(e^1) \otimes \alpha^2(e^2) \\ &= (w^{-1}\alpha^{-1}(a))\alpha(e^1) \otimes \alpha^2(e^2) \end{aligned}$$

$$
\begin{aligned}
&= \alpha(w^{-1})(\alpha^{-1}(a)e^1) \otimes \alpha(\alpha(e^2))\\
&= \alpha(w^{-1})\alpha(e^1) \otimes \alpha(e^2\alpha^{-1}(a))\\
&= \alpha(w^{-1}e^1) \otimes \alpha(e^2)a,
\end{aligned}
$$

which shows that $\mathcal{R}$ is (A,α)-central. Moreover, the normalizing separability condition holds: $(w^{-1}e^1)\alpha(e^2) = w^{-1}(e^1e^2) = 1_A$. Thus (A,α) is a separable monoidal Hom-algebra. ∎

We have seen that, for a monoidal Hom-algebra (A,α), any (A,α)-central element $\mathcal{R} \in A \otimes A$ is a solution of the HFS-equation. Conversely, in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, any $\alpha^{\otimes 2}$-invariant solution of the HFS-equation arises in this way.

THEOREM 4.4. *Let (A,α) be a monoidal Hom-algebra, and suppose an $\alpha^{\otimes 2}$-invariant element $\mathcal{R} = \sum \mathcal{R}^1 \otimes \mathcal{R}^2 \in A \otimes A$ is a solution of the HFS-equation.*

1. *There exists a monoidal sub-Hom-algebra $\mathcal{A}(\mathcal{R})$ of (A,α) such that $\mathcal{R} \in \mathcal{A}(\mathcal{R}) \otimes \mathcal{A}(\mathcal{R})$ and $\mathcal{R}$ is $\mathcal{A}(\mathcal{R})$-central.*
2. *$(\mathcal{A}(\mathcal{R}),\mathcal{R})$ satisfies the following universal property: if (B,β) is a monoidal Hom-algebra, and $e \in B \otimes B$ is (B,β)-central, then any Hom-algebra map $f : (B,\beta) \to (A,\alpha)$ with $(f \otimes f)(e) = \mathcal{R}$ factors through a Hom-algebra map $\widetilde{f} : B \to \mathcal{A}(\mathcal{R})$.*
3. *If $\mathcal{R} \in A \otimes A$ is a solution of the HS-equation (resp. HF-equation), then $\mathcal{A}(\mathcal{R})$ is a separable (resp. Frobenius) monoidal Hom-algebra.*

Proof. (1) Let $\mathcal{A}(\mathcal{R}) = \{a \in A \mid a \triangleright \mathcal{R} = \mathcal{R} \triangleleft a\}$ and let the automorphism on $\mathcal{A}(\mathcal{R})$ be α restricted to $\mathcal{A}(\mathcal{R})$. Obviously, $1_A \in \mathcal{A}(\mathcal{R})$. For any $a, b \in \mathcal{A}(\mathcal{R})$, we have

$$
\begin{aligned}
(ab) \triangleright \mathcal{R} &= (ab)\mathcal{R}^1 \otimes \alpha(\mathcal{R}^2) = \alpha(a)(b\alpha^{-1}(\mathcal{R}^1)) \otimes \alpha(\mathcal{R}^2)\\
&= \alpha(a)(b\mathcal{R}^1) \otimes \alpha(\alpha(\mathcal{R}^2)) = \alpha(a)\alpha(\mathcal{R}^1) \otimes \alpha(\mathcal{R}^2 b)\\
&= \alpha(a\mathcal{R}^1) \otimes \alpha(\mathcal{R}^2)\alpha(b) = \alpha(\alpha(\mathcal{R}^1)) \otimes (\mathcal{R}^2 a)\alpha(b)\\
&= \alpha^2(\mathcal{R}^1) \otimes \alpha(\mathcal{R}^2)(ab) = \alpha(\mathcal{R}^1) \otimes \mathcal{R}^2(ab) = \mathcal{R} \triangleleft (ab),
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha(a) \triangleright \mathcal{R} &= \alpha(a)\mathcal{R}^1 \otimes \alpha(\mathcal{R}^2) = \alpha(a\mathcal{R}^1) \otimes \alpha(\alpha(\mathcal{R}^2))\\
&= \alpha(\alpha(\mathcal{R}^1)) \otimes \alpha(\mathcal{R}^2 a) = \alpha(\mathcal{R}^1) \otimes \mathcal{R}^2\alpha(a)\\
&= \mathcal{R} \triangleleft \alpha(a),
\end{aligned}
$$

