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# SPECTRAL DISTRIBUTION OF THE FREE JACOBI PROCESS ASSOCIATED WITH ONE PROJECTION

BY

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**Abstract.** Given an orthogonal projection $P$ and a free unitary Brownian motion $Y = (Y_t)_{t\geq 0}$ in a $W^\star$-non commutative probability space such that $Y$ and $P$ are $\star$-free in Voiculescu's sense, we study the spectral distribution $\nu_t$ of $J_t = PY_tPY_t^\star P$ in the compressed space. To this end, we focus on the spectral distribution $\mu_t$ of the unitary operator $SY_tSY_t^\star$, $S = 2P-1$, whose moments are related to those of $J_t$ via a binomial-type expansion already obtained by Demni et al. [Indiana Univ. Math. J. 61 (2012)]. In this connection, we use free stochastic calculus in order to derive a partial differential equation for the Herglotz transform $\mu_t$. Then, we exhibit a flow $\psi(t,\cdot)$ valued in $[-1,1]$ such that the composition of the Herglotz transform with the flow is governed by both the ones of the initial and the stationary distributions $\mu_0$ and $\mu_\infty$. This enables us to compute the weights $\mu_t\{1\}$ and $\mu_t\{-1\}$ which together with the binomial-type expansion lead to $\nu_t\{1\}$ and $\nu_t\{0\}$. Fatou's theorem for harmonic functions in the upper half-plane shows that the absolutely continuous part of $\nu_t$ is related to the nontangential extension of the Herglotz transform of $\mu_t$ to the unit circle. In the last part of the paper, we use combinatorics of noncrossing partitions in order to analyze the term corresponding to the exponential decay $e^{-nt}$ in the expansion of the $n$th moment of $\mu_t$.

**1. Reminder and main results.** Let $(\mathscr{A},\tau)$ be a $W^\star$-noncommutative probability space with unit $\mathbf{1}$ and adjoint operation $\star$: $\mathscr{A}$ is a von Neumann algebra endowed with a faithful tracial state $\tau$. In the recent paper [DHH], we started the spectral study of the free Jacobi process: this is a family of positive operators $J = (J_t)_{t\geq 0}$ valued in the compressed noncommutative probability space

$$\left(P\mathscr{A}P, \tau_P \triangleq \frac{1}{\tau(P)}\tau\right)$$

where $P \in \mathscr{A}$ is an orthogonal projection. Actually, the operator $J_t$ is defined [De1] by

$$J_t = PY_tQY_t^\star P,$$

where $Q \in \mathscr{A}$ is another orthogonal projection and $Y = (Y_t)_{t\geq 0} \subset \mathscr{A}$ is a Lévy process with respect to the free unitary multiplicative convolution,

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referred to as the free unitary Brownian motion [Bi1]. In this definition, it is also assumed that $\{P, Q\}$ and $\{Y, Y^\star\}$ are free families in Voiculescu's sense [NS]. When $P = Q$ and $\tau(P) = 1/2$, we proved in [DHH] that the spectral distribution of $J_t$ in $(P\mathscr{A}P, \tau_P)$ fits that of the positive operator

$$\frac{Y_{2t} + Y_{2t}^\star + 2\mathbf{1}}{4}$$

in $(\mathscr{A}, \tau)$. In particular, if $U \in \mathscr{A}$ is a Haar unitary random variable [NS] then the spectral distribution of

$$\frac{U + U^\star + 2\mathbf{1}}{4}$$

is the arcsine distribution in $(0, 1)$ [De1]. Two proofs leading to this result were given in [DHH]. One of them relies on the following binomial-type expansion. Let $S \triangleq 2P - 1$. Then $\tau(S) = 0$ and

$$\tau[(PY_tPY_t^\star)^n] = \frac{1}{2^{2n+1}}\binom{2n}{n} + \frac{1}{2^{2n}}\sum_{k=1}^{n}\binom{2n}{n-k}\tau((SY_tSY_t^\star)^k).$$

The description of the spectral distribution of $J_t$ then follows from the additional fact that if $\tau(S) = 0$ then $SY_tSY_t^\star$ and $Y_{2t}$ share the same spectral distribution [DHH, Lemma 1]. In particular, if $S$ and $U$ are $\star$-free in $(\mathscr{A}, \tau)$ then $SUSU^\star$ is again a Haar unitary random variable (this also follows obviously from the freeness of $S$ and $USU^\star$, see [NS]).

In this paper, we investigate the spectral distribution $\nu_t^\theta \in (0, 1)$ of $J_t$ when the projection $P$ has arbitrary rank $\tau(P) = \theta$. The key ingredient is the general form of the binomial-type expansion written above:

$$\tau[(PY_tPY_t^\star)^n] = \frac{1}{2^{2n+1}}\binom{2n}{n} + \frac{2\theta - 1}{2} + \frac{1}{2^{2n}}\sum_{k=1}^{n}\binom{2n}{n-k}\tau((SY_tSY_t^\star)^k), \tag{1.1}$$

which carries our investigations to the spectral distribution, say $\mu_t^\theta$, of $SY_tSY_t^\star$ in $(\mathscr{A}, \tau)$. As we shall see below, one cannot expect a description of $\mu_t^\theta$, $\theta \in (0, 1)$, similar to that of $\mu_t^{1/2}$. Indeed, the multivariate free stochastic calculus developed in [BL] allows one to derive a recursive time-dependent relation for the moments

$$r_n^\theta(t) \triangleq \tau[(SY_tSY_t^\star)^n] = \int_{\mathbb{T}} z^n\, \mu_t^\theta(dz),$$

$\mathbb{T}$ being the unit circle. Moreover, this relation is then transformed into a partial differential equation for the Herglotz transform of $\mu_t^\theta$:

$$H^\theta(t, z) \triangleq 1 + 2\sum_{n\geq 1} r_n^\theta(t) z^n, \qquad |z| < 1,$$

which is a perturbation of the pde satisfied by $H^{1/2}$,

$$2[\tau(S)]^2\frac{z(1+z)}{(1-z)^3} = 2(2\theta-1)^2\frac{z(1+z)}{(1-z)^3},$$

while keeping the same initial data

$$H^\theta(0,z) = H^{1/2}(0,z) = \frac{1+z}{1-z}.$$

Nonetheless, we shall prove using the method of characteristics that there exists a flow $(t,z) \mapsto \psi^\theta(t,z)$ on an open subset of $\mathbb{R}_+ \times [-1,1]$ such that

$$(1.2) \qquad [H^\theta_\infty(\psi^\theta(t,z))]^2 - [H^\theta_\infty(z)]^2 = [H^\theta(t,\psi^\theta(t,z))]^2 - [H^\theta(0,z)]^2.$$

In (1.2), $H^\theta_\infty$ is the Herglotz transform of the spectral distribution $\mu^\theta_\infty$ of $SUSU^\star$. Equivalently, $\mu^\theta_\infty$ is the weak limit as $t\to\infty$ of $\mu^\theta_t$ and is a deformation of the Haar distribution $\mu^{1/2}_\infty$ on $\mathbb{T}$ since $SY_tSY_t^\star$ and $Y_{2t}$ are equally distributed when $\theta = 1/2$. To the best of our knowledge, no description of $\mu^\theta_\infty$ has shown up yet in the literature. For that reason and with regard to (1.2), we shall supply here a full description of $\mu^\theta_\infty$ relying on an explicit expression for $H^\theta_\infty$. In particular, $\mu^\theta_\infty$ admits an absolutely continuous part whose support consists of two symmetric (with respect to the real axis) arcs that join at $z = \pm 1$ if and only if $\tau(S) = 0$. As to its discrete part, it consists of the single point $z = 1$ with weight $|\tau(S)|$. However, rather than using the analytic machinery of multiplicative convolution of probability distributions on $\mathbb{T}$ [VDN], we found it more convenient to write down $H^\theta_\infty$ by taking the limit in (1.1) as $t\to\infty$ and by using the moment generating function of $PUPU^\star P$ in $(P\mathscr{A}P,\tau_P)$, already computed in [De2].

Coming back to the flow, we shall express it through a conformal one-to-one map $\alpha$ from $\mathbb{C}\setminus[1,\infty[$ onto the open unit disc, its inverse function $\alpha^{-1}$ and the inverse function $\xi_{2t}$ of the Herglotz transform of the spectral distribution of $Y_{2t}$ [Bi1]. This new expression has at least two advantages. Firstly, we can easily find $z^\theta_t \in (0,1)$ such that $\psi^\theta(t,z^\theta_t) = 1$. In this way, the equality $\mu^\theta_t\{1\} = |\tau(S)|$ follows after taking a radial limit in (1.2) as $z \to z_t$. Similarly, we easily see that

$$\lim_{z\to -1,\, z>-1} \psi^\theta(t,z) = -1,$$

whence $\mu^\theta_t\{-1\} = 0$, and if $t > 2$ then we can also find a real number lying in $(-1,0)$ where the same limit holds. Secondly, the above immediately shows that the maximal range of $\psi(t,\cdot)$ is $\mathbb{D}$ together with the lower semicircle, unless $\tau(P) = 1/2$ for which the range is $\overline{\mathbb{D}}$ [Bi2].

