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CONSTRUCTION OF AUSLANDER–GORENSTEIN LOCAL RINGS AS FROBENIUS EXTENSIONS

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Abstract. Starting from an arbitrary ring R we provide a systematic construction of $\mathbb{Z}/n\mathbb{Z}$-graded rings A which are Frobenius extensions of R, and show that under mild assumptions, A is an Auslander–Gorenstein local ring if and only if so is R.

1. Introduction. Although commutative Gorenstein local rings have been studied extensively (see e.g. [Ma]), there is a lack of study of Auslander–Gorenstein local rings, the class of which contains commutative Gorenstein local rings. This is because we know very few examples of Auslander–Gorenstein local rings which are not commutative, despite the fact that Auslander–Gorenstein rings appear in various fields of current research in mathematics. For instance, regular 3-dimensional algebras of type A in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite-dimensional Lie algebras and Sklyanin algebras are Auslander–Gorenstein rings (see [ATV], [Bj1], [Bj2] and [TV], respectively).

In this note, starting from an arbitrary Auslander–Gorenstein local ring we will provide a systematic construction of Auslander–Gorenstein local rings as Frobenius extensions, a notion we recall in Section 1.

We fix a set of integers $I = \{0, 1, \dots, n-1\}$ with $n \geq 2$, and a cyclic permutation

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

of I. Note that the law of composition $I \times I \to I$, $(i,j) \mapsto \pi^j(i)$, makes I into a cyclic group with 0 the unit element. Note also that if $A = F[X]$ is the polynomial ring in one variable X over a ring F, and $R = F[X^n]$ is a subring of A, then A can be considered as an I-graded ring over R. In this note, starting from an arbitrary ring R, we provide a systematic

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way to construct I-graded rings A so that the ring extensions A/R are split Frobenius extensions of second kind. Namely, we will define an appropriate multiplication on a free right R-module A with a basis $\{e_i\}_{i\in I}$ using the following two data: certain pairs (q,χ) of an integer q and a mapping $\chi : I \to \mathbb{Z}$; and certain triples (σ, c, t) of $\sigma \in \mathrm{Aut}(R)$ and $c, t \in R$. Our main results state that if either $t \in \mathrm{rad}(R)$, or $c \in \mathrm{rad}(R)$ and $n\chi(i) > iq$ for all $i \neq 0$, then A is an Auslander–Gorenstein local ring if and only if so is R (Theorems 3.6 and 3.7). Also, in the final section, we will provide a way to obtain every pair (q,χ) mentioned above.

2. Preliminaries. For a ring R we denote by $\mathrm{rad}(R)$ the Jacobson radical of R, by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R, by $\mathrm{Aut}(R)$ the group of ring automorphisms of R, for $\sigma \in \mathrm{Aut}(R)$ by R^σ the subring of R consisting of all $x \in R$ with $\sigma(x) = x$, and for $n \geq 2$ by $\mathrm{M}_n(R)$ the ring of $n\times n$ matrices over R. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by $\mathrm{inj\,dim}\, R$ (resp., $\mathrm{inj\,dim}\, R^{\mathrm{op}}$) the injective dimension of R as a right (resp., left) R-module, and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}).

We start by recalling the notion of Auslander–Gorenstein ring.

Proposition 2.1 (Auslander; see e.g. [FGR, Theorem 3.7]). *Let R be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent:*

(1) *In a minimal injective resolution $I^\bullet$ of R in* Mod-R*,* $\mathrm{flat\,dim}\, I^i \leq i$ *for all $0 \leq i \leq n$.*
(2) *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}*,* $\mathrm{flat\,dim}\, J^i \leq i$ *for all $0 \leq i \leq n$.*
(3) *For any $1 \leq i \leq n+1$, any $M \in$* mod-R *and any submodule X of* $\mathrm{Ext}^i_R(M,R) \in$ mod-R^{op} *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X,R) = 0$ *for all $0 \leq j < i$.*
(4) *For any $1 \leq i \leq n+1$, any $X \in$* mod-R^{op} *and any submodule M of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X,R) \in$ mod-R *we have* $\mathrm{Ext}^j_R(M,R) = 0$ *for all $0 \leq j < i$.*

Definition 2.2 ([Bj2]). A right and left noetherian ring R is said to satisfy the *Auslander condition* if it satisfies the equivalent conditions in Proposition 2.1 for all $n \geq 0$, and to be an *Auslander–Gorenstein ring* if it satisfies the Auslander condition and $\mathrm{inj\,dim}\, R = \mathrm{inj\,dim}\, R^{\mathrm{op}} < \infty$.