so $ab, \alpha(a) \in \mathcal{A}(\mathcal{R})$, which implies that $(\mathcal{A}(\mathcal{R}), \alpha|_{\mathcal{A}(\mathcal{R})})$ is a monoidal sub-Hom-algebra of (A,α).

Next, we have to show that $\mathcal{R} \in \mathcal{A}(\mathcal{R}) \otimes \mathcal{A}(\mathcal{R})$. Consider the map $\varphi : A \to A \otimes A^{\mathrm{op}}$ given by

$$
\varphi(a) = a \triangleright \mathcal{R} - \mathcal{R} \triangleleft a = a\mathcal{R}^1 \otimes \alpha(\mathcal{R}^2) - \alpha(\mathcal{R}^1) \otimes \mathcal{R}^2 a.
$$

Then we find that $\mathcal{A}(\mathcal{R}) = \ker\varphi$. Since A is flat as a k-module, we have

$$\mathcal{A}(\mathcal{R}) \otimes A = \ker(\varphi \otimes \alpha).$$

Now,

$$\begin{aligned}(\varphi\otimes\alpha)(\mathcal{R}) &= \sum \varphi(\mathcal{R}^1)\otimes\alpha(\mathcal{R}^2)\\ &= \sum \mathcal{R}^1 r^1 \otimes \alpha(r^2)\otimes\alpha(\mathcal{R}^2) - \alpha(r^1)\otimes r^2\mathcal{R}^1\otimes\alpha(\mathcal{R}^2)\\ &= \mathcal{R}^{13}\mathcal{R}^{12} - \mathcal{R}^{12}\mathcal{R}^{23} = 0,\end{aligned}$$

where $r = \sum r^1 \otimes r^2$ is another copy of $\mathcal{R}$. Thus $\mathcal{R} \in \ker(\varphi\otimes\alpha) = \mathcal{A}(\mathcal{R})\otimes A$. Similarly, we also have $\mathcal{R} \in A\otimes\mathcal{A}(\mathcal{R})$ from the fact $\mathcal{R}^{12}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}$. So $\mathcal{R} \in \mathcal{A}(\mathcal{R})\otimes\mathcal{A}(\mathcal{R})$. Thus $\mathcal{R}$ is an $\mathcal{A}(\mathcal{R})$-central element of $\mathcal{A}(\mathcal{R})\otimes\mathcal{A}(\mathcal{R})$.

(2) For any $b \in B$, applying $f\otimes f$ to the equality $b \triangleright e = e \triangleleft b$, we have $f(b)\triangleright\mathcal{R} = \mathcal{R}\triangleleft f(b)$ since $(f\otimes f)(e) = \mathcal{R}$. So $\operatorname{Im} f \subseteq \mathcal{A}(\mathcal{R})$, which yields the commutative diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\widetilde{f}"'] & A \\ \mathcal{A}(\mathcal{R}) \arrow[ur, "i"'] & \end{tikzcd}$$

that is, we obtain the universal property of $(\mathcal{A}(\mathcal{R}), \mathcal{R})$.