With the help of this information, we arrive at the following description of $\nu_t$:

THEOREM 1.1. *Let $t > 0$. Then:*

(1) *The discrete part of $\nu_t^\theta$ is given by*

$$\nu_t^\theta\{1\} = \frac{1}{\theta}\max\{(2\theta - 1), 0\}.$$

(2) *At any point $x \in (0,1)$ where the distribution function of $\nu_t^\theta$ is differentiable,* $\mathrm{Re}(H_t)$ *admits a nontangential limit at $\alpha(1/x)$ and the density of $\nu_t^\theta$ is given by*

$$\frac{1}{\pi\sqrt{x(1-x)}}\mathrm{Re}\{H_t\}(\alpha(1/x)) = \frac{1}{\pi\sqrt{x(1-x)}}\mathrm{Re}\{H_t\}(e^{i2\arccos(\sqrt{x})}).$$

The paper is organized as follows. We first supply a full description of $\mu_\infty^\theta$ and derive a closed formula for its moments through Jacobi polynomials. The time-dependent recursive relation for the moments $r_n^\theta(t)$, $n \geq 1$, of $\mu_t^\theta$ comes next and is an instance of a general formula derived in [BL]. The relation is then transformed into a pde satisfied by $H^\theta$ whose dynamics is analyzed using the method of characteristics, leading to the flow $\psi^\theta$. Once this has been completed, we put the obtained expression in the compact form we mentioned above and prove the existence of $z_t^\theta$ at any time $t > 0$. Doing so allows us to compute $\mu_t^\theta\{1\}$, whence we deduce $\nu_t^\theta\{1\}$ after proving that the limit as $n \to \infty$ of the RHS of (1.1) depends only on the weight of $\mu_t^\theta\{1\}$ (Lebesgue's convergence theorem clearly implies that the LHS does so after normalizing by $1/\tau(P)$). As to $\nu_t^\theta\{0\}$, we relate the Cauchy–Stieltjes transform $G_t^\theta$ of $\nu_t^\theta$ to $H_t^\theta \circ \alpha$, then prove the equality $2\tau(P)\nu_t^\theta\{0\} = \mu_t^\theta\{-1\} = 0$. Coming to the absolutely continuous part of $\nu_t$, we first relate $\mathrm{Im}[G^\theta(x+iy)]$, $x \in (0,1)$, to $\mathrm{Re}\, H_t^\theta$ along the curve $y \mapsto \alpha[1/(x+iy)]$ and then prove that $\alpha[1/(x+iy)] \to \alpha(1/x)$ nontangentially as $y \to 0^+$. As a matter fact, the last statement of Theorem 1.1 follows from Fatou's theorem for positive harmonic functions in the upper half-plane [Don] and from [BV, Lemma 5.11].

We close the paper by analyzing the term corresponding to the exponential decay $e^{-nt}$ in $r_n^\theta(t)$. When $\theta = 1/2$, $r_n^{1/2}$ is expressed through a Laguerre polynomial since $SY_tSY_t^\star$ and $Y_{2t}$ are equally distributed ([Bi1], [DHH]). For general $\theta \in (0,1)$, we shall see that the term with fastest decay in $r_n^\theta$ satisfies the same equation as $r_n^{1/2}$, yet has a different initial value at $t = 0$. Using combinatorics of noncrossing partitions, we prove that this value is the $n$th even moment of the $1/2$-fold convolution of the self-adjoint operator $a_1 - a_2$, where $a_1, a_2 \in \mathscr{A}$ are two free copies of $S$. Of course, the resulting convolution is not necessarily a probability measure for general $\theta \in (0,1]$, while this is obviously true when $\theta = 1/2$ (it reduces simply to the spectral distribution of $S$ since $a_1$ and $-a_2$ have the symmetric Bernoulli distribution). Using the $R$-transform machinery [VDN], we can see that the

Cauchy–Stieltjes transform of the 1/2-convolution of $a_1 - a_2$ is a root of a third degree polynomial that one can express using Gauss hypergeometric functions.

Henceforth, we shall omit the dependence of our notation on $\theta$ for the sake of clarity.

**2. The stationary distribution $\mu_\infty$.** This section is devoted to the Lebesgue decomposition of the spectral distribution $\mu_\infty$ of $SUSU^\star$, where we recall that $U \in \mathscr{A}$ is a Haar unitary operator and $S$ and $U$ are $\star$-free. More precisely, we show that $\mu_\infty$ splits into an absolutely continuous part and a singular discrete one supported in $\{1\}$ with weight $|\tau(S)|$. To proceed, we shall write down its Herglotz transform $H_\infty$:

$$H_\infty(z) = \int_{\mathbb{T}} \frac{w+z}{w-z}\,\mu_\infty(dw) = \int_{\mathbb{T}} \frac{1+zw}{1-zw}\,\mu_\infty(dw) = 1 + 2\sum_{n\geq 1} r_n z^n$$

where we set

$$r_n \triangleq r_n(\infty) = \tau((SUSU^\star)^n), \qquad n \geq 1,$$

and the second equality follows from the invariance of $\mu_\infty$ under the conjugate mapping. This may be done using the free multiplicative convolution of the unitary operators $S$ and $USU^\star$ (see [NS]) whose common spectral distribution is given by

$$\theta\delta_1 + (1-\theta)\delta_{-1}.$$

However, we found it more convenient to deduce $H_\infty$ from (1.1) and from the knowledge of the moment generating function of $PUPU^\star P$ in $P\mathscr{A}P$ [De2]. The result of our computations is

LEMMA 2.1. *Set $\kappa \triangleq 2\theta - 1 = \tau(S)$. Then*

$$H_\infty(z) = \sqrt{1 + 4\kappa^2 \frac{z}{(1-z)^2}}$$

*in some neighborhood of the origin. The equality extends analytically to the open unit disc.*

*Proof.* Define

$$m_n \triangleq \frac{1}{\tau(P)}\tau[(PUPU^\star P)^n], \quad n \geq 1, \qquad m_0 = 1.$$

These are the moments of the stationary free Jacobi process with parameters $\lambda = 1$, $\theta \in (0,1]$ (see [De2] for notation, see also [Co]). From [De2, equation (1), p. 108], we already know that

$$\sum_{n\geq 1} m_n z^n = \frac{(2\theta-1) + \sqrt{1-4\theta(1-\theta)z}}{2\theta(1-z)} - 1, \ |z| < 1. \tag{2.1}$$

On the other hand, after summing (1.1) over $n \geq 1$ we get

$$\sum_{n\geq 1} m_n z^n = \frac{1}{2\theta}\left[\frac{1}{\sqrt{1-z}} - 1 + \frac{(2\theta-1)z}{1-z}\right] + \frac{1}{\theta}\sum_{n\geq 1}\frac{z^n}{2^{2n}}\sum_{k=1}^{n}\binom{2n}{n-k} r_k \tag{2.2}$$
$$= \frac{1}{2\theta}\left[\frac{1}{\sqrt{1-z}} - 1 + \frac{(2\theta-1)z}{1-z}\right] + \frac{1}{\theta}\sum_{k\geq 1} r_k \frac{z^k}{2^{2k}}\sum_{n\geq 0}\binom{2n+2k}{n}\frac{z^n}{2^{2n}}.$$

Using the identity (see [DHH, p. 1360])

$$\sum_{n\geq 0}\binom{2n+2k}{n}\frac{z^n}{2^{2n}} = \frac{2^{2k}}{\sqrt{1-z}}\left(1+\sqrt{1-z}\right)^{-2k}, \qquad |z|<1,$$

and equating (2.1) and (2.2), we get

$$2\sum_{n\geq 1} r_n[\alpha(z)]^n = \frac{\sqrt{1-4\theta(1-\theta)z}}{\sqrt{1-z}} - 1,$$

where (1)

$$\alpha(z) \triangleq \frac{z}{(1+\sqrt{1-z})^2} = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}. \tag{2.3}$$

Now $\alpha$ is a bi-holomorphism from $\mathbb{C}\setminus[1,\infty[$ onto $\mathbb{D}$, where its inverse function is given by

$$\alpha^{-1}(z) = \frac{4z}{(1+z)^2}.$$

As a result, for any $z \in \mathbb{D}$,

$$H_\infty(z) = 1 + 2\sum_{n\geq 1} r_n z^n = \frac{\sqrt{1-4\theta(1-\theta)\alpha^{-1}(z)}}{\sqrt{1-\alpha^{-1}(z)}}$$
$$= \sqrt{\frac{1+z^2+2z(1-8\theta(1-\theta))}{(1-z)^2}} = \sqrt{1+4\kappa^2\frac{z}{(1-z)^2}}. \blacksquare$$

COROLLARY 2.2. *The Lebesgue decomposition of the spectral measure $\mu_\infty$ of $SUSU^\star$ is given by*

$$\mu_\infty = |\kappa|\delta_1 + \sqrt{1-\frac{\kappa^2}{\sin^2\psi}}\,\mathbf{1}_{\{|\sin\psi|\geq|\kappa|\}}\,d\psi.$$

*Here $d\psi$ denotes the Lebesgue measure on $[0,2\pi]$.*

(1) We consider the principal determination of the square root.

