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THE DUALITY THEOREM FOR TWISTED SMASH PRODUCTS OF HOPF ALGEBRAS AND ITS APPLICATIONS

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Abstract. Let $A\#_T H$ denote the twisted smash product of an arbitrary algebra A and a Hopf algebra H over a field. We present an analogue of the celebrated Blattner–Montgomery duality theorem for $A\#_T H$, and as an application we establish the relationship between the homological dimensions of $A\#_T H$ and A if H and its dual H^* are both semisimple.

1. Introduction and preliminaries. In the theory of Hopf algebras, the Blattner–Montgomery duality theorem [5] is a celebrated result, and several versions of it have appeared (see [1] and [6] for example). These duality theorems play an important role in actions of Hopf algebras (see [14]), and also offer us a method to investigate the homological dimensions of some algebraic structures over Hopf algebras.

The notion of the twisted smash product $A\#_T H$ of an arbitrary algebra A and a Hopf algebra H was introduced in [12], and the relationship between the homological dimensions of $A\#_T H$ and A was first investigated. However, the duality theorem for twisted smash products has not been given yet, because of the abstract twisting map T. Fortunately, we have now succeeded in obtaining a duality theorem for twisted smash products. As applications, we study the homological dimensions of twisted smash products.

Throughout this paper, we fix a field k, and by an algebra and a Hopf algebra we mean a k-algebra and a Hopf k-algebra. We freely use the Hopf algebras and coalgebras terminology introduced in [9], [14], [16], [19] and [20]. H will always denote a Hopf algebra with multiplication μ, comultiplication Δ, counit ε and antipode S. For the comultiplication of H, we write

$$\Delta(h)=\sum h_1\otimes h_2 \quad \text{for } h\in H,$$

and denote the structure of right H-comodule of M by

$$\rho(m)=\sum m_{(0)}\otimes m_{(1)} \quad \text{for } m\in M.$$

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Let H be a Hopf algebra and A an algebra. Recall from [20] the "finite dual" H^0 of H, defined by

(1) $H^0 = \{f \in H^* \mid \operatorname{Ker} f$ contains an ideal of H of finite codimension$\}$.

Then the operations of H dualize to turn H^0 into a Hopf algebra, whose multiplication, comultiplication, counit and antipode will also be denoted by μ, $\triangle$, ε and S, respectively.

Let $f \in H^0$ and $h \in H$. Define a left action of H^0 on H by $f \rightharpoonup h = \sum f(h_2)h_1$, and a right action of H^0 on H by $h \leftharpoonup f = \sum f(h_1)h_2$. Then H is both a left and right H^0-module algebra (see [5]).

We recall from [5] and [14] that an ordinary smash product $A \# H$ of an algebra A and a Hopf algebra H is an algebra with unit $1_A \# 1_H$ defined on the k-space $A \otimes H$ with multiplication given by

$$(a \# h)(b \# g) = \sum a(h_1 \cdot b) \# h_2 g \tag{2}$$

for all $a, b \in A$ and $h, g \in H$.

The reader is also referred to the recent papers [2], [3] and [21] for a study of the representation type properties and the uniseriality of special smash products, namely, for twisted group algebras $S^\lambda G$ with respect to a 2-cocycle $\lambda : G \times G \to S^*$, and for local weak crossed product orders.

2. The duality theorem for twisted smash products. In this section, we give the duality theorem for twisted smash products.

Definition 2.1. Let A and B be two algebras with units.

(a) A *normal twist* on the tensor product $B \otimes A$ is defined to be a k-linear map

$$T : B \otimes A \to A \otimes B$$

satisfying the following two conditions:

$$T(x \otimes 1_A) = 1_A \otimes x \quad \text{for any } x \in B,$$
$$T(1_B \otimes a) = a \otimes 1_B \quad \text{for any } a \in A.$$

The normal twist T is called *quasi-triangular* if it satisfies the following two conditions:

$$T(m \otimes \mathrm{id}) = (\mathrm{id} \otimes m)T_{12}T_{23}, \qquad T(\mathrm{id} \otimes m) = (m \otimes \mathrm{id})T_{23}T_{12},$$

where m is the product map of the algebras A and B, $T_{12} = T \otimes \mathrm{id}$ and $T_{23} = \mathrm{id} \otimes T$.