It should be noted that for a right and left noetherian ring R we have $\mathrm{inj\,dim}\, R = \mathrm{inj\,dim}\, R^{\mathrm{op}}$ whenever $\mathrm{inj\,dim}\, R < \infty$ and $\mathrm{inj\,dim}\, R^{\mathrm{op}} < \infty$ (see [Za, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [NT1, NT2], which we modify as follows (cf. [AH, HKK]).

DEFINITION 2.3 ([HKK]). A ring A is said to be an *extension* of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be *Frobenius* if the following conditions are satisfied:

(F1) A is finitely generated as a left R-module;
(F2) A is finitely generated projective as a right R-module;
(F3) $A \cong \operatorname{Hom}_R(A, R)$ as right A-modules.

PROPOSITION 2.4 ([HKK]). *Let A/R be a ring extension, and let $\phi : A \xrightarrow{\sim} \operatorname{Hom}_R(A, R)$ be an isomorphism in* Mod-A. *Then:*

1. *There exists a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$.*
2. *Let $\phi' : A \xrightarrow{\sim} \operatorname{Hom}_R(A, R)$ be another isomorphism in* Mod-A, *and let $\theta' : R \to A$ be the associated ring homomorphism such that $x\phi(1) = \phi(1)\theta'(x)$ for all $x \in R$. Then there exists $u \in A^\times$ such that $\phi'(1) = \phi(1)u$ and $\theta'(x) = u^{-1}\theta(x)u$ for all $x \in R$.*
3. *ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*

DEFINITION 2.5 (cf. [NT1, NT2]). A Frobenius extension A/R is said to be *of first kind* if $A \cong \operatorname{Hom}_R(A, R)$ as R-A-bimodules, and to be *of second kind* if there exists an isomorphism $\phi : A \xrightarrow{\sim} \operatorname{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind.

PROPOSITION 2.6 ([HKK, Proposition 1.6]). *If A/R is a Frobenius extension of second kind, then A is projective as a left R-module.*

PROPOSITION 2.7 ([HKK, Proposition 1.7]). *For any Frobenius extensions Λ/A, A/R the following hold:*

1. *Λ/R is a Frobenius extension.*
2. *Assume Λ/A is of first kind. If A/R is of second (resp., first) kind, then so is Λ/R.*

DEFINITION 2.8 ([AH]). A ring extension A/R is said to be *split* if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

PROPOSITION 2.9 ([HKK, Proposition 1.9]). *For any Frobenius extension A/R the following hold:*

1. *If R is an Auslander–Gorenstein ring, then so is A with* $\operatorname{inj\,dim} A \leq \operatorname{inj\,dim} R$.

(2) *Assume A is projective as a left R-module and A/R is split. If A is an Auslander–Gorenstein ring, then so is R with* $\operatorname{inj\,dim} R = \operatorname{inj\,dim} A$.

3. Construction. Throughout the rest of this note, we fix a set of integers $I = \{0, 1, \dots, n-1\}$ with $n \geq 2$, and a cyclic permutation

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

of I. Note that the law of composition $I \times I \to I$, $(i, j) \mapsto \pi^j(i)$, makes I a cyclic group with 0 the unit element. Note also that if $A = F[X]$ is the polynomial ring in one variable X over a ring F, and $R = F[X^n]$ is a subring of A, then A can be considered as an I-graded ring over R. In the following, we will provide a systematic construction of I-graded local rings starting from an arbitrary local ring.

Throughout this section, we fix a pair (q, χ) of an integer q and a mapping $\chi : I \to \mathbb{Z}$ satisfying the following conditions:

(X1) $q - \chi(n-j+i) \leq \chi(j) - \chi(i) \leq \chi(j-i)$ for all $i, j \in I$ with $i < j$;
(X2) $\chi(i) + \chi(n-i-1) = \chi(n-1)$ for all $i \in I$.