(3) The first statement follows from the definition of separable monoial Hom-algebra and the second one is also true by Proposition 4.2. ∎

Let (A, α) be a monoidal Hom-algebra. If A is finite-dimensional, then we can describe the monoidal Hom-algebra $\mathcal{A}(\mathcal{R})$ using generators and relations. Let $\{m_1, \ldots, m_n\}$ be a basis of a finite-dimensional vector space M in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. Suppose that $\mathcal{R}$ is a solution of the HFS-equation satisfying (3.3). Identifying $\mathrm{End}_k(M)$ with $\widetilde{\mathcal{H}}(M_n(k))$, we will write $\mathcal{A}(n, \mathcal{R})$ for the monoidal sub-Hom-algebra of $\widetilde{\mathcal{H}}(M_n(k))$ corresponding to $\mathcal{A}(\mathcal{R})$.

Now we have the main result of this paper.

THEOREM 4.5. *Let (A, α) be an n-dimensional monoidal Hom-algebra. Then the following statements are equivalent:*

1. *A^α is a separable (resp. Frobenius) monoidal Hom-algebra, where $A^\alpha = \{a \mid \alpha(a) = a\}$.*
2. *There exists a Hom-algebra isomorphism*

$$A^\alpha \cong \mathcal{A}(n, \mathcal{R}),$$

where $\mathcal{R} = (x^{ij}_{uv}) = \widetilde{\mathcal{H}}(M_n(k))\otimes\widetilde{\mathcal{H}}(M_n(k)) \cong \mathrm{End}_k(A)\otimes\mathrm{End}_k(A)$ is a solution of the HFS-equation.

Proof. The implication (2)⇒(1) is a consequence of the final statement of Theorem 4.4.

(1)⇒(2). It follows from the above discussion that both separable and Frobenius monoidal Hom-algebras are characterized by the existence of an (A,α)-central element with different normalizing conditions. Let now $e = \sum e^1 \otimes e^2$ be such an (A,α)-central element. Then the map

$$\mathcal{R} = \mathcal{R}_e : A \otimes A \to A \otimes A, \qquad a \otimes b \mapsto \sum e^1 a \otimes e^2 b,$$

is a solution of the HFS-equation. Since A is finite-dimensional, we view $\mathcal{R}_e \in \mathrm{End}_k(A \otimes A) \cong \mathrm{End}_k(A) \otimes \mathrm{End}_k(A)$. Consequently, we can construct a monoidal Hom-algebra $\mathcal{A}(\mathcal{R}) \subseteq \mathrm{End}_k(A)$.

First, we consider the injection $i : A^\alpha \to \mathrm{End}_k(A)$ given by $i(a)(b) = ab$ for all $a \in A^\alpha$ and $b \in A$. We claim that $\mathcal{A}(\mathcal{R}) \subseteq \mathrm{Im}(i)$, when (A,α) is a separable (resp. Frobenius) Hom-algebra, where

$$\mathcal{A}(\mathcal{R}) = \{f \in \mathrm{End}_k(A) \mid (f \otimes \alpha) \circ \mathcal{R} = \mathcal{R} \circ (\alpha \otimes f)\},$$

with the twisting map $\varpi : \mathcal{A}(\mathcal{R}) \to \mathcal{A}(\mathcal{R})$, $\varpi(f)(a) = f(\alpha(a)) = \alpha(f(a))$. Indeed, if $f \in \mathcal{A}(\mathcal{R})$, then $(f \otimes \alpha) \circ \mathcal{R} = \mathcal{R} \circ (\alpha \otimes f)$, and evaluating this equality at $1_a \otimes a$, we have

$$\sum f(\alpha(e^1)) \otimes \alpha(e^2 a) = \sum \alpha(e^1) \otimes e^2 f(a). \tag{4.1}$$

Now, suppose that A is separable, so $\sum e^1 e^2 = 1_A$. Applying m_A to (4.1), we obtain

$$\sum f(\alpha(e^1))\alpha(e^2 a) = \sum \alpha(e^1)(e^2 f(a)) = \sum (e^1 e^2)\alpha(f(a)) = \alpha^2(f(a)).$$

So,

$$\begin{aligned} f(a) &= \sum \alpha^{-2}(f(\alpha(e^1)))\alpha^{-1}(e^2 a) \\ &= \sum \alpha^{-1}(\alpha^{-2}(f(\alpha(e^1)))e^2)a \\ &= \sum (f(\alpha^{-1}(e^1))e^2)a \end{aligned}$$

for all $a \in A$. The $\alpha^{\otimes 2}$-invariance of e implies that $\sum f(\alpha^{-1}(e^1))e^2 \in A^\alpha$. Thus, $f = i(\sum f(\alpha^{-1}(e^1))e^2)$.