(b) Given a normal twist $T : B \otimes A \to A \otimes B$, we define $A \#_T B$ to be the k-vector space $A \#_T B := A \otimes B$ with multiplication defined by

$$(a \#_T x)(b \#_T y) = \sum ab_T \#_T x_T y,$$

where $T(x \otimes b)$ is denoted by $\sum b_T \otimes x_T$.

(c) The k-vector space $A \#_T B$ is defined to be a *twisted smash product* (also called "T-smashed product" in [12]) of algebras A and B if the multiplication defined in (b) defines an associative algebra structure on $A \#_T B$ with unit $1_A \#_T 1_B$.

(d) The twisted smash product $A \#_T B$ is said to be *strong* if the twist T is invertible.

By [11, Proposition 1.3], $A \#_T B$ is strong if and only if $B\#_{T^{-1}}A$ is strong.

The following lemma proved in [12] is very useful.

LEMMA 2.2. *Under the notations introduced above, the k-space $A \#_T B$ is a normal twisted smash product if and only if the normal twist T satisfies the following two quasi-triangularity conditions:*

$$T(xy \otimes a) = \sum a_{Tt} \otimes x_t y_T, \tag{3}$$

$$T(x \otimes ab) = \sum a_T b_t \otimes x_{Tt}, \tag{4}$$

for all $a, b \in A$ and $x, y \in B$, where $\sum a_t \otimes x_t$ is the copy of $\sum a_T \otimes x_T \equiv T(x \otimes a)$.

Assume that $A \#_T B$ is a twisted smash product. Then it is easy to see that A and B are subalgebras of $A \#_T B$.

Let H be a Hopf algebra and A an algebra. Then it is easy to see that $A \otimes H$ is a right H-comodule via

$$\rho_{A\otimes H}(a \otimes h) = \sum a \otimes h_1 \otimes h_2,$$

and $H \otimes A$ is a right H-comodule via

$$\rho_{H\otimes A}(h \otimes a) = \sum h_1 \otimes a \otimes h_2,$$

for all $a \in A$ and $h \in H$.

LEMMA 2.3. *Let H be a Hopf algebra, A an algebra, and $T : H \otimes A \to A \otimes H$ a normal twist. Then $A \#_T H$ is a right H-comodule algebra if and only if T is a right H-comodule homomorphism.*

Proof. For any $a, b \in A$ and $x, y \in H$, if T is a right H-comodule map, then

$$\sum b_T \#_T (h_T)_1 \otimes (h_T)_2 = \sum b_T \#_T h_{1T} \otimes h_2. \tag{5}$$

Then

$$\begin{aligned}\rho_{A\#_T H}((a \#_T h)(b \#_T g)) &= \rho_{A\#_T H}(\textstyle\sum ab_T \#_T h_T g)\\ &= \textstyle\sum ab_T \#_T (h_T)_1 g_1 \otimes (h_T)_2 g_2 = \sum ab_T \#_T h_{1T} g_1 \otimes h_2 g_2\\ &= \textstyle\sum (a \#_T h_1 \otimes h_2)(b \#_T g_1 \otimes g_2) = \rho_{A\#_T H}(a \#_T h)\rho_{A\#_T H}(b \#_T g).\end{aligned}$$

Conversely, if $A \#_T H$ is a right H-comodule algebra, then for any $b \in A$ and $h \in H$ we have

$$\rho_{A\#_T H}((1_A \#_T h)(b \#_T 1_H)) = \rho_{A\#_T H}(1_A \#_T h)\rho_{A\#_T H}(b \#_T 1_H),$$

that is, (5) holds, so the map T is H-colinear. ■

DEFINITION 2.4. Let H be a Hopf algebra and A an algebra. A normal twist $T : H \otimes A \to A \otimes H$ is defined to be an *H-comodule twist* if T is a homomorphism of right H-comodules. In this case we call $A \#_T H$ the twisted smash product of A and H *along the H-comodule twist T*.