These are obviously satisfied if $q \leq n$ and $\chi(i) = i$ for all $i \in I$. We set

$$\omega(i,j) = \begin{cases} \chi(i) + \chi(j) - \chi(\pi^j(i)) & \text{if } i+j < n, \\ \chi(i) + \chi(j) - \chi(\pi^j(i)) - q & \text{if } i+j \geq n, \end{cases}$$

for $i, j \in I$.

LEMMA 3.1. *The following hold:*

(1) $\omega(i,j) \geq 0$ *for all* $i, j \in I$.
(2) $\omega(0,i) = \omega(i,0) = \chi(0) = 0$ *for all* $i \in I$.
(3) $\omega(i, n-i-1) = 0$ *for all* $i \in I$.

Proof. (1) If $i + j < n$, then setting $j' = i + j$ we have $i, j' \in I$ with $i < j'$ and

$$\omega(i,j) = \chi(j'-i) - \{\chi(j') - \chi(i)\}.$$

If $i + j \geq n$, then setting $i' = i + j - n$ we have $i', j \in I$ with $i' < j$ and

$$\omega(i,j) = \{\chi(j) - \chi(i')\} - \{q - \chi(n-j+i')\}.$$

Consequently, the assertion follows from (X1).

(2) By definition we have $\omega(0,i) = \omega(i,0) = \chi(0)$, and by (X2), $\chi(0) = 0$.

(3) Immediate by (X2). ■

In the following, we fix a ring R together with a triple (σ, c, t) of $\sigma \in \operatorname{Aut}(R)$ and $c, t \in R$ satisfying the following condition:

$(*)$ $\qquad c, t \in R^\sigma \quad \text{and} \quad xc = c\sigma(x), xt = t\sigma^q(x)$ for all $x \in R$.

This is obviously satisfied if either $\sigma = \mathrm{id}_R$ and $c, t \in \mathrm{Z}(R)$, or σ is arbitrary and $c = t = 0$. Note also that $ct = tc$. As usual, we require $c^0 = 1$ even if $c = 0$.

Let A be a free right R-module with a basis $\{e_i\}_{i\in I}$, and $\{\delta_i\}_{i\in I}$ the dual basis of $\{e_i\}_{i\in I}$ for the free left R-module $\mathrm{Hom}_R(A, R)$, i.e., $a = \sum_{i\in I} e_i\delta_i(a)$ for all $a \in A$. According to Lemma 3.1(1), we can define a multiplication on A subject to the following axioms:

(M1) $e_ie_j = e_{\pi^j(i)}c^{\omega(i,j)}$ if $i + j < n$, and $e_ie_j = e_{\pi^j(i)}tc^{\omega(i,j)}$ if $i + j \geq n$;
(M2) $xe_i = e_i\sigma^{\chi(i)}(x)$ for all $x \in R$ and $i \in I$.

We will see that A is an associative ring with $1 = e_0$, and the mapping

$$\phi : A \to \mathrm{Hom}_R(A, R), \qquad a \mapsto \delta_{n-1}a,$$

is an isomorphism in Mod-A with $\sigma^{\chi(n-1)}(x)\phi(1) = \phi(1)x$ for all $x \in R$.

Lemma 3.2. *The following hold:*

(1) *For any $a, b \in A$ we have*

$$ab = \sum_{i+j<n} e_{\pi^j(i)}c^{\omega(i,j)}\sigma^{\chi(j)}(\delta_i(a))\delta_j(b) + \sum_{i+j\geq n} e_{\pi^j(i)}tc^{\omega(i,j)}\sigma^{\chi(j)}(\delta_i(a))\delta_j(b)$$

and $\delta_0(ab) = \delta_0(a)\delta_0(b) + \sum_{i\neq 0} tc^{\omega(i,n-i)}\sigma^{\chi(n-i)}(\delta_i(a))\delta_{n-i}(b)$.

(2) *For any $a \in A$ and $i, j \in I$ we have*

$$\delta_i(ae_j) = \begin{cases} c^{\omega(\pi^{-j}(i),j)}\sigma^{\chi(j)}(\delta_{\pi^{-j}(i)}(a)) & \text{if } i \geq j, \\ tc^{\omega(\pi^{-j}(i),j)}\sigma^{\chi(j)}(\delta_{\pi^{-j}(i)}(a)) & \text{if } i < j. \end{cases}$$

Proof. (1) Straightforward.