If (A,α) is Frobenius, then there exists $\varepsilon : A \to k$ such that $\sum \varepsilon(e^1)e^2 = \sum e^1 \varepsilon(e^2) = 1_A$. Applying $\varepsilon \otimes \mathrm{id}$ to (4.1), we get

$$\sum \varepsilon(f(\alpha(e^1)))\alpha(e^2 a) = \varepsilon(e^1)e^2 f(a).$$

So $f(a) = (\sum \varepsilon(f(\alpha(e^1)))e^2)a$ for all $a \in A$. In addition, $\sum \varepsilon(f(\alpha(e^1)))e^2$ is in A^α, thus $f = i(\sum \varepsilon(f(\alpha(e^1)))e^2)$, proving that $\mathcal{A}(\mathcal{R}) \subseteq \mathrm{Im}(i)$.

Conversely, $\mathrm{Im}(i) \subseteq \mathcal{A}(\mathcal{R})$. That is,

$$(i(a) \otimes \alpha) \circ \mathcal{R} = \mathcal{R} \circ (\alpha \otimes i(a))$$

for all $a \in A^\alpha$. In fact, for all $b, c \in A$,

$$\begin{aligned}(i(a) \otimes \alpha) \circ \mathcal{R}(b \otimes c) &= \sum (i(a) \otimes \alpha)(e^1 b \otimes e^2 c) \\ &= \sum a(e^1 b) \otimes \alpha(e^2 c) = \sum \alpha(a)(e^1 b) \otimes \alpha(e^2 c) \\ &= \sum (ae^1)\alpha(b) \otimes \alpha(e^2 c) \overset{(3.3)}{=} \sum \alpha(e^1)\alpha(b) \otimes (e^2 a)\alpha(c) \\ &= \sum \alpha(e^1)\alpha(b) \otimes \alpha(e^2)(ac) = \sum e^1 \alpha(b) \otimes e^2(ac) \\ &= \mathcal{R}(\alpha(b) \otimes ac) = \mathcal{R} \circ (\alpha \otimes i(a))(b \otimes c).\end{aligned}$$

So $\mathrm{Im}(i) = \mathcal{A}(\mathcal{R})$, proving that A^α is isomorphic to $\mathcal{A}(\mathcal{R})$. ∎

In the following, we will consider a Hom-coalgebra version of a separable monoidal Hom-algebra.

DEFINITION 4.6. Let (C, γ) be a monoidal Hom-coalgebra.

(1) A map $\sigma : C \otimes C \to k$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called an *HFS-map* if

$$\sigma(\gamma^{-1}(c) \otimes d_1)d_2 = \sigma(c_2 \otimes \gamma^{-1}(d))c_1 \tag{4.2}$$

for all $c, d \in C$. In addition, if σ satisfies the normalizing condition

$$\sigma(c_1 \otimes c_2) = \varepsilon(c),$$

then σ is called a *coseparability idempotent.*

(2) If there exists an element $e \in C$ such that the HFS-map satisfies the normalizing condition

$$\sigma(e \otimes c) = \sigma(c \otimes e) = \varepsilon(c)$$

for all $c \in C$, then we call (σ, e) an *HF-map.*

(3) A monoidal Hom-coalgebra (C, γ) is called *coseparable* if there exists a coseparability idempotent.

Since σ is $\gamma^{\otimes 2}$-invariant, (4.2) is equivalent to

$$\sigma(c \otimes \gamma(d_1))\gamma(d_2) = \sigma(\gamma(c_2) \otimes d)\gamma(c_1) \tag{4.3}$$

for all $c, d \in C$, which will be used later.