Throughout this paper, we assume that H is a Hopf algebra, A an algebra, $T : H \otimes A \to A \otimes H$ an H-comodule twist, and $A \#_T H$ the twisted smash product of A and H along the H-comodule twist T.

By Lemma 2.3, $A \#_T H$ is a left module over H^0 (see (1)) with the left H^0-module structure defined by

$$f \cdot (a \#_T h) = a \#_T (f \rightharpoonup h)$$

for all $a \in A$, $h \in H$ and $f \in H^0$. Consequently, we can form the ordinary smash product $(A \#_T H) \# H^0$.

DEFINITION 2.5. Let H be a Hopf algebra, and let U be a Hopf subalgebra of H^0. Then U is said to satisfy the *RL-condition with respect to H* in [5] if

$$\varrho_{H,U}(U) \subseteq \lambda_{H,U}(H \# U),$$

where $\varrho_{H,U}$ is the algebra anti-homomorphism

$$\varrho_{H,U} : U \to \mathrm{End}(H), \quad \varrho_{H,U}(f)(h) = h \leftharpoonup f,$$

and $\lambda_{H,U}$ is the algebra homomorphism

$$\lambda_{H,U} : H \# U \to \mathrm{End}(H), \quad \lambda_{H,U}(h \# f)(g) = h(f \rightharpoonup g).$$

Here, $\mathrm{End}(H)$ denotes the set of k-maps from H to H.

DEFINITION 2.6. Let $A \#_T H$ be a twisted smash product, and let U be a Hopf subalgebra of H^0. Then A is said to be *U-locally finite* if, for any $a \in A$, there exist $f_1, \ldots, f_r \in U$ such that

$$(\mathrm{id}_A \otimes \varepsilon)\Big(\sum a_T \otimes \Big(\bigcap_{j=1}^{r} \mathrm{Ker}\, f_j\Big)_T\Big) = 0.$$

LEMMA 2.7. *Let $A \#_T H$ be a twisted smash product, and U a Hopf subalgebra of H^0. Then A is U-locally finite if and only if, for every $a \in A$, there exist $f_1, \ldots, f_r \in U$ and $a_1, \ldots, a_r \in A$ such that*

$$\sum \varepsilon(h_T)a_T = \sum_{j=1}^{r} f_j(h)a_j \tag{6}$$

for all $h \in H$.

Proof. Suppose A is U-locally finite. Choose $f_1, \ldots, f_r \in U$ as in Definition 2.6. We may assume $f_1, \ldots, f_r$ to be linearly independent. Choose $h_1, \ldots, h_r \in H$ such that $f_i(h_j) = \delta_{ij}$ for $1 \le i, j \le r$. Then $h - \sum_{j=1}^r f_j(h)h_j$ is in $\operatorname{Ker} f_j$ for all $h \in H$ such that, for any $a \in A$,

$$\sum \varepsilon(h_T)a_T = \sum_{j=1}^{r} f_j(h)\varepsilon((h_j)_T)a_T.$$

Let $a_j = \sum \varepsilon((h_j)_T)a_T$. Then the desired relation holds. The converse is obvious. ∎

Following [5], we define k-linear maps

$$\alpha : (A \#_T H) \# U \to \operatorname{End}(A \#_T H) \quad \text{and} \quad \beta : A \otimes (H \# U) \to \operatorname{End}(A \#_T H)$$

by setting $\alpha = \lambda_{A\#_T H,U}$ and $\beta = L \otimes \lambda_{H,U}$, where $L : A \to \operatorname{End}(A)$ is the left regular representation. In other words, we have

$$\alpha((a \#_T h) \# f)(b \#_T g) = (a \#_T h)(b \#_T (f \rightharpoonup g)),$$
$$\beta(a \otimes (h \# f))(b \#_T g) = ab \#_T h(f \rightharpoonup g),$$

for all $a, b \in A$, $h, g \in H$ and $f \in U$.