*Proof.* From the previous lemma, $H_\infty$ admits a pole at $z = 1$, therefore $\mu_\infty$ assigns a weight to $z = 1$ given by (see [CMR])

$$\frac{1}{2}\lim_{z\to 1^-}(1-z)H_\infty(z) = \frac{1}{2}\lim_{z\to 1}\sqrt{(1-z)^2 + 4\kappa^2 z} = |\kappa|.$$

As to the remaining parts of $\mu_\infty$, we first discard the value $\kappa = 0$. Indeed, we know that $\mu_\infty$ reduces to the Haar distribution on $\mathbb{T}$ when $\kappa = 0$ or equivalently $\theta = 1/2$. Now if $|\kappa| \in (0,1)$ then we claim that $\mu_\infty - |\kappa|\delta_1$ is absolutely continuous with respect to the Haar distribution in $\mathbb{T}$. Indeed, the Herglotz transform of $\mu_\infty - |\kappa|\delta_1$ is given by

$$H_\infty(z) - |\kappa|\frac{1+z}{1-z} = \frac{(1-\kappa^2)(1-z)}{\sqrt{z^2 + 2(2\kappa^2 - 1)z + 1} + |\kappa|(1+z)}$$

and admits a continuous extension to the boundary $\mathbb{T}$, since

$$z \mapsto z^2 + 2(2\kappa^2 - 1)z + 1$$

does not take negative values (its roots lie on $\mathbb{T}$) and since the denominator does not vanish on the closed unit disc. Thus our claim follows from the Poisson representation of positive harmonic functions in the open unit disc extending continuously to $\mathbb{T}$ [Ru, Ch. 11]. Finally, the density of $\mu_\infty$ is

$$\begin{aligned}\mathrm{Re}\left[H_\infty(e^{i\psi}) - |\kappa|\frac{1+e^{i\psi}}{1-e^{i\psi}}\right] &= \mathrm{Re}\left[\sqrt{1 + 4\kappa^2\frac{e^{i\psi}}{(1-e^{i\psi})^2}}\right] \\ &= \sqrt{1 - \frac{\kappa^2}{\sin^2\psi}}\,\mathbf{1}_{\{|\sin\psi| > |\kappa|\}}. \quad \blacksquare\end{aligned}$$

We close this section with the following closed form of the moments $r_n$, $n \geq 1$, showing that these are somehow averages over $(0, |\kappa|)$ of special polynomials:

PROPOSITION 2.3. *For any* $\kappa \in (-1, 1)$,

$$r_n = \kappa\int_0^\kappa P_{n-1}^{1,0}(1 - 2s^2)\, ds,$$

*where* $P_n^{1,0}$ *is the nth Jacobi polynomial with parameters* $(1,0)$ [Ra, p. 254].

*Proof.* Using the generalized binomial theorem [Ra, p. 47], we write

$$\begin{aligned}\sqrt{1 + \frac{4\kappa^2 z}{(1-z)^2}} - 1 &= \sum_{k\geq 1}\frac{(-1/2)_k}{k!}\left[-\frac{4\kappa^2 z}{(1-z)^2}\right]^k \\ &= \sum_{k\geq 1}\frac{(-1/2)_k}{k!}(-4\kappa^2 z)^k \sum_{n\geq 0}\frac{(2k)_n}{n!}z^n\end{aligned}$$

where for $x \in \mathbb{R}$, $(x)_k = x(x+1)\dots(x+k-1)$ is the Pochhammer symbol [Ra, p. 45]. Inverting the order of summation and identifying coefficients of $z^n$, one obtains

$$r_n = \frac{1}{2}\sum_{k=1}^{n} \frac{(-1/2)_k}{k!}\,\frac{(2k)_{n-k}}{(n-k)!}(-4\kappa^2)^k, \qquad n \geq 1.$$

Writing $(2k)_{n-k} = \Gamma(n+k)/\Gamma(2k)$, $k \geq 1$, using the Legendre duplication formula [Erd]

$$\Gamma(2k) = 2^{2k-1}(k-1)!(1/2)_k,$$

and since

$$(-1/2)_k = -\frac{1}{2k-1}(1/2)_k,$$

one gets

$$\begin{aligned}
r_n &= (n-1)!\sum_{k=1}^{n} \frac{(-1/2)_k}{(1/2)_k}\,\frac{(n)_k}{(n-k)!(k-1)!}\,\frac{(-\kappa^2)^k}{k!} \\
&= -(n-1)!\sum_{k=1}^{n} \frac{1}{2k-1}\,\frac{(n)_k}{(n-k)!(k-1)!}\,\frac{(-\kappa^2)^k}{k!} \\
&= -(n-1)!\sum_{k=0}^{n-1} \frac{1}{2k+1}\,\frac{(n)_{k+1}}{(n-1-k)!(k+1)!}\,\frac{(-\kappa^2)^{k+1}}{k!} \\
&= n\sum_{k=0}^{n-1} \frac{(1-n)_k}{2k+1}\,\frac{(n+1)_k}{(k+1)!}\,\frac{(\kappa^2)^{k+1}}{k!} \\
&= n\kappa\sum_{k=0}^{n-1} \frac{(1-n)_k}{2k+1}\,\frac{(n+1)_k}{(2)_k}\,\frac{(\kappa)^{2k+1}}{k!}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{d}{d\kappa}\sum_{k=0}^{n-1} \frac{(1-n)_k}{2k+1}\,\frac{(n+1)_k}{(2)_k}\,\frac{(\kappa)^{2k+1}}{k!} &= \sum_{k=0}^{n-1}(1-n)_k\frac{(n+1)_k}{(2)_k}\,\frac{(\kappa)^{2k}}{k!} \\
&= \frac{(n-1)!}{(2)_{n-1}}P_{n-1}^{1,0}(1-2\kappa^2) = \frac{1}{n}P_{n-1}^{1,0}(1-2\kappa^2),
\end{aligned}$$

where the second equality follows from [Ra, p. 255]. ∎

**3. The time-dependent regime.** Now, we proceed to the study of $\mu_t$ and we start with

**3.1. Time-dependent recursive relation.** This subsection is devoted to the proof via free stochastic calculus of the following result:

PROPOSITION 3.1. *Let*

$$s_n(t) \triangleq e^{nt}\tau((SY_tSY_t^\star)^n) = e^{nt}r_n(t), \qquad n \geq 1.$$

*Then*

$$s_1(t) = \kappa^2 e^t + (1-\kappa^2),$$
$$\partial_t s_n(t) = -n\sum_{j=1}^{n-1} s_j(t)s_{n-j}(t) + \kappa^2 n^2 e^{nt}, \qquad n \geq 2.$$

*Proof.* The proof follows the lines of that of Proposition 1 in [DHH], with minor modifications due to the cancellations $S^2 = \mathbf{1}$ rather than $P^2 = P$. For the reader's convenience, we give the whole proof and first recall Theorem 3.4 of [BL]:

THEOREM 3.2. *Let $n \geq 1$ and define*

$$f_{2n}(a_1, \ldots, a_{2n}, t) \triangleq e^{nt}\tau(a_1Y_ta_2Y_t^\star \ldots a_{2n-1}Y_ta_{2n}Y_t^\star)$$

*where $\{a_1, \ldots, a_{2n}\} \in \mathscr{A}$ is $\star$-free with $Y$. Set $f_0(A,t) \triangleq \tau(A)$ for any $A \in \mathscr{A}$. Then*

$$\begin{aligned}
&\partial_t f_{2n}(a_1, \ldots, a_{2n}, t) \\
&\quad = -\sum_{\substack{1\leq k<l\leq 2n \\ l-k\equiv 0\,[2]}} f_{2n-(l-k)}(a_1, \ldots, a_k, a_{l+1}, \ldots, a_{2n}, t) f_{l-k}(a_{k+1}, \ldots, a_l, t) \\
&\qquad + e^t \sum_{\substack{1\leq k<l\leq 2n \\ l-k-1\equiv 0\,[2]}} \big[f_{2n-(l-k)-1}(a_1, \ldots, a_{k-1}, a_ka_{l+1}, a_{l+2}, \ldots, a_{2n}, t) \\
&\qquad\qquad \times f_{l-k-1}(a_la_{k+1}, a_{k+2}, \ldots, a_{l-1}, t)\big].
\end{aligned}$$