PROPOSITION 4.7. *Let (C, γ) be a monoidal Hom-coalgebra, (M, μ) a right (C, γ)-Hom-comodule, and $\sigma : C \otimes C \to k$ an HFS-map. Then the map*

$$\mathcal{R} = \mathcal{R}_\sigma : M \otimes M \to M \otimes M, \mathcal{R}_\sigma(m \otimes n) = \sigma(m_{(1)} \otimes n_{(1)})\mu^2(m_{(0)}) \otimes \mu^2(n_{(0)}),$$

is a solution of the HFS-equation in $\mathrm{End}_k(M \otimes M \otimes M)$.

Proof. For all $l, m, n \in M$, we have

$$\begin{aligned}
\mathcal{R}^{12}\mathcal{R}^{23}(l \otimes m \otimes n) &= \mathcal{R}^{12}(\sigma(m_{(1)} \otimes n_{(1)})\mu(l) \otimes \mu^2(m_{(0)}) \otimes \mu^2(n_{(0)}))\\
&= \sigma(\gamma(l_{(1)}) \otimes \gamma^2(m_{(0)(1)}))\sigma(m_{(1)} \otimes n_{(1)})\mu^3(l_{(0)}) \otimes \mu^4(m_{(0)(0)}) \otimes \mu^3(n_{(0)})\\
&= \sigma(l_{(1)} \otimes \gamma(m_{(1)1}))\sigma(\gamma(m_{(1)2}) \otimes n_{(1)})\mu^3(l_{(0)}) \otimes \mu^3(m_{(0)}) \otimes \mu^3(n_{(0)})\\
&\overset{(4.3)}{=} \sigma(\gamma(l_{(1)1}) \otimes n_{(1)})\sigma(\gamma(l_{(1)2}) \otimes m_{(1)})\mu^3(l_{(0)}) \otimes \mu^3(m_{(0)}) \otimes \mu^3(n_{(0)})\\
&= \sigma(\gamma(l_{(0)(1)}) \otimes n_{(1)})\sigma(l_{(1)} \otimes m_{(1)})\mu^4(l_{(0)(0)}) \otimes \mu^3(m_{(0)}) \otimes \mu^3(n_{(0)})\\
&= \mathcal{R}^{13}(\sigma(l_{(1)} \otimes m_{(1)})\mu^2(l_{(0)}) \otimes \mu^2(m_{(0)}) \otimes \mu(n)) = \mathcal{R}^{13}\mathcal{R}^{12}(l \otimes m \otimes n),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}^{12}\mathcal{R}^{23}(l \otimes m \otimes n) &= \mathcal{R}^{12}(\sigma(m_{(1)} \otimes n_{(1)})\mu(l) \otimes \mu^2(m_{(0)}) \otimes \mu^2(n_{(0)}))\\
&= \sigma(\gamma(l_{(1)}) \otimes \gamma^2(m_{(0)(1)}))\sigma(m_{(1)} \otimes n_{(1)})\mu^3(l_{(0)}) \otimes \mu^4(m_{(0)(0)}) \otimes \mu^3(n_{(0)})\\
&= \sigma(l_{(1)} \otimes \gamma(m_{(1)1}))\sigma(\gamma(m_{(1)2}) \otimes n_{(1)})\mu^3(l_{(0)}) \otimes \mu^3(m_{(0)}) \otimes \mu^3(n_{(0)})\\
&\overset{(4.3)}{=} \sigma(l_{(1)} \otimes \gamma(n_{(1)2}))\sigma(m_{(1)} \otimes \gamma(n_{(1)1}))\mu^3(l_{(0)}) \otimes \mu^3(m_{(0)}) \otimes \mu^3(n_{(0)})\\
&= \sigma(l_{(1)} \otimes n_{(1)})\sigma(m_{(1)} \otimes \gamma(n_{(0)(1)}))\mu^3(l_{(0)}) \otimes \mu^3(m_{(0)}) \otimes \mu^4(n_{(0)(0)})\\
&= \mathcal{R}^{23}(\sigma(l_{(1)} \otimes n_{(1)})\mu^2(l_{(0)}) \otimes \mu(m) \otimes \mu^2(n_{(0)})) = \mathcal{R}^{23}\mathcal{R}^{13}(l \otimes m \otimes n).
\end{aligned}$$