LEMMA 2.8. *Let H be a Hopf algebra with bijective antipode S, U a Hopf subalgebra of H^0, and A an algebra. Suppose that $A \#_T H$ is a twisted smash product. Then α and β are injective algebra homomorphisms.*

Proof. Since $\lambda_{A\#_T H,U}$, $\lambda_{H,U}$ and L are algebra homomorphisms, it follows that α and β are also algebra homomorphisms.

Define

$$\Phi : \operatorname{End}(A \#_T H) \to \operatorname{End}(A \#_T H),$$
$$\Phi(\sigma)(b \#_T g) = \sum [\sigma(b \#_T g_2)](1_A \#_T g_1).$$

Then Φ is injective with a left inverse given by

$$\Psi : \operatorname{End}(A \#_T H) \to \operatorname{End}(A \#_T H),$$
$$\Psi(\sigma)(b \#_T g) = \sum [\sigma(b \#_T g_2)](1_A \#_T S^{-1}(g_1)).$$

Indeed,

$$\begin{aligned}
\Psi \circ \Phi(\sigma)(b \#_T g) &= \sum [\Phi(\sigma)(b \#_T g_2)](1_A \#_T S^{-1}(g_1)) \\
&= \sum [\sigma(b \#_T g_3)](1_A \#_T g_2)(1_A \#_T S^{-1}(g_1)) \\
&= \sum [\sigma(b \#_T g_3)]((1_A)_T \#_T g_{2T}S^{-1}(g_1)) \\
&= \sum [\sigma(b \#_T g_3)](1_A \#_T g_2S^{-1}(g_1)) = \sigma(b \#_T g).
\end{aligned}$$

Thus Ψ is a left inverse for Φ. Define

$$\widetilde{\alpha} : (A \#_T H) \# U \to \operatorname{End}(A \#_T H),$$
$$\widetilde{\alpha}((a \#_T h) \# f)(b \#_T g) = f(g)(a \#_T h)(b \#_T 1_H).$$

It follows easily that the diagram

$$\begin{tikzcd}
& \mathrm{End}(A \#_T H) \arrow[dr, "\Phi"] & \\
(A \#_T H) \# U \arrow[ur, "\widetilde{\alpha}"] \arrow[rr, "\alpha"] & & \mathrm{End}(A \#_T H)
\end{tikzcd}$$

is commutative. Indeed, let $\sigma = \widetilde{\alpha}((a \#_T h) \# f)$. Then

$$\begin{aligned}
\Phi(\sigma)(b \#_T g) &= \sum [\sigma(b \#_T g_2)](1_A \#_T g_1) \\
&= \sum (a \#_T h)(b \#_T 1_H)(1_A \#_T f(g_2)g_1) \\
&= (a \#_T h)(b \#_T (f \rightharpoonup g)) = \alpha((a \#_T h) \# f)(b \#_T g),
\end{aligned}$$

which shows that $\Phi \circ \widetilde{\alpha} = \alpha$, as desired.

Hence, to show α is injective, we have to prove that $\widetilde{\alpha}$ is injective. Let $u \in \operatorname{Ker} \widetilde{\alpha}$, and write $u = \sum_{j=1}^{r} v_j \# f_j$, where $v_j \in A \#_T H$ and $\{f_1, \dots, f_r\}$ is a linearly independent subset of U. Choose $h_1, \dots, h_r \in H$ such that $f_i(h_j) = \delta_{ij}, 1 \le i, j \le r$. Then $0 = \widetilde{\alpha}(u)(1_A \#_T h_j) = \sum_{i=1}^{r} f_i(h_j) v_i = v_j$ for all j, so that $u = 0$. Thus $\widetilde{\alpha}$ is injective.