Before passing to computations, we stress that some terms in the second sum may seem ambiguous, mainly when $l = 2n, k = 1$ or when $l = k+1$. For that reason, we refer the reader to the proof of Theorem 3.2 in order to avoid any ambiguity. Now, we specialize Theorem 3.2 to $a_k = S$ for all $1 \leq k \leq n$ so that $f_{2n} = s_n$ and consider $n \geq 2$ (for $n = 1$, the result is derived for instance from [BL, p. 923]). Since both indices $k, l$ in the first (resp. second) sum in Theorem 3.2 have the same (resp. different) parity, it follows that $k$ and $l+1$ in the second (resp. first) sum have the same (resp. different) parity and so do $l$ and $k+1$. Accordingly, the first sum does not contain terms $f_0(\cdot, t)$ while the second does: they correspond to the indices $l = 2n, k = 1$ and to $l = k+1, 1 \leq k \leq 2n-1$. Since $S^2 = \mathbf{1}$ and since $\tau$ is a trace, the contribution of the indices $k = 1$, $l = 2n$ is

$$\kappa^2 e^{nt}.$$

For $l = k+1$, $1 \leq k \leq 2n-1$, we distinguish between two cases: the contribution of $1 \leq k \leq 2n-2$ is

$$(2n-2)\kappa^2 e^{nt},$$

while that of $k = 2n-1$, $l = 2n$ is $\kappa^2 e^{nt}$. Thus, the whole contribution of the indices $k = 1$, $l = 2n$ and of $1 \leq k \leq 2n-1$, $l = k+1$ is

$$2n\kappa^2 e^{nt}. \tag{3.1}$$

Next we write $l = k+2s+1$ for positive integer values of $s$ and we distinguish between $n = 2$ and $n \geq 3$. If $n = 2$ then there is no additional term in the second sum, while if $n \geq 3$ we separate $k = 1$ and $2 \leq k \leq 2n-3$. By the same properties of $a, \tau$ mentioned above, the contribution of $k = 1$, $l = 2s+2$, $1 \leq s \leq n-2$ is

$$(n-2)\kappa^2 e^{nt}. \tag{3.2}$$

For the remaining values of $2 \leq k \leq 2n-3$, we distinguish even from odd $k$'s: the contribution of the indices $k = 2j$, $1 \leq j \leq n-2$, $l = 2j+2s+1$ is

$$\sum_{j=1}^{n-2}\sum_{s=1}^{n-j-1} \kappa^2 e^{nt} = \kappa^2 e^{nt}\frac{(n-1)(n-2)}{2}, \tag{3.3}$$

while for $k = 2j+1$, $1 \leq j \leq n-2$, $l = 2s+2j+2$ we distinguish between $1 \leq s \leq n-j-2$ and $s = n-j-1$. When $1 \leq s \leq n-j-2$ we get

$$\sum_{j=1}^{n-2}\sum_{s=1}^{n-j-2} \kappa^2 e^{nt} = \kappa^2 e^{nt}\frac{(n-2)(n-3)}{2}, \tag{3.4}$$

while for $s = n-j-1$ we get

$$\sum_{j=1}^{n-2} \kappa^2 e^{nt} = (n-2)\kappa^2 e^{nt}. \tag{3.5}$$

Coming to the first sum in the statement of the theorem, its contribution is the same as in [DHH, Lemma 1]:

$$-n\sum_{k=1}^{n-1} s_{n-k}(t)s_k(t). \tag{3.6}$$

The proposition is proved by summing (3.1)–(3.5). ■

**3.2. Dynamics of the Herglotz transform.** Here, we transform the time-dependent recursive relation into a pde governing the Herglotz transform $z \mapsto H(t,z)$ of $\mu_t$. Recall that

$$H(t,z) = \int_{\mathbb{T}} \frac{w+z}{w-z}\, d\mu_t(w)$$

and that the moments

$$r_n(t) = \tau[(SY_tSY_t^{\star})^n], \qquad n \geq 1,$$

are the coefficients of the expansion of $H(t,\cdot)$ as an analytic function ($\mu_t$ is invariant under conjugation):

$$H(t,z) = 1 + 2\sum_{n\geq 1} r_n(t)z^n, \qquad |z| < 1.$$

Using Proposition 3.1, we readily get:

PROPOSITION 3.3. *The Herglotz transform $H$ satisfies the equation*

$$\partial_t H + \frac{z}{2}\partial_z H^2 = 2\kappa^2\frac{z(1+z)}{(1-z)^3}, \qquad H(0,z) = \frac{1+z}{1-z}, \qquad |z| < 1. \tag{3.7}$$

*Proof.* Elementary computations show that the sequence $(r_n(t))_{n\geq 1}$ satisfies

$$\begin{aligned}
\partial_t r_1(t) &= -r_1(t) + \kappa^2,\\
\partial_t r_n(t) &= -nr_n(t) - n\sum_{j=1}^{n-1} r_j(t)r_{n-j}(t) + \kappa^2 n^2, \quad n \geq 2.
\end{aligned}$$

Since $|r_n(t)| \leq 1$ we have $|\partial_t r_n(t)| \leq Cn^2$, $n \geq 1$, for some positive constant $C$. Thus we can interchange differentiation and summation, which leads to

$$\begin{aligned}
\partial_t H &= 2\sum_{n\geq 1}\partial_t r_n(t)z^n\\
&= 2\kappa^2\sum_{n\geq 1} n^2z^n - 2\sum_{n\geq 1} nr_n(t)z^n - 2\sum_{n\geq 2} n\sum_{j=1}^{n-1} r_j(t)r_{n-j}(t)z^n\\
&= 2\kappa^2\frac{z(1+z)}{(1-z)^3} - z\partial_z H - 2\sum_{j\geq 1} r_j(t)z^j\sum_{n\geq j+1} nr_{n-j}(t)z^{n-j}\\
&= 2\kappa^2\frac{z(1+z)}{(1-z)^3} - z\partial_z H - 4\frac{H-1}{2}\sum_{j\geq 1} jr_j(t)z^j\\
&= 2\kappa^2\frac{z(1+z)}{(1-z)^3} - zH\partial_z H. \ \blacksquare
\end{aligned}$$

REMARK 3.4. Equation (3.7) is a nonhomogeneous Burgers equation. It allows one to retrieve the expression of $H_\infty$ already obtained in the previous section. Indeed, any stationary solution of (3.7) is a solution of $\partial_t H = 0$, that is, $H(t,z) = H(z)$ solves the first-order ordinary differential equation

$$\partial_z(H^2) = 4\kappa^2\frac{1+z}{(1-z)^3}.$$

After integrating and taking into account $H(0) = 1$, we get

$$H^2(z) = 4\kappa^2\frac{z}{(1-z)^2} + H^2(0) = H_\infty^2(z).$$

**4. Properties of Herglotz transform.** In this section we intend to solve (3.7) by using the method of characteristics which transforms this pde into two coupled ordinary differential equations. As an application we shall give the explicit values of the weights $\mu_t\{\pm 1\}$ and $\nu_t\{\pm 1\}$.

**4.1. Resolution of the Burgers equation.** We shall prove that the dynamics of the Herglotz transform $H(t,\cdot)$ is completely determined around the origin $z = 0$ by its initial value $H(0,\cdot)$, the long-time behavior $H_\infty$ and some explicit curves $\{z \mapsto \psi(t,z)\}$ called characteristics. To make the computations easier, we first use the Möbius transform

$$z \mapsto y = \frac{1+z}{1-z},$$

which realizes a one-to-one map between the open unit disc and the right half-plane $\{\operatorname{Re} z > 0\}$. Indeed, this transform replaces the fraction on the RHS of (3.7) by a cubic polynomial. To see this, set

$$F(t,y) \triangleq H(t,z), \qquad y = \frac{1+z}{1-z}.$$

Then $F$ satisfies the equation

$$\partial_t F + \frac{1}{4}(y^2-1)\partial_y F^2 = \frac{\kappa^2}{2} y(y^2-1), \qquad F(0,y) = y. \tag{4.1}$$

Observe that after this change of variable, the stationary solution $H_\infty$ reads

$$H_\infty(z) \triangleq F_\infty(y) = \sqrt{(1-\kappa^2)+\kappa^2 y^2}.$$

The description of the solution of (4.1) will be the subject of the next theorem, but first we introduce the functions

$$a \triangleq \kappa^2 + (1-\kappa^2)y^2, \qquad \alpha^{-1}(z) = \frac{4z}{(1+z)^2}, \qquad \xi_t(u) = \frac{u-1}{u+1}e^{tu}.$$

Recall that $\alpha^{-1}$ is the inverse of $\alpha$ (see the proof of Lemma 2.1) and $\xi_t$ is the inverse of the Herglotz transform of the spectral distribution of $Y_{2t}$ [Bi2].

THEOREM 4.1. *Let $F$ be the solution of the nonlinear equation* (4.1). *Then*

$$F(t,\phi(t,y)) = \sqrt{\kappa^2\phi^2(t,y) + (1-\kappa^2)y^2} \tag{4.2}$$

*with*

$$\phi(t,y) = \frac{1}{\sqrt{1 - \frac{a}{a-\kappa^2}\alpha^{-1}(\xi_t(\sqrt{a}))}}, \qquad y \in \left(0, \left(\frac{u_t^2-\kappa^2}{1-\kappa^2}\right)^{1/2}\right),$$

*where $u_t > 1$ is the unique solution of the equation*

$$\xi_t(u) = \frac{u-|\kappa|}{u+|\kappa|}.$$

*In the z-configuration this reads*

$$H(t,\psi(t,z)) = \sqrt{\kappa^2\phi^2\Big(t,\frac{1+z}{1-z}\Big) + (1-\kappa^2)\Big(\frac{1+z}{1-z}\Big)^2} \tag{4.3}$$

*with*

$$\psi(t,z) = \alpha\Big(\frac{a}{a-\kappa^2}\alpha^{-1}(\xi_t(\sqrt{a}))\Big), \qquad z = \frac{y-1}{y+1} \in (-1, z_t),$$

*and* $z_t \in (0,1)$ *is defined by*

$$u_t = \sqrt{(1-\kappa^2)\Big(\frac{1+z_t}{1-z_t}\Big)^2 + \kappa^2}.$$

*In addition, we have* $\psi(t, z_t) = 1$.