So, $\mathcal{R}$ is a solution of the HFS-equation. ∎

If (C, γ) is finitely generated and projective, and $(C^*, (\gamma^*)^{-1})$ is its dual Hom-algebra, then there is a one-to-one correspondence between HFS-maps $\sigma : C \otimes C \to k$ and C^*-central elements $f = \sum f^1 \otimes f^2 \in C^* \otimes C^*$. The correspondence is given by the formula

$$\sigma(c \otimes d) = \sum \langle f^1, c\rangle \langle f^2, d\rangle$$

for all $c, d \in C$. In fact, for all $x \in C^*$,

$$\sum \langle xf^1, c\rangle \langle (\gamma^*)^{-1}(f^2), d\rangle = \sum \langle (\gamma^*)^{-1}(f^1), c\rangle \langle f^2 x, d\rangle$$

if and only if

$$\sum \langle x, c_1\rangle \langle f^1, c_2\rangle \langle f^2, \gamma^{-1}(d)\rangle = \sum \langle f^1, \gamma^{-1}(c)\rangle \langle f^2, d_1\rangle \langle x, d_2\rangle,$$

which is equivalent to

$$\sum \langle x, c_1\rangle \sigma(c_2 \otimes \gamma^{-1}(d)) = \sigma(\gamma^{-1}(c) \otimes d_1)\langle x, d_2\rangle,$$

that is,

$$\sum \langle x, c_1 \sigma(c_2 \otimes \gamma^{-1}(d))\rangle = \langle x, \sigma(\gamma^{-1}(c) \otimes d_1) d_2\rangle.$$

So f is C^*-central if and only if σ is an HFS-map.

If (M,μ) is a right (C,γ)-Hom-comodule with the coaction $\rho : m \mapsto m_{(0)} \otimes m_{(1)}$, then it is also a left $(C^*, (\gamma^*)^{-1})$-Hom-module (see [13, Lemma 4.2]), and the action is given by

$$f^i \cdot m = \langle f^i, \gamma(m_{(1)})\rangle \mu^2(m_{(0)})$$

for any $f^i \in C^*, m \in M$.

In this situation, the map $\mathcal{R}_e$ defined in Proposition 3.3 is equivalent to the $\mathcal{R}_\sigma$ given in Proposition 4.7. Indeed, for all $m,n \in M$,

$$\begin{aligned}\mathcal{R}_\sigma(m\otimes n) &= \sum \sigma(m_{(1)} \otimes n_{(1)})\mu^2(m_{(0)}) \otimes \mu^2(n_{(0)})\\ &= \sum \sigma(\gamma(m_{(1)}) \otimes \gamma(n_{(1)}))\mu^2(m_{(0)}) \otimes \mu^2(n_{(0)})\\ &= \sum \langle f^1, \gamma(m_{(1)})\rangle\langle f^2, \gamma(n_{(1)})\rangle \mu^2(m_{(0)}) \otimes \mu^2(n_{(0)})\\ &= f^1 \cdot m \otimes f^2 \cdot n = \mathcal{R}_e(m\otimes n).\end{aligned}$$

5. Nakayama automorphisms of Frobenius monoidal Hom-algebras. In this section, we consider the Nakayama automorphisms of Frobenius monoidal Hom-algebras, which generalize the Nakayama automorphisms of Frobenius algebras in [27].