Define

$$\begin{aligned}
&\widetilde{\beta} : A \otimes (H \# U) \to \mathrm{End}(A \#_T H), \\
&\widetilde{\beta}(a \otimes (h \# f))(b \#_T g) = f(g)(ab \#_T h).
\end{aligned}$$

Then the diagram

$$\begin{tikzcd}
& \mathrm{End}(A \#_T H) \arrow[dr, "\Phi"] & \\
A \otimes (H \# U) \arrow[ur, "\widetilde{\beta}"] \arrow[rr, "\beta"] & & \mathrm{End}(A \#_T H)
\end{tikzcd}$$

is commutative. Indeed, let $\sigma = \widetilde{\beta}(a \otimes (h \# f))$. Then

$$\begin{aligned}
\Phi(\sigma)(b \#_T g) &= \sum [\sigma(b \#_T g_2)](1_A \#_T g_1) \\
&= [\widetilde{\beta}((a \#_T h) \# f)(b \#_T g_2)](1_A \#_T g_1) \\
&= \sum (ab \#_T h)(1_A \#_T f(g_2)g_1) = \sum ab(1_A)_T \#_T h_T(f \rightharpoonup g) \\
&= ab \#_T h(f \rightharpoonup g) = \beta(a \otimes (h \# f))(b \#_T g).
\end{aligned}$$

This shows that $\Phi \circ \widetilde{\beta} = \beta$, as desired.

Hence, to show that β is injective, we have to prove that $\widetilde{\beta}$ is injective. Let $u \in \operatorname{Ker} \widetilde{\beta}$, and write $u = \sum_{j=1}^{r} a_j \otimes (g_j \# f_j)$, where $\{f_1, \dots, f_r\}$ is a linearly independent subset of U. Choose $h_1, \dots, h_r \in H$ such that $f_i(h_j) = \delta_{ij}$, $1 \le i, j \le r$. Then $0 = \widetilde{\beta}(u)(1_A \#_T h_j) = \sum_{i=1}^{r} f_i(h_j) a_i \#_T g_i = a_j \#_T g_j$ for all j, so that $u = 0$. Thus $\widetilde{\beta}$ is injective. ∎

Lemma 2.9. *The map*

$$\gamma : A \#_T H \to A \#_T H, \qquad \gamma(a \#_T h) = \sum \varepsilon(S^{-1}(h_1)_T) a_T \#_T h_2,$$

is invertible with inverse ν *given by*

$$\nu : A \#_T H \to A \#_T H, \qquad \nu(a \#_T h) = \sum \varepsilon(h_{1T}) a_T \#_T h_2,$$

for all $a \in A$ *and* $h \in H$.

Proof. Indeed, for any $a \in A$ and $h \in H$,

$$\begin{aligned}
\gamma(\nu(a \#_T h)) &= \gamma\big(\sum \varepsilon(h_{1T}) a_T \#_T h_2\big) = \sum \varepsilon(h_{1T})\varepsilon(S^{-1}(h_2)_t) a_{Tt} \#_T h_3 \\
&\overset{(3)}{=} \sum \varepsilon((S^{-1}(h_2)h_1)_T) a_T \#_T h_3 = a \#_T h, \\
\nu(\gamma(a \#_T h)) &= \nu\big(\sum \varepsilon(S^{-1}(h_1)_T) a_T \#_T h_2\big) \\
&= \sum \varepsilon(S^{-1}(h_1)_T)\varepsilon(h_{2t}) a_{Tt} \#_T h_3 \\
&= \sum \varepsilon((h_2 S^{-1}(h_1))_T) a_T \#_T h_3 = a \#_T h. \ \blacksquare
\end{aligned}$$

The next two lemmas show the map of γ conjugating $\beta(A \otimes (H \# U))$ is onto $\alpha((A \#_T H) \# U)$. We now compute $\nu \circ \beta(1_A \otimes (h \# f)) \circ \gamma$, $\nu \circ \beta(a \otimes (1_H \# \varepsilon)) \circ \gamma$ and $\gamma \circ \alpha((a \#_T 1_H) \# \varepsilon) \circ \nu$ for all $a \in A$, $h \in H$ and $f \in U$, respectively.

Lemma 2.10. $\nu \circ \beta(1_A \otimes (h \# f)) \circ \gamma = \alpha((1_A \#_T h) \# f)$.