Before going into the details of the proof, some remarks are in order.

Remarks.

- We can easily check the validity of (4.2) and (4.3) at $t = 0$.
- Observe that $\phi(0,y) = y$ so that (4.2) can be rewritten in the form
$$\widetilde{F}(t,\phi(t,y)) = \widetilde{F}(0,y), \qquad \widetilde{F}(t,y) \triangleq F(t,y) - \kappa^2 y^2.$$
This means that $\widetilde{F}$ is constant along characteristics.
- Keeping in mind the expressions for $H_\infty$ and $H(0,\cdot)$, (4.3) is easily seen to be equivalent to (1.2).

*Proof of Theorem 4.1.* Let $\phi$ be the solution of the ordinary differential equation (hereafter ODE)

$$\partial_t\phi = \tfrac{1}{2}(\phi^2 - 1)F(t,\phi), \qquad \phi(0,y) = y. \tag{4.4}$$

Since the Herglotz transform is analytic inside $\mathbb{D}$, the function $F$ should be also analytic in the half-plane $\{\operatorname{Re} y > 0\}$. Thus one can for example use the Cauchy–Lipschitz theorem to deduce the local well-posedness for this ODE. Now differentiating the function $F_1 : (t,y) \mapsto F(t,\phi(t,y))$ with respect to $t$ yields

$$\partial_t F_1(t,y) = \tfrac{1}{2}\kappa^2\phi(t,y)(\phi^2(t,y) - 1).$$

Consequently, solving the pde (4.1) reduces to the study of two coupled ODEs:

$$\begin{cases} \partial_t\phi = \tfrac{1}{2}(\phi^2 - 1)F_1, \\ \partial_t F_1 = \tfrac{1}{2}\kappa^2\phi(\phi^2 - 1), \\ \phi(0,y) = y, \quad F_1(0,y) = y. \end{cases} \tag{4.5}$$

It is clear that (4.5) entails

$$F_1\partial_t F_1 - \kappa^2\phi\partial_t\phi = 0.$$

Hence, integrating with respect to $t$ yields (4.2):

$$F_1^2(t,y) - \kappa^2\phi^2(t,y) = (1-\kappa^2)y^2,$$

which in turn leads to

$$\begin{cases} \partial_t\phi = \frac{1}{2}(\phi^2-1)\sqrt{(1-\kappa^2)y^2+\kappa^2\phi^2}, \\ \phi(0,y) = y. \end{cases} \tag{4.6}$$

Now, we shall solve (4.6) for fixed $y > 0$, which is equivalent to $z \in (-1,1)$. First, observe that $\phi(t,y) = \pm 1$ are stationary solutions of (4.6) for any $\kappa$, and by uniqueness of solution of (4.6), we deduce that if $0 < y < 1$ then $\phi$ is global in time and ([2])

$$|\phi(t,y)| < 1, \qquad \forall t \in \mathbb{R}_+.$$

More precisely, assume for instance that there exist $T > 0$ and $0 < y < 1$ such that $\phi(T,y) = 1$. Then $\phi$ and the constant function $t \mapsto 1$ solve the Cauchy problem corresponding to the data $\phi(T,y) = 1$. Necessarily, $\phi = 1$, which contradicts $0 < y < 1$. As a matter of fact, $t \mapsto \phi(t,y)$ is nonincreasing and crosses the right half-plane with limit $-1$ as $t \to \infty$. Similar arguments show that

$$y > 1 \;\Rightarrow\; \phi(t,y) > 1, \forall t \in ]0,T^\star[.$$

Next, we need to compute the indefinite integral

$$2\int \frac{dx}{(1-x^2)\sqrt{(1-\kappa^2)y^2+\kappa^2x^2}}$$

for real positive $x$ (which is equivalent to $\phi \in (-1,1)$). First, we perform the change of variable $u = 1-x^2 \in (-\infty,1)$ to transform the integral to

$$-\int \frac{du}{u\sqrt{a-bu+cu^2}}$$

where we set

$$\begin{aligned} a &= (1-\kappa^2)y^2+\kappa^2, \\ b &= 2\kappa^2+(1-\kappa^2)y^2, \\ c &= \kappa^2. \end{aligned}$$

Note that $b^2-4ac = (1-\kappa^2)^2y^4$ and that the roots of $a-bu+cu^2 = 0$ lie in $[1,\infty]$. Next, we perform the change of variable

$$\sqrt{a}(1-vu) = \sqrt{a-bu+cu^2}$$

and we easily get

$$u = \frac{2av-b}{av^2-c}, \qquad du = -2a\frac{av^2-bv+c}{(av^2-c)^2}dv.$$

([2]) If $y = 1$ then $\phi \equiv 1$.

As a result,

$$\int \frac{du}{u\sqrt{a-bu+cu^2}} = 2\sqrt{a}\int \frac{dv}{2av-b} = \frac{1}{\sqrt{a}} \ln\left|\frac{2a-bu-2\sqrt{a}\sqrt{a-bu+cu^2}}{u}\right|.$$

But $u<1$ so that $2a-bu>2a-b=(1-\kappa^2)y^2>0$ and

$$(2a-bu)^2-4a(a-bu+cu^2)=(b^2-4ac)u^2>0.$$

Consequently,

$$\int \frac{du}{u\sqrt{a-bu+cu^2}} = \frac{1}{\sqrt{a}} \ln \frac{2a-bu-2\sqrt{a}\sqrt{a-bu+cu^2}}{|u|}$$

and if $U(t,y) \triangleq 1-\phi^2(t,y)$, then

$$\frac{1}{\sqrt{a}} \ln \frac{2a-bU(t,y)-2\sqrt{a}\sqrt{a-bU(t,y)+cU^2(t,y)}}{|U(t,y)|} = t+A$$

for some $A=A(y,\kappa)$. Equivalently,

$$\frac{2a-bU(t,y)-2\sqrt{a}\sqrt{a-bU(t,y)+cU^2(t,y)}}{|U(t,y)|} = \lambda e^{\sqrt{a}t}$$

where $\lambda = e^{\sqrt{a}A}$. Writing this equality as

$$2a-[b+\epsilon\lambda e^{\sqrt{a}t}]U(t,y) = 2\sqrt{a}\sqrt{a-bU(t,y)+cU^2(t,y)}$$

and squaring we get

$$\big((b+\epsilon\lambda e^{\sqrt{a}t})^2-4ac\big)U(t,y) = 4a(b+\epsilon\lambda e^{\sqrt{a}t})-4ab = 4\epsilon a\lambda e^{\sqrt{a}t}, \tag{4.7}$$

$\epsilon \in \{-1,1\}$ being the sign of $U$. From the observation made before, the sign of $U$ does not change in time. To find the value of $\lambda$ we check the preceding equation for $t=0$:

$$\lambda^2+2\epsilon\left(b-\frac{2a}{1-y^2}\right)\lambda+b^2-4ac=0.$$

- *Case* $y^2 \le 1$. This corresponds to $\epsilon=1$ and the above equation becomes

$$\lambda^2+2\left(b-\frac{2a}{1-y^2}\right)\lambda+b^2-4ac=0. \tag{4.8}$$

The discriminant of this polynomial is

$$\begin{aligned}\Delta &= \frac{16a}{(1-y^2)^2}\big(a-b(1-y^2)+c(1-y^2)^2\big)\\ &= \frac{16ay^2}{(1-y^2)^2}(b-2c+cy^2) = \frac{16ay^4}{(1-y^2)^2}.\end{aligned}$$

Therefore the only solution of (4.8) which is not singular at $y = 1$ is

$$\begin{aligned}\lambda &= -b + \frac{2a}{1-y^2} - \frac{2y^2\sqrt{a}}{1-y^2} = -b + 2 + \frac{2(a-1)}{1-y^2} - \frac{2y^2(\sqrt{a}-1)}{1-y^2}\\ &= -b + 2\kappa^2 + 2(1-\kappa^2)\frac{y^2}{1+\sqrt{a}} = (1-\kappa^2)y^2\left(-1 + \frac{2}{1+\sqrt{a}}\right)\\ &= \frac{(1-\kappa^2)^2 y^2}{(1+\sqrt{\kappa^2 + (1-\kappa^2)y^2})^2}(1-y^2).\end{aligned}$$

- *Case* $y^2 \geq 1$. Reproducing the same computation yields

$$\lambda = \frac{(1-\kappa^2)^2 y^2}{(1+\sqrt{\kappa^2 + (1-\kappa^2)y^2})^2}(y^2-1).$$

Hence in both cases we get

$$\epsilon\lambda = \frac{(1-\kappa^2)^2 y^2}{(1+\sqrt{\kappa^2 + (1-\kappa^2)y^2})^2}(1-y^2).$$

Finally, (4.7) yields

$$U(t,y) = \frac{4a\lambda e^{\sqrt{a}t}}{(b+\lambda e^{\sqrt{a}t})^2 - 4ac} = 1 - \phi^2(t,y)$$

where we have performed the change $\epsilon\lambda \to \lambda$.