(2) Obviously, the equality holds for $j = 0$. Let $j \neq 0$. For any $a \in A$ and $k \in I$ we have

$$e_k\delta_k(a) \cdot e_j = \begin{cases} e_{\pi^j(k)}c^{\omega(k,j)}\sigma^{\chi(j)}(\delta_k(a)) & \text{if } k + j < n, \\ e_{\pi^j(k)}tc^{\omega(k,j)}\sigma^{\chi(j)}(\delta_k(a)) & \text{if } k + j \geq n. \end{cases}$$

If $k + j < n$, then setting $i = k + j$ we have

$$e_{i-j}\delta_{i-j}(a) \cdot e_j = e_ic^{\omega(i-j,j)}\sigma^{\chi(j)}(\delta_{i-j}(a))$$

and $\delta_i(ae_j) = c^{\omega(i-j,j)}\sigma^{\chi(j)}(\delta_{i-j}(a))$. If $k + j \geq n$, then setting $i = k + j - n$ we have

$$e_{i-j+n}\delta_{i-j+n}(a) \cdot e_j = e_itc^{\omega(i-j+n,j)}\sigma^{\chi(j)}(\delta_{i-j+n}(a))$$

and $\delta_i(ae_j) = tc^{\omega(i-j+n,j)}\sigma^{\chi(j)}(\delta_{i-j+n}(a))$. ∎

In the following, we write

$$e_{i+kn} = e_it^k \quad \text{and} \quad \chi_q(i + kn) = \chi(i) + kq$$

for $i \in I$ and $k \in \mathbb{Z}_+$, the set of non-negative integers, and define

$$\omega_q(k,l) = \chi_q(k) + \chi_q(l) - \chi_q(k+l)$$

for $k, l \in \mathbb{Z}_+$. Obviously, $\chi_q|_I = \chi$ and $\omega_q|_{I\times I} = \omega$. Also, it is not difficult to check the following:

(a) $e_i e_j = e_{i+j} c^{\omega_q(i,j)}$ for all $i, j \in I$;
(b) $x e_k = e_k \sigma^{\chi_q(k)}(x)$ for all $x \in R$ and $k \in \mathbb{Z}_+$;
(c) $\omega_q(i,j) = \chi_q(i) + \chi_q(j) - \chi_q(i+j)$ for all $i, j \in I$;
(d) $\omega_q(i+j,k) + \omega_q(i,j) = \omega_q(i,j+k) + \omega_q(j,k)$ for all $i, j, k \in I$.

Proposition 3.3. *The following hold:*

(1) *A is an associative ring with $1 = e_0$, and contains R as a subring via the injective ring homomorphism $R \to A$, $x \mapsto e_0 x$, i.e., setting $A_i = e_i R$ for $i \in I$, $A = \bigoplus_{i\in I} A_i$ is an I-graded ring with $A_0 = R$.*
(2) *ϕ is an isomorphism in* Mod-A *with $\sigma^{\chi(n-1)}(x)\phi(1) = \phi(1)x$ for all $x \in R$, i.e., A/R is a split Frobenius extension of second kind.*

Proof. (1) It follows from Lemma 3.1(2) that $e_0 \cdot e_i x = e_i x = e_i x \cdot e_0$ for all $i \in I$ and $x \in R$. Let $i, j, k \in I$ and $x, y, z \in R$. By (a), (b) we have

$$\begin{aligned}(e_i x \cdot e_j y)\cdot e_k z &= e_{i+j} c^{\omega_q(i,j)} \sigma^{\chi_q(j)}(x) y \cdot e_k z\\ &= e_{i+j+k} c^{\omega_q(i+j,k)} \sigma^{\chi_q(k)}(c^{\omega_q(i,j)} \sigma^{\chi_q(j)}(x) y) z\\ &= e_{i+j+k} c^{\omega_q(i+j,k)} c^{\omega_q(i,j)} \sigma^{\chi_q(k)+\chi_q(j)}(x) \sigma^{\chi_q(k)}(y) z\\ &= e_{i+j+k} c^{\omega_q(i+j,k)+\omega_q(i,j)} \sigma^{\chi_q(k)+\chi_q(j)}(x) \sigma^{\chi_q(k)}(y) z,\end{aligned}$$