Let (A,α) be a finite-dimensional Frobenius monoidal Hom-algebra with the nondegenerate Hom-associative bilinear form $B : A \times A \to k$ defined in [11, Proposition 5.2]. Note that there are two isomorphisms $(A,\alpha) \cong (A^*, \alpha^{*-1})$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, namely,

$$a \mapsto \lambda(a-) = B(a,-) \quad \text{and} \quad a \mapsto \lambda(-a) = B(-,a),$$

where λ is the Frobenius structure of (A,α) defined in [11, Proposition 5.2]. So for any $a \in A$, the function $\lambda(a-) : (A,\alpha) \to k$ can be represented as $\lambda(-a') : (A,\alpha) \to k$ for some $a' \in A$. This defines an automorphism $\sigma : (A,\alpha) \to (A,\alpha)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ given by $a \mapsto a'$. Then $B(a,b) = B(b,\sigma(a))$ for any $a,b \in A$. Furthermore, for $x \in A$,

$$\begin{aligned}B(ax,1) &= B(\alpha(a),\alpha(x)) = B(\alpha(x),\sigma(\alpha(a)))\\ &= B(\alpha(x),\sigma(a)1) = B(x\sigma(a),1).\end{aligned}$$

So we have

$$\begin{aligned}B((ab)x,1) &= B(\alpha(a)(b\alpha^{-1}(x)),1) = B((b\alpha^{-1}(x))\sigma(\alpha(a)),1)\\ &= B(\alpha(b)(\alpha^{-1}(x)\sigma(a)),1) = B((\alpha^{-1}(x)\sigma(a))\sigma(\alpha(b)),1)\\ &= B(x(\sigma(a)\sigma(b)),1).\end{aligned}$$

On the other hand, $B((ab)x,1)$ is also equal to $B(x\sigma(ab),1)$, giving $\sigma(ab) = \sigma(a)\sigma(b)$. So σ is a Hom-algebra automorphism, called the *Nakayama automorphism of* (A,α).

Instead of the form B we can also use the map $\lambda : (A, \alpha) \to k$ defined in Section 3 to define σ as follows: for any $a, b \in A$,

$$\lambda(ab) = \lambda(b\sigma(a)).$$

Then we have the main result of this section.

THEOREM 5.1. *Let (A, α) be a finite-dimensional Frobenius monoidal Hom-algebra. Then the Nakayama automorphism σ is independent of the field k.*

Proof. Let k_1 and k_2 be two fields, associated with two monoidal categories $\widetilde{\mathcal{H}}(\mathcal{M}_{k_1})$ and $\widetilde{\mathcal{H}}(\mathcal{M}_{k_2})$, in which (A, α) is a finite-dimensional monoidal Hom-algebra. And assume that the Nakayama automorphism of (A, α) is σ_1 with respect to $\widetilde{\mathcal{H}}(\mathcal{M}_{k_1})$. Then σ_1 arises from a map $\lambda_1 : (A, \alpha) \to k_1$ via

$$\lambda_1(ab) = \lambda_1(b\sigma_1(a))$$

for any $a, b \in A$.

Hence, $C := \{\sum(a_i b_i - b_i\sigma_1(a_i)) \mid a_i, b_i \in A\} \subseteq \ker \lambda_1$. It is easy to check that C is closed under the twisting automorphism α and Hom-multiplication by any element from the center $Z(A)$. Note that $(C, \alpha|_C)$ is indeed a subobject of (A, α) both in $\widetilde{\mathcal{H}}(\mathcal{M}_{k_1})$ and $\widetilde{\mathcal{H}}(\mathcal{M}_{k_2})$, since σ_1 is a Hom-algebra morphism.

As in the non-Hom case, (A, α) is the only principal indecomposable left (A, α)-Hom-module via multiplication, and so (A, α) has a simple socle S by [18, Theorem 16.4]. Then $S \nsubseteq \ker \lambda_1$, so $S \nsubseteq C$.