We shall now give compact formulae for $\phi$ and $\psi$. To this end, write

$$\phi^2(t,y) = 1 - \frac{4a\lambda e^{\sqrt{a}t}}{(a+\kappa^2+\lambda e^{\sqrt{a}t})^2 - 4\kappa^2 a} = \frac{(\kappa^2 - a + \lambda e^{\sqrt{a}t})^2}{(\kappa^2 - a + \lambda e^{\sqrt{a}t})^2 + 4a\lambda e^{\sqrt{a}t}}$$

and

$$\lambda = \frac{(1-a)(a-\kappa^2)}{(1+\sqrt{a})^2}.$$

Hence, for any $\kappa \in (-1,1)$ we get

$$\phi^2(t,y) = \frac{(1+\xi_t(\sqrt{a}))^2}{(\xi_t(\sqrt{a})+1)^2 - 4\frac{a}{a-\kappa^2}\xi_t(\sqrt{a})} = \frac{1}{1 - \frac{a}{a-\kappa^2}\alpha^{-1}(\xi_t(\sqrt{a}))}.$$

Accordingly, the domain of definition of $\phi$ in $\{y > 0\}$ is submitted to the constraint

$$\alpha^{-1}(\xi_t(\sqrt{a})) < \frac{a-\kappa^2}{a}\cdot$$

Since $\alpha$ is nondecreasing in $(-\infty, 1)$, the last inequality reduces to

$$\xi_t(\sqrt{a}) < \alpha(1-\kappa^2/a) = \frac{\sqrt{a}-|\kappa|}{\sqrt{a}+|\kappa|}\cdot \tag{4.9}$$

But the variations of the functions $\xi_t$ and $u \mapsto (u-|\kappa|)(u+|\kappa|)$ show that the equation

$$\xi_t(u) = \frac{u-|\kappa|}{u+|\kappa|}$$

has a unique positive solution $u_t > 1$. Thus the inequality (4.9) is satisfied provided that $\sqrt{a} < u_t$ and we obtain the formula

$$\phi(t,y) = \frac{1}{\sqrt{1-\frac{a}{a-\kappa^2}\alpha^{-1}(\xi_t(\sqrt{a}))}}.$$

Finally, the condition $\sqrt{a} < u_t$ is equivalent to $0 < y < \sqrt{(u_t^2-\kappa^2)/(1-\kappa^2)}$ and the expression for $\psi$ follows from elementary computations. ∎

REMARK 4.2. From the recursive relation satisfied by $r_n(t)$ and the inequality $|r_n(t)| \le 1$ , we readily see that $|r_n'(t)| \le 2n^2$, which implies that the map $t \mapsto H(t,z)$ is $C^1$. More generally, one proves for each $k \ge 1$, using the Leibniz rule, that $|r_n^{(k)}(t)| \le c(k)n^{2k}, n \ge 1$, for some constant $c(k)$ depending only on $k$ and such that $c(k+1) \ge c(k)$. Thus $t \mapsto H(t,z)$ is even an analytic map and the Cauchy–Kowalevski Theorem applies to (4.4).

We close this section with the following discussion about the monotonicity of the lifespans of the solutions to (4.6) with respect to the initial data.

PROPOSITION 4.3. *Let $1 < y_1 < y_2$ and denote by $T_i^\star$ the lifespan of the trajectory $t \mapsto \phi(t,y_i)$. Then $T_2^\star \le T_1^\star$ and*

$$1 < \phi(t,y_1) < \phi(t,y_2), \qquad \forall t \in [0,T_2^\star[.$$

*Proof.* Let $t \mapsto \phi_i(t)$ be the trajectory associated to $y_i$. Then $\phi_i > 1$ and

$$\begin{cases} \frac{d\phi_i}{dt} = \frac{1}{2}(\phi_i^2-1)\sqrt{(1-\kappa^2)y_1^2+\kappa^2\phi_i^2}, \\ \phi_2(0) > \phi_1(0). \end{cases} \tag{4.10}$$

Assume now that $\phi_1$ and $\phi_2$ intersect at some (first) time $T$. Then

$$\forall t \in [0,T[,\ \phi_1(t) < \phi_2(t), \quad \text{and} \quad \phi_1(T) = \phi_2(T),$$

which implies from (4.10) that $\phi_2'(T) = \phi_1'(T)$, and using the backward uniqueness of the Cauchy problem we conclude that the functions $\phi_1$ and $\phi_2$ must agree everywhere in $[0,T]$, which is a contradiction. Hence, the two trajectories do not intersect and the inequality $T_2^\star \le T_1^\star$ follows from the blow up criterion. ∎

**4.2. Computing $\mu_t\{1\}$ and $\nu_t\{1\}$.** To compute $\mu_t\{1\}$ and $\nu_t\{1\}$, we need the following lemma which gives the limit of the RHS of (1.1) as $n \to \infty$.

LEMMA 4.4. *The following assertions hold true:*

- *Let $\phi \in [0, 2\pi]$. Then*

$$\frac{1}{2^{2n}} \sum_{k=1}^{n} \binom{2n}{n-k} (e^{ik\phi} + e^{-ik\phi}) = \cos^{2n}(\phi/2) - \frac{1}{2^{2n}} \binom{2n}{n}.$$

- *Let $\mu$ be a probability distribution on the unit circle $\mathbb{T}$. Then*

$$\lim_{n\to\infty} \frac{1}{2^{2n}} \sum_{k=1}^{n} \binom{2n}{n-k} \int_{\mathbb{T}} (z^k + \overline{z}^k)\, \mu(dz) = \mu(\{1\}).$$

- *Recall the spectral distribution $\mu_t$ of the unitary operator $SY_tSY_t^\star$. Then*

$$\lim_{n\to\infty} \tau[(PY_tPY_t^\star)^n] = \frac{1}{2}[2\theta - 1 + \mu_t(\{1\})].$$

*Proof.* • Using the fact that $\binom{2n}{n-k} = \binom{2n}{n+k}$ we write

$$\begin{aligned}
\sum_{k=1}^{n} \binom{2n}{n-k} (e^{ik\phi} + e^{-ik\phi}) &= \sum_{k=1}^{n} \binom{2n}{n-k} e^{ik\phi} + \sum_{k=-n}^{-1} \binom{2n}{n+k} e^{ik\phi} \\
&= \sum_{k=-n}^{n} \binom{2n}{n+k} e^{ik\phi} - \binom{2n}{n} \\
&= e^{-in\phi} \sum_{k=0}^{2n} \binom{2n}{k} e^{ik\phi} - \binom{2n}{n} \\
&= 2^{2n} \cos^{2n}(\phi/2) - \binom{2n}{n}.
\end{aligned}$$

• Identifying $\mu$ with its image under the map $z \mapsto \arg(z) \in (-\pi, \pi]$ we readily get

$$\begin{aligned}
\frac{1}{2^{2n}} \sum_{k=1}^{n} \binom{2n}{n-k} \int_{\mathbb{T}} (z^k + \overline{z}^k)\, \mu(dz) &= \frac{1}{2^{2n}} \sum_{k=1}^{n} \binom{2n}{n-k} \int_{-\pi}^{\pi} (e^{ik\phi} + e^{-ik\phi})\, \mu(d\phi) \\
&= \int_{-\pi}^{\pi} \cos^{2n}(\phi/2)\, \mu(d\phi) - \frac{1}{2^{2n}} \binom{2n}{n}.
\end{aligned}$$

The result follows from the Stirling formula

$$\lim_{n\to\infty} \frac{1}{2^{2n+1}} \binom{2n}{n} = \lim_{n\to\infty} \frac{1}{\sqrt{\pi n}} = 0$$

and from the Lebesgue convergence theorem.

• Due to the trace property of $\tau$, the spectral distributions of $SY_tSY_t^\star$ and of $Y_t^\star SY_tS$ coincide so that $\mu_t$ is invariant under $z \mapsto \overline{z}$. Hence

$$2\int_{\mathbb{T}} z^k \, \mu_t(dz) = \int_{\mathbb{T}} (z^k + \overline{z}^k) \, \mu_t(dz),$$

and the desired limit follows from (1.1). ∎

Corollary 4.5. *For any $t > 0$,*

$$\lim_{z \to 1^-} (1-z)H(t,z) = 2|\kappa|.$$

*Consequently,*

$$\nu_t\{1\} = \frac{1}{2\theta}[2\theta - 1 + |\kappa|] = \frac{1}{\theta}\max\{2\theta - 1, 0\}.$$

*Proof.* Fix $t > 0$. There exists $z_t \in (0,1)$ such that $\psi(t, z_t) = 1$ or equivalently there exists $y_t = (1+z_t)/(1-z_t) > 1$ such that

$$\lim_{y \to y_t^-} \phi(t,y) = \infty.$$

We deduce from Proposition 4.3 that the lifespan of the trajectory starting at $y \in \,]1, y_t[$ is larger than the lifespan $t$ of the trajectory $\phi(\cdot, y_t)$. Keeping in mind $F(t,y) = H(t,z)$, we get

$$\lim_{z \to 1^-} (1-z)H(t,z) = 2 \lim_{y \to \infty} \frac{F(t,y)}{y} = 2 \lim_{y \to y_t^-} \frac{F(t, \phi(t,y))}{\phi(t,y)}.$$