$$\begin{aligned}e_i x \cdot (e_j y \cdot e_k z) &= e_i x \cdot e_{j+k} c^{\omega_q(j,k)} \sigma^{\chi_q(k)}(y) z\\ &= e_{i+j+k} c^{\omega_q(i,j+k)} \sigma^{\chi_q(j+k)}(x) c^{\omega_q(j,k)} \sigma^{\chi_q(k)}(y) z\\ &= e_{i+j+k} c^{\omega_q(i,j+k)} c^{\omega_q(j,k)} \sigma^{\omega_q(j,k)}(\sigma^{\chi_q(j+k)}(x)) \sigma^{\chi_q(k)}(y) z\\ &= e_{i+j+k} c^{\omega_q(i,j+k)+\omega_q(j,k)} \sigma^{\omega_q(j,k)+\chi_q(j+k)}(x) \sigma^{\chi_q(k)}(y) z.\end{aligned}$$

It then follows from (c), (d) that $(e_i x \cdot e_j y)\cdot e_k z = e_i x \cdot (e_j y \cdot e_k z)$. The last assertion is obvious.

(2) It follows from (M2) that $\delta_i x = \sigma^{\chi(i)}(x)\delta_i$ for all $x \in R$ and $i \in I$. In particular, $\{\delta_i\}_{i\in I}$ is a basis for the right R-module $\mathrm{Hom}_R(A,R)$. Also, for any $i \in I$, by Lemma 3.1(3), $e_i e_{n-i-1} = e_{n-1}$ and hence $\delta_{n-1} e_i = \delta_{n-i-1}$. It follows that $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A,R)$, $a \mapsto \delta_{n-1} a$, in Mod-A. Obviously, A is a free left R-module with a basis $\{e_i\}_{i\in I}$. Thus, since $\delta_{n-1} x = \sigma^{\chi(n-1)}(x)\delta_{n-1}$ for all $x \in R$, the associated ring homomorphism is just $\sigma^{-\chi(n-1)} : R \xrightarrow{\sim} R$, and hence A/R is a Frobenius extension of second kind. Also, by (1), A/R is split. ∎

In the following, we set

$$\varepsilon(i) = \sum_{k=1}^{n-1} \omega_q(i, ki)$$

for $i \in I$. By Lemma 3.1(1), $\varepsilon(i) \geq 0$ for all $i \in I$. Also, for any $i \in I$ we have $\chi_q(in) = iq$, and hence

$$\begin{aligned}\varepsilon(i) &= \sum_{k=1}^{n-1} \{\chi_q(i) + \chi_q(ki) - \chi_q((k+1)i)\} \\ &= n\chi_q(i) - \chi_q(ni) = n\chi(i) - iq.\end{aligned}$$

Lemma 3.4. *The following hold:*

(1) *$e_i e_j = e_j e_i$ for all $i, j \in I$, and $e_i^n = e_0 t^i c^{\varepsilon(i)}$ for all $i \in I$.*
(2) *If $t \in \operatorname{rad}(R)$, then $\delta_0(a) \in R^\times$ for all $a \in A^\times$.*

Proof. (1) For any $i, j \in I$, by (a) we have $\omega_q(i, j) = \omega_q(j, i)$ and $e_i e_j = e_j e_i$. Next, by induction we see that $e_i^r = e_{ir} c^{\omega_q(i,i)+\cdots+\omega_q(i,(r-1)i)}$ for all $r \geq 2$, so that $e_i^n = e_{in} c^{\varepsilon(i)} = e_0 t^i c^{\varepsilon(i)}$.

(2) Let $a \in A^\times$. By Lemma 3.2(1) we have

$$\delta_0(aa^{-1}) = \delta_0(a)\delta_0(a^{-1}) + \sum_{i \neq 0} tc^{\omega(i,n-i)} \sigma^{\chi(n-i)}(\delta_i(a))\delta_{n-i}(a^{-1}).$$

Since $tc^{\omega(i,n-i)} \in \operatorname{rad}(R)$ for all $i \neq 0$, and since $\delta_0(aa^{-1}) = 1$, it follows that $\delta_0(a)\delta_0(a^{-1}) \in R^\times$ and $\delta_0(a)$ has a right inverse. Similarly, $\delta_0(a)$ has a left inverse. ∎

Proposition 3.5. *If $t \in \operatorname{rad}(R)$, then $R/\operatorname{rad}(R) \xrightarrow{\sim} A/\operatorname{rad}(A)$ canonically.*

Proof. Setting $\mathfrak{m} = \operatorname{rad}(R)$, we will see that $\operatorname{rad}(A) = e_0\mathfrak{m} \oplus \bigoplus_{i \neq 0} e_i R$. We divide the proof into several steps.