Since $(S, \alpha|_S)$ and $(C, \alpha|_C)$ are both subobjects of (A, α) in $\widetilde{\mathcal{H}}(\mathcal{M}_{k_2})$, we define a map $\lambda_2 : (A, \alpha) \to k_2$ in $\widetilde{\mathcal{H}}(\mathcal{M}_{k_2})$ that is 0 on C but not on S. Then since $S \nsubseteq \ker \lambda_2$, $\ker \lambda_2$ contains no nonzero Hom-ideals, and the Nakayama automorphism σ_2 of (A, α) in $\widetilde{\mathcal{H}}(\mathcal{M}_{k_2})$ is given by

$$\lambda_2(ab) = \lambda_2(b\sigma_2(a))$$

for any $a, b \in A$. That is, $\sigma_2(a)$ is uniquely defined by

$$ab - b\sigma_2(a) \in \ker \lambda_2, \qquad \forall b \in A.$$

In addition, for any $b \in A$ we have $ab - b\sigma_1(a) \in \ker \lambda_2 \in C$, so $\sigma_1(a) = \sigma_2(a)$ for any $a \in A$, as required. ∎

In particular, if a Frobenius monoidal Hom-algebra (A, α) is symmetric, then for any choice of the Frobenius structure the Nakayama automorphism is of the form $a \mapsto b^{-1}(ab)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ for any $a, b \in A$.

6. Concluding remarks and a future work. After the acceptance of the paper, Professor D. Simson has pointed out to us that one can simplify part of the calculations in the paper if one uses the functorial isomorphism between our monoidal category $\mathcal{H}(\mathcal{M}_k)$ defined in Section 2 and the category

$\mathcal{M}_{k[t,t^{-1}]}$ of all modules over the k-algebra $k[t,t^{-1}]$ of all polynomials in one indeterminate t, with coefficients in the field k, localized at the multiplicative system $\{1,t,t^2,\ldots\}$. More precisely, we have the following useful fact.

PROPOSITION 6.1. *Let $k[t,t^{-1}]$ be the localized polynomial k-algebra defined above and $\mathcal{H}(\mathcal{M}_k)$ be the monoidal category defined in Section 1. Then we have the following assertions:*

(1) *There exists an exact category isomorphism*
$$\Phi : \mathcal{H}(\mathcal{M}_k) \to \mathcal{M}_{k[t,t^{-1}]}.$$

(2) *The monoidal category $\mathcal{H}(\mathcal{M}_k)$ is a full exact subcategory of the category $\operatorname{Rep}_k Q$ of all k-linear representations of the quiver Q with one vertex and one loop.*

Proof. (1) Given an object (M,μ) in the monoidal category $\mathcal{H}(\mathcal{M}_k)$, with $\mu \in \operatorname{Aut}_k M$, we define $\Phi(M)$ to be the k-module M equipped with the $K[t,t^{-1}]$-action $\cdot_\mu : M \times k[t,t^{-1}] \to M$ defined by $m \cdot_\mu t = \mu(m)$ and $m \cdot_\mu t^{-1} = \mu^{-1}(m)$, for all $m \in M$. In other words, we set $\Phi(M) = (M, \cdot_\mu)$. It is easy to see that, given a morphism $f : (M,\mu) \to (M',\mu')$ in $\mathcal{H}(\mathcal{M}_k)$, the underlying k-module homomorphism $f : M \to M'$ is a homomorphism of $k[t,t^{-1}]$-modules by the fact $f \circ \mu = \mu' \circ f$, and we set $\Phi(f) := f : \Phi(M) \to \Phi(M')$. A standard checking shows that Φ is an exact functor and is a category isomorphism.

(2) Apply the well known results in Section 14.1-4 of [31] and [33]. ∎

Motivated by Professor D. Simson's suggestions, we intend to study "Hom-structures" in the more natural category $\widetilde{\mathcal{H}}(\mathcal{M}_{k[t,t^{-1}]})$ via twisting the associator and unitors. Explicitly, some questions (such as Galois extensions as well as dual theory) of Hom-Hopf algebras may be considered in the category $\widetilde{\mathcal{H}}(\mathcal{M}_{k[t,t^{-1}]})$.

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