But formula (4.2) entails

$$\lim_{y \to y_t^-} \frac{F^2(t, \phi(t,y))}{\phi^2(t,y)} = \kappa^2,$$

whence we deduce

$$\lim_{z \to 1^-} (1-z)H(t,z) = 2|\kappa|.$$

Now, it is a general fact that the discrete part of $\mu_t$ corresponds exactly to the poles of $H(t,\cdot)$. Moreover, the weight that $\mu_t$ assigns to a given pole can be recovered using radial limits [CMR]. In particular, $\mu_t\{1\} = |\kappa|$ and $\nu_t\{1\}$ follows from the last assertion of Lemma 4.4 after normalizing (1.1) by $1/\tau(P) = 1/\theta$ since

$$\lim_{n \to \infty} \frac{1}{\tau(P)} \tau[(PY_tPY_t^\star)^n] = \lim_{n \to \infty} \int_0^1 x^n \, \nu_t(dx) = \nu_t\{1\}. \;\blacksquare$$

**4.3. Relating $\mu_t\{0\}$ and $\nu_t\{-1\}$.** The relation between $\mu_t\{1\}$ and $\nu_t\{0\}$ was established using (1.1). Imitating the computations in the proof of Lemma 2.1, we find that this binomial-type expansion relates as well the moment generating function of $\nu_t$ to $H_t$. To see this, set

$$m_n(t) \triangleq \frac{1}{\tau(P)}\tau((PY_tPY_t^\star)^n) = \int_0^1 x^n\,\nu_t(dx),$$
$$M_t(z) \triangleq \sum_{n\geq 0} m_n(t)z^n, \qquad |z|<1.$$

Using (1.1) and similar computations to those leading to (2.2), we find the identity

$$M_t(z) = \frac{1}{(1+\tau(S))\sqrt{1-z}}\left[H_t(\alpha(z)) + \frac{\tau(S)}{\sqrt{1-z}}\right]. \tag{4.11}$$

Note that since $\alpha$ is conformal from $\mathbb{C}\setminus[1,\infty)$ onto the open disc $\mathbb{D}$, (4.11) entails that $M_t$ admits a holomorphic extension to $\mathbb{C}\setminus[1,\infty)$. Now we relate $\nu_t\{0\}$ and $\mu_t\{1\}$ as follows.

Proposition 4.6. *Let*

$$G_t(z) \triangleq \frac{1}{z}M_t\left(\frac{1}{z}\right) = \int_0^1 \frac{1}{z-x}\,\nu_t(dx), \qquad z\in\mathbb{C}\setminus[0,1],$$

*be the Cauchy–Stieltjes transform of $\nu_t$. Then*

$$\nu_t\{0\} = -\lim_{h\to 0^+}\operatorname{Im}[hG_t(ih)] = \frac{1}{1+\tau(S)}\mu_t\{-1\} = 0.$$

*Proof.* By the very definition of $G_t$ and from (4.11), it follows that

$$\begin{aligned}-\lim_{h\to 0^+}\operatorname{Im}[hG_t(ih)] &= \lim_{h\to 0^+}\operatorname{Re}\left[M_t\left(\frac{1}{ih}\right)\right]\\ &= \frac{1}{(1+\tau(S))}\lim_{h\to 0^+}\operatorname{Re}\left[\frac{H_t(\alpha(1/(ih)))}{\sqrt{1-1/(ih)}}\right].\end{aligned}$$

But

$$\frac{H_t(\alpha(z))}{\sqrt{1-z}} = \frac{1+\alpha(z)}{1-\alpha(z)}H_t(\alpha(z))$$

and

$$\alpha(1/(ih)) = i(\sqrt{h}-\sqrt{h+i})^2 = 2ih-1-2i\sqrt{h}\sqrt{h+i}$$

tends (nontangentially) to $-1$, which yields

$$\lim_{h\to 0^+}\frac{|1+\alpha(1/(ih))|}{1-|\alpha(1/(ih))|} = \sqrt{2}.$$

Now

$$\frac{1}{|\zeta-\alpha(1/(ih))|} \leq \frac{1}{1-|\alpha(1/(ih))|}, \qquad |\zeta|=1,$$

shows that

$$h \mapsto \frac{\zeta + \alpha(1/(ih))}{\zeta - \alpha(1/(ih))}$$

is bounded for small positive values of $h$. Hence, the Lebesgue convergence theorem implies

$$\lim_{h \to 0^+} \frac{1 + \alpha(1/(ih))}{1 - \alpha(1/(ih))} H_t(\alpha(1/ih)) = \mu_t\{-1\},$$

and consequently

$$-\lim_{h \to 0^+} \operatorname{Im}[hG_t(ih)] = \frac{1}{1 + \tau(S)} \mu_t\{-1\}.$$

Finally, the expression for $\psi$ readily shows that

$$\lim_{z \to -1^+} \psi(t, z) = -1.$$

For any $t > 0$, since

$$2\mu_t\{-1\} = \lim_{z \to -1^+} (1 + z) H_t(z),$$

(1.2) leads to $\mu_t(\{-1\}) = 0$, and the proposition is proved. ∎

**5. End of the proof of Theorem 1.1.** So far, we have determined the discrete part of the Lebesgue decomposition of $\nu_t$ relying on both (1.1) and (1.2). In this section, we prove the second statement of Theorem 1.1 which provides the partial description of the density of $\nu_t$. From Fatou's theorem for harmonic functions on the upper half-plane [Don], this is given by

$$-\frac{1}{\pi} \lim_{y \to 0^+} \operatorname{Im}[G_t(x + iy)]$$

whenever the distribution function of $\nu_t$ is differentiable at $x \in [0, 1]$. But (4.11) shows that for any $x \in (0, 1)$,

$$\begin{aligned} -\lim_{y \to 0^+} \operatorname{Im}[G_t(x + iy)] &= \frac{1}{\sqrt{x(1 - x)}} \lim_{y \to 0^+} \operatorname{Re}\{H_t[\alpha(1/(x + iy))]\} \\ &= \frac{1}{\sqrt{x(1 - x)}} \lim_{y \to 0^+} \{\operatorname{Re}[H_t]\}[\alpha(1/(x + iy))]. \end{aligned}$$

Now we need to investigate how $\alpha(1/(x+iy))$ approaches $\alpha(1/x)$ as $y \to 0^+$. This is summarized in the following lemma:

LEMMA 5.1. *For any $x \in [0, 1]$, $\alpha(1/(x+iy)) \to \alpha(1/x)$ nontangentially as $y \to 0^+$.*

*Proof.* Let $x \in [0, 1]$. Then the statement is equivalent to the argument of the complex number

$$1 - \frac{\alpha(1/(x + iy))}{\alpha(1/x)}$$

being in $]-\pi/2, \pi/2[$ for small positive values of $y$. Hence, it suffices to compute the derivative at $y = 0$ of the curve

$$y \mapsto 1 - \frac{\alpha(1/(x+iy))}{\alpha(1/x)}.$$

But (2.3) shows again that the sought derivative equals

$$\frac{i}{x^2}\,\frac{1}{\alpha(1/x)}\alpha'(1/x) = \frac{1}{\sqrt{x(1-x)}},$$

therefore it is real positive. The lemma is proved. ∎

According to this lemma, $\mathrm{Re}[H_t]$ has a nontangential limit along the curve $y \mapsto \alpha(1/(x+iy))$ at any point $x \in (0,1)$ where the density of $\nu_t$ exists. By [BV, Lemma 5.11], $\mathrm{Re}[H_t]$ tends nontangentially to $\alpha(1/x)$ at any such $x$. Finally, the density of $\nu_t$ is given for all $x \in (0,1)$ by

$$\begin{aligned} -\lim_{y\to 0^+} \mathrm{Im}[G_t(x+iy)] &= \frac{1}{\sqrt{x(1-x)}}\{\mathrm{Re}[H_t]\}[\alpha(1/x)] \\ &= \frac{1}{\sqrt{x(1-x)}}\{\mathrm{Re}[H_t]\}[e^{i\arccos(2x-1)}] \\ &= \frac{1}{\sqrt{x(1-x)}}\{\mathrm{Re}[H_t]\}[e^{i2\arccos(\sqrt{x})}]. \end{aligned}$$

The proof of Theorem 1.1 is complete. ∎

Remarks. 1. The main result proved in [Don] shows that the symmetric derivative of the distribution function of $\mu_t$ exists and is finite at any $x$ where the distribution function of $\nu_t$ is differentiable.

2. Unfortunately, we have not been able to prove the extension of $\psi(t,\cdot)$ to the unit circle when $\kappa \neq 0$. When $\kappa = 0 \Leftrightarrow \tau(P) = 1/2$, $H_t$ has a continuous extension to the closed unit disc for any time $t > 0$ [Bi2] and one retrieves the description of $\nu_t$ derived in [DHH].