Claim 1. *There exists an injective ring homomorphism*

$$\rho : A \to \mathrm{M}_n(R), \qquad a \mapsto (\delta_i(ae_j))_{i,j \in I},$$

such that for any $a \in A$ if $\rho(a) \in \mathrm{M}_n(R)^\times$ then $a \in A^\times$.

Proof. We have an injective ring homomorphism $A \to \operatorname{End}_R(A)$, $a \mapsto (b \mapsto ab)$, and a ring isomorphism $\varphi : \operatorname{End}_R(A) \xrightarrow{\sim} \mathrm{M}_n(R)$, $f \mapsto (\delta_i(f(e_j)))_{i,j \in I}$, so that their composite yields an injective ring homomorphism $\rho : A \to \mathrm{M}_n(R)$ such that $\rho(a) = (\delta_i(ae_j))_{i,j \in I}$ for all $a \in A$. Next, for any $a \in A$ with $\rho(a) \in \mathrm{M}_n(R)^\times$, since $(b \mapsto ab) = \varphi^{-1}(\rho(a)) \in \operatorname{End}_R(A)^\times$, we have $A \xrightarrow{\sim} A$, $b \mapsto ab$, and hence $a \in A^\times$. ∎

Claim 2. *$A\mathfrak{m} = \bigoplus_{i \in I} e_i \mathfrak{m}$ is a two-sided ideal of A with $A\mathfrak{m} \subseteq \operatorname{rad}(A)$.*

Proof. Obviously, $A\mathfrak{m}$ is a left ideal. Since $A\mathfrak{m}$ consists of all $a \in A$ with $\delta_i(a) \in \mathfrak{m}$ for all $i \in I$, and since $\sigma(\mathfrak{m}) = \mathfrak{m}$, it follows from Lemma 3.2(1) that $A\mathfrak{m}$ is a two-sided ideal. Let $a \in A\mathfrak{m}$. We claim that $a \in \mathrm{rad}(A)$. Since $\delta_i(1-a) = -\delta_i(a) \in \mathfrak{m}$ for $i \neq 0$ and $\delta_0(1-a) = 1 - \delta_0(a) \in R^\times$, it follows from Lemmas 3.1(2) and 3.2(2) that $\rho(1-a)_{ii} \in R^\times$ for all i, and $\rho(1-a)_{ij} \in \mathfrak{m}$ unless $i = j$. Note that $\mathrm{rad}(\mathrm{M}_n(R))$ consists of all matrices with the entries in $\mathfrak{m}$ (see e.g. [Ka, Chapter 1, Proposition 7.22]). Thus $\rho(1-a) \in \mathrm{M}_n(R)^\times$, and by Claim 1 we have $1 - a \in A^\times$, so that $a \in \mathrm{rad}(A)$. ∎

Claim 3. *$\mathfrak{n} = e_0\mathfrak{m} \oplus \bigoplus_{i\neq 0} e_i R$ is a two-sided ideal of A with $\mathfrak{n} \subseteq \mathrm{rad}(A)$.*

Proof. Obviously, $\mathfrak{n}$ is a subgroup of A. It then follows from Lemma 3.2(1) that $\mathfrak{n}$ is a two-sided ideal of A. Next, since $t^i c^{\varepsilon(i)} \in \mathfrak{m}$ for all $i \neq 0$, by Lemma 3.4(1) there exists $m \geq 1$ such that $a^m \in A\mathfrak{m}$ for all $a \in \mathfrak{n}$, i.e., $\mathfrak{n}/A\mathfrak{m}$ is a two-sided ideal of $A/A\mathfrak{m}$ consisting only of nilpotent elements. Thus $\mathfrak{n}/A\mathfrak{m} \subseteq \mathrm{rad}(A/A\mathfrak{m})$. It follows from Claim 2 that $\mathfrak{n} \subseteq \mathrm{rad}(A)$. ∎

Claim 4. $\mathrm{rad}(A) \subseteq \mathfrak{n}$.

Proof. Let $a \in \mathrm{rad}(A)$. For any $x \in R$ we have $1 - a(e_0 x) \in A^\times$, and by Lemma 3.4(2), $1 - \delta_0(a)x = \delta_0(1 - a(e_0x)) \in R^\times$. Thus $\delta_0(a) \in \mathfrak{m}$ and $a \in \mathfrak{n}$. ∎

This finishes the proof of Proposition 3.5. ∎

Now, by Propositions 2.6, 2.9, 3.3 and 3.5 we have the following.

Theorem 3.6. *Assume $t \in \mathrm{rad}(R)$. Then A is an Auslander–Gorenstein local ring if and only if so is R.*

In Lemma 3.4(2) the assumption $t \in \mathrm{rad}(R)$ can be replaced by the condition that $c \in \mathrm{rad}(R)$ and $\omega(i, n-i) > 0$ for all $i \neq 0$. Similarly, in Claim 3 in the proof of Proposition 3.5 the assumption $t \in \mathrm{rad}(R)$ can be replaced by the condition that $c \in \mathrm{rad}(R)$ and $\varepsilon(i) > 0$ for all $i \neq 0$. Note also that

$$\varepsilon(i) + \varepsilon(n-i) = n\omega(i, n-i)$$

for all $i \neq 0$. Consequently, we have the following.

Theorem 3.7. *Assume $c \in \mathrm{rad}(R)$ and $n\chi(i) > iq$ for all $i \neq 0$. Then A is an Auslander–Gorenstein local ring if and only if so is R.*

4. Classification. In this section, we will provide a way to obtain every pair (q, χ) satisfying conditions (X1) and (X2).

Let q be an integer and $\chi : I \to \mathbb{Z}$ a mapping satisfying condition (X2).

Lemma 4.1. *The following hold:*

(1) $\chi(i) - \chi(i-1) = \chi(n-i) - \chi(n-i-1)$ *for all* $1 \leq i \leq n-1$.

(2) *If* $n = 2m$ *with* $m \geq 1$*, then there exist* $p_1, \ldots, p_{m+1} \in \mathbb{Z}$ *such that*

$$\chi(i) = \begin{cases} 0 & \text{if } i = 0, \\ p_1 + \cdots + p_i & \text{if } 1 \leq i \leq m, \\ p_1 + \cdots + p_m + p_{m-1} + \cdots + p_{n-i} & \text{if } m+1 \leq i \leq n-1, \end{cases}$$

and $q = 2\{p_1 + \cdots + p_{m-1}\} + p_m + p_{m+1}$.

(3) *If* $n = 2m + 1$ *with* $m \geq 1$*, then there exist* $p_1, \ldots, p_{m+1} \in \mathbb{Z}$ *such that*

$$\chi(i) = \begin{cases} 0 & \text{if } i = 0, \\ p_1 + \cdots + p_i & \text{if } 1 \leq i \leq m, \\ p_1 + \cdots + p_m + p_m + \cdots + p_{n-i} & \text{if } m+1 \leq i \leq n-1, \end{cases}$$

and $q = 2\{p_1 + \cdots + p_m\} + p_{m+1}$.

Proof. (1) For any $1 \leq i \leq n-1$, $\chi(i) + \chi(n-i-1) = \chi(n-1) = \chi(i-1) + \chi(n-i)$, and hence $\chi(i) - \chi(i-1) = \chi(n-i) - \chi(n-i-1)$.

(2)&(3) Since $\chi(0) = 0$, we have $\chi(i) = \sum_{k=1}^{i}\{\chi(k) - \chi(k-1)\}$ for all $1 \leq i \leq n-1$. Thus, on setting $p_i = \chi(i) - \chi(i-1)$ for $1 \leq i \leq m$ and $p_{m+1} = q - \chi(n-1)$, the assertions follow by (1). ∎

Proposition 4.2. *Let* $n = 2m$ *with* $m \geq 1$*. Then* (q, χ) *satisfies condition* (X1) *if and only if:*

(1) $p_i \geq p_{m+1}$ *for all* $1 \leq i \leq m$.