**6. Analysis of the moments.** Since we do not have at our disposal an explicit expression of the moments $r_n(t)$, we perform an analysis of the term corresponding to the fastest decay $e^{-nt}$. To proceed, set $c \triangleq \kappa^2 = [\tau(S)]^2$ and recall from Proposition 3.1 that $s_n(t) = e^{nt}r_n(t)$, $n \geq 1$, satisfy

$$\begin{aligned} \partial_t s_n(t) &= -n\sum_{j=1}^{n-1} s_j(t)s_{n-j}(t) + cn^2e^{nt}, \\ s_1(t) &= ce^t + (1-c). \end{aligned}$$

When $c = 0$, we already know that [DHH]

$$s_n(t) = \frac{1}{n}L_{n-1}^{(1)}(2nt)$$

where $L_n^{(1)}$ is the $n$th Laguerre polynomial [Ra]. For general $c \in (0,1)$, it is easy to see by induction that

$$s_n(t) = P_n(t) + \sum_{k=1}^{n} e^{kt} \times \text{polynomial of degree } k-1,$$

where $P_n(t) \equiv P_n(c,t)$ is a polynomial in $t$ of degree $n-1$ and depending on $c$. Now, observe that compared to the equation satisfied by $s_n(t)$ when $c = 0$, the deformation comes with the factor $e^{nt}$. As a matter of fact, the polynomials $P_n(t), n \geq 1$, still satisfy

$$\begin{aligned}\partial_t P_n(t) &= -n \sum_{k=1}^{n-1} P_j(t) P_{n-j}(t), \qquad n \geq 2,\\ P_1(t) &= 1 - c.\end{aligned}$$

However, one needs to compute $P_n(0)$ in order to determine the polynomials $P_n(t)$, $n \geq 2$. We shall see that while $P_n(0) = s_n(0) = 1$ when $c = 0$, $P_n(0)$ changes drastically when $c \in (0,1)$. Indeed, expand $s_n(t)$ as [NS, Theorem 14.4]

$$\begin{aligned}s_n(t) &= e^{nt}\tau((SY_tSY_t^\star)^n)\\ &= e^{nt} \sum_{\pi \in \mathrm{NC}(2n)} c_\pi(S,\dots,S) m_{K(\pi)}(Y_t, Y_t^\star, \dots, Y_t, Y_t^\star).\end{aligned}$$

Here $\mathrm{NC}(2n)$ is the lattice of noncrossing partitions of size $2n$, $K(\pi) \in \mathrm{NC}(2n)$ denotes the Kreweras complement of $\pi$, $c_\pi(S,\dots,S)$ is the free cumulant of the $2n$-tuple $(S,\dots,S)$ associated with $\pi$, and $m_{K(\pi)}$ is the mixed moment of the $2n$-tuple $(Y_t, Y_t^\star, \dots, Y_t, Y_t^\star)$ [NS, Chapter XI]. But since the polynomial $P_n$ comes without any exponential factor, we only need to focus exactly on partitions $\pi \in \mathrm{NC}(2n)$ whose Kreweras complement $K(\pi)$ is non-parity-alternating, that is, each block of $K(\pi)$ lies either in $\{1,3,\dots,2n-1\}$ or in $\{2,4,\dots,2n\}$ (we identify $K(\pi) \approx \{1,\dots,2n\}$). More precisely, the $k$th moment of $Y_t$ is given by (see [Bi1])

$$e^{-kt/2}\frac{1}{k}L_{k-1}^{(1)}(kt), \qquad k \geq 1,$$

so that the polynomial $P_n$ corresponds to $m_{K(\pi)}$ for which there is no cancellation between $Y$ and $Y^\star$. According to [NS, Exercise 9.42, pp. 153–154], the partition $\pi$ runs over the set $\mathrm{NCE}(2n)$ of noncrossing even partitions (each block of $\pi$ has an even number of elements). Moreover, since the constant term of

$$\frac{1}{k}L_{k-1}^{(1)}(kt)$$

equals 1 for any $k \geq 1$, we end up with

$$P_n(0) = \sum_{\pi \in \mathrm{NCE}(2n)} c_\pi(S, \dots, S).$$

We can write this sum as

$$P_n(0) = \sum_{\pi \in \mathrm{NC}(2n)} \frac{1}{2^{|\pi|}} c_\pi(a_1 - a_2, \dots, a_1 - a_2)$$

where $a_1, a_2 \in \mathscr{A}$ are two free copies of $S$ and $|\pi|$ is the number of blocks of $\pi$. Indeed, by freeness of $a_1$ and $a_2$ and multilinearity of free cumulants, one has

$$c_V(a_1 - a_2, \dots, a_1 - a_2) = c_V(a_1, \dots, a_1) + c_V(-a_2, \dots, -a_2)$$

for any block $V \in \pi$, whence the equality follows. This new way of expressing $P_n(0)$ hints at the even moments of the $1/2$-fold free convolution of the spectral distribution of $a_1 - a_2$ [NS]. Note that if $c = 0$ then $a_1, a_2, -a_2$ are distributed according to the symmetric Bernoulli distribution

$$\tfrac{1}{2}[\delta_1 + \delta_{-1}],$$

hence the $1/2$-fold free convolution of the spectral distribution of $a_1 - a_2$ is still the symmetric Bernoulli distribution. Accordingly, we retrieve $P_n(0, 0)$:

$$P_n(0) = \int x^{2n} \frac{1}{2}[\delta_1 + \delta_{-1}](dx) = 1.$$

However, when $c \neq 0$ the situation becomes rather cumbersome: the spectral distribution of $a_1$ is given by

$$\theta\delta_1 + (1 - \theta)\delta_{-1},$$

while that of $-a_2$ is given by

$$(1 - \theta)\delta_1 + \theta\delta_{-1}.$$

Equivalently, the $R$-transform of $a_1$ reads

$$R_{a_1}(u) = \frac{\sqrt{1 + 4u(u + \kappa)} - 1}{2u},$$

while that of $-a_2$ reads

$$R_{-a_2}(u) = \frac{\sqrt{1 + 4u(u - \kappa)} - 1}{2u}$$

near $u = 0$. It follows that the $R$-transform of the $1/2$-fold free convolution of $a_1 - a_2$ is given by

$$R_{a_1 - a_2}(u) = \frac{1}{2}[R_{a_1}(u) + R_{-a_2}(u)] = \frac{\sqrt{1 + 4u(u + \kappa)} + \sqrt{1 + 4u(u - \kappa)} - 2}{4u}$$

and that its $K$-transform is given by

$$K_{a_1-a_2}(u) \triangleq R_{a_1-a_2}(u) + \frac{1}{u} = \frac{\sqrt{1+4u(u+\kappa)} + \sqrt{1+4u(u-\kappa)} + 2}{4u}.$$

Inverting $K$ (in composition sense) leads to the cubic polynomial equation

$$u^3 - h_1(v)y^2 + h_2(v)y - h_3(v) = 0,$$

where

$$h_1(v) = \frac{2v^2-1}{v(v^2-1)},$$
$$h_2(v) = \frac{5v^2+c-1}{4v^2(v^2-1)},$$
$$h_3(v) = \frac{1}{4v(v^2-1)}.$$

After elementary transformations, we can express the solutions of this equation through Gauss hypergeometric functions ${}_2F_1$ [Hi, pp. 265–266].

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## REFERENCES

[BL] F. Benaych-Goerges and T. Lévy, *A continuous semigroup of notions of independence between the classical and the free one*, Ann. Probab. 39 (2011), 904–938.

[BV] H. Bercovici and D. V. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J. 42 (1993), 733–773.

[Bi1] P. Biane, *Free Brownian motion, free stochastic calculus and random matrices*, in: Fields Inst. Comm. 12, Amer. Math. Soc., Providence, RI, 1997, 1–19.

[Bi2] P. Biane, *Segal–Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems*, J. Funct. Anal. 144 (1997), 232–286.

[CMR] J. Cima, A. L. Matheson and W. T. Ross, *The Cauchy Transform*, Math. Surveys Monogr. 125, Amer. Math. Soc., 2006.

[Co] B. Collins, *Product of random projections, Jacobi ensembles and universality problems arising from free probability*, Probab. Theory Related Fields 133 (2005), 315–344.

[CK] B. Collins and T. Kemp, *Liberating projections*, J. Funct. Anal. 266 (2014), 1988–2052.

[De1] N. Demni, *Free Jacobi process*, J. Theoret. Probab. 21 (2008), 118–143.

[De2] N. Demni, *Free martingale polynomials for stationary Jacobi processes*, in: Quantum Probability and Related Topics, QP-PQ Quantum Probab. White Noise Anal. 23, World Sci., 2008, 107–119.

[DHH] N. Demni, T. Hamdi and T. Hmidi, *Spectral distribution of the free Jacobi process*, Indiana Univ. Math. J. 61 (2012), 1351–1368.

[Don] W. F. Donoghue, *A theorem of the Fatou type*, Monatsh. Math. 67 (1963), 225–228.

[Erd] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions, Vol. I*, McGraw-Hill, New York, 1981.

[Hi] E. Hille, *Analytic Function Theory, Vol. 1*, Ginn, 1959.

[NS] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, London Math. Soc. Lecture Note Ser. 335, Cambridge Univ. Press, 2006.

[Ra] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.

[Ru] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1966.

[VDN] D. V. Voiculescu, K. J. Dykema and A. Nica, *Free Random Variables*, CRM Monogr. Ser. 1, Amer. Math. Soc., 1992.

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