(2) *If* $m \geq 2$, $1 \leq r \leq m/2$ *and* $1 \leq s \leq m - 2r + 1$*, then*

$$(p_1 + \cdots + p_r) - (p_{s+r} + \cdots + p_{s+2r-1}) \geq 0.$$

(3) *If* $m \geq 5$, $3 \leq r \leq (2m-1)/3$ *and* $1 \leq s \leq (r-1)/2$*, then*

$$(p_1 + \cdots + p_r) - (p_{m-s} + \cdots + p_m + p_{m-1} + \cdots + p_{m+s-r+1}) \geq 0.$$

(4) *If* $m \geq 3$, $1 \leq r \leq (m-1)/2$ *and* $1 \leq s \leq m - 2r$*, then*

$$-(p_1 + \cdots + p_r) + (p_{s+r} + \cdots + p_{s+2r}) \geq p_{m+1}.$$

(5) *If* $m \geq 4$, $2 \leq r \leq (2m-2)/3$ *and* $1 \leq s \leq r/2$*, then*

$$-(p_1 + \cdots + p_r) + (p_{m-s} + \cdots + p_m + p_{m-1} + \cdots + p_{m+s-r}) \geq p_{m+1}.$$

Proof. For convenience, we set $\chi(n) = q$. Then condition (X1) is equivalent to

$$\chi(j-i) - \{\chi(j) - \chi(i)\} \geq 0 \quad \text{and} \quad \{\chi(j) - \chi(i)\} - \{\chi(n) - \chi(n-j+i)\} \geq 0$$

for all $0 \leq i < j \leq n-1$. In the case $i = 0$, the first inequality is trivial and

$$\begin{aligned} \chi(j) - \{\chi(n) - \chi(n-j)\} &= \chi(j) + \chi(n-j) - \chi(n) \\ &= \chi(j) + \{\chi(n-1) - \chi(j-1)\} - \chi(n) \\ &= \{\chi(j) - \chi(j-1)\} - \{\chi(n) - \chi(n-1)\} \end{aligned}$$

for all $1 \le j \le n-1$. Let $1 \le i < j \le n-1$. Setting $r = j - i$ and $s = i$, we have $r, s \ge 1$ with $r + s \le n - 1$ and

$$\chi(j-i) - \{\chi(j) - \chi(i)\} = \sum_{k=1}^{r}\{\chi(k) - \chi(k-1)\} - \sum_{l=s+1}^{r+s}\{\chi(l) - \chi(l-1)\},$$

$$\{\chi(j) - \chi(i)\} - \{q - \chi(n-j+i)\} = \sum_{k=s+1}^{r+s}\{\chi(k) - \chi(k-1)\} - \sum_{l=n-r+1}^{n}\{\chi(l) - \chi(l-1)\}.$$

Consequently, canceling common terms yields the assertion. ∎

PROPOSITION 4.3. *Let $n = 2m + 1$ with $m \ge 1$. Then (q, χ) satisfies condition* (X1) *if and only if:*

(1) $p_i \ge p_{m+1}$ *for all* $1 \le i \le m$.
(2) *If* $m \ge 2$, $1 \le r \le m/2$ *and* $1 \le s \le m - 2r + 1$, *then*

$$(p_1 + \cdots + p_r) - (p_{s+r} + \cdots + p_{s+2r-1}) \ge 0.$$

(3) *If* $m \ge 3$, $2 \le r \le 2m/3$ *and* $1 \le s \le r/2$, *then*

$$(p_1 + \cdots + p_r) - (p_{m-s+1} + \cdots + p_m + p_m + \cdots + p_{m+s-r+1}) \ge 0.$$

(4) *If* $m \ge 3$, $1 \le r \le (m-1)/2$ *and* $1 \le s \le m - 2r$, *then*

$$-(p_1 + \cdots + p_r) + (p_{s+r} + \cdots + p_{s+2r}) \ge p_{m+1}.$$

(5) *If* $m \ge 2$, $1 \le r \le (2m-1)/3$ *and* $1 \le s \le (r+1)/2$, *then*

$$-(p_1 + \cdots + p_r) + (p_{m-s+1} + \cdots + p_m + p_m + \cdots + p_{m+s-r}) \ge p_{m+1}.$$

Proof. Similar to Proposition 4.2. ∎

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