Proof. For any $b \in A$ and $g \in H$, since $\triangle(f \rightharpoonup g) = \sum g_1 \otimes (f \rightharpoonup g_2)$ by [5, Lemma 1.1], we have

$$\begin{aligned}
&[\nu\circ\beta(1_A\otimes(h\# f))\circ\gamma](b\#_T g) = \sum [\nu\circ\beta(1_A\otimes(h\# f))]\big(\varepsilon(S^{-1}(g_1)_T) b_T \#_T g_2\big) \\
&= \sum \nu\big(\varepsilon(S^{-1}(g_1)_T) b_T \#_T h(f \rightharpoonup g_2)\big) \\
&= \sum \varepsilon(S^{-1}(g_1)_T)\varepsilon((h_1(f \rightharpoonup g_2)_1)_t) b_{Tt} \#_T h_2(f \rightharpoonup g_2)_2 \\
&= \sum \varepsilon(S^{-1}(g_1)_T)\varepsilon((h_1 g_2)_t) b_{Tt} \#_T h_2(f \rightharpoonup g_3) \\
&= \sum \varepsilon((h_1 g_2)_t S^{-1}(g_1)_T) b_{Tt} \#_T h_2(f \rightharpoonup g_3) \\
&\overset{(3)}{=} \sum \varepsilon((h_1 g_2 S^{-1}(g_1))_T) b_T \#_T h_2(f \rightharpoonup g_3) = \sum \varepsilon(h_{1T}) b_T \#_T h_2(f \rightharpoonup g) \\
&\overset{(5)}{=} \sum \varepsilon((h_T)_1) b_T \#_T (h_T)_2(f \rightharpoonup g) = \sum b_T \#_R h_T(f \rightharpoonup g) \\
&= (1_A \#_T h)(b \#_T (f \rightharpoonup g)) = \alpha((1_A \#_T h) \# f)(b \#_T g). \ \blacksquare
\end{aligned}$$

Fix $a \in A$. Choose $f_1, \dots, f_r \in U$ and $a_1, \dots, a_r \in A$ as in Lemma 2.7. In the following, by using the fact that A is a U-locally finite algebra, we get

Lemma 2.11. *If the antipode of* U *is bijective, then:*

(i) $\nu\circ\beta(a\otimes(1_H\#\varepsilon))\circ\gamma = \sum_{j=1}^r \alpha((a_j\#_T 1_H)\#\varepsilon)\circ\nu\circ(\mathrm{id}_A\otimes\varrho_{H,U}(f_j))\circ\gamma$,

(ii) $\gamma\circ\alpha((a\#_T 1_H)\#\varepsilon)\circ\nu = \sum_{j=1}^r \beta(a_j\otimes(1_H\#\varepsilon))\circ(\mathrm{id}_A\otimes\varrho_{H,U}(S^{-1}(f_j)))$.

Proof. (i) For any $b \in A$, $g \in H$, $f \in H^0$, since $\triangle(g \leftharpoonup f) = \sum (g_1 \leftharpoonup f) \otimes g_2$ by [5, Lemma 1.2], we have

$$
\begin{aligned}
&\nu \circ \beta(a \otimes (1_H \# \varepsilon)) \circ \gamma(b \#_T g) \\
&\quad = \sum \nu \circ \beta(a \otimes (1_H \# \varepsilon))\big(\varepsilon(S^{-1}(g_1)_T) b_T \#_T g_2\big) \\
&\quad = \sum \nu\big(a\varepsilon(S^{-1}(g_1)_T) b_T \#_T (\varepsilon \rightharpoonup g_2)\big) = \sum \nu\big(a\varepsilon(S^{-1}(g_1)_T) b_T \#_T g_2\big) \\
&\quad = \sum \varepsilon(g_{2t})\varepsilon(S^{-1}(g_1)_T)(ab_T)_t \#_T g_3 \overset{(4)}{=} \sum \varepsilon(g_{2tt'} S^{-1}(g_1)_T) a_t b_{Tt'} \#_T g_3 \\
&\quad \overset{(3)}{=} \sum \varepsilon[(g_{2t} S^{-1}(g_1))_T] a_t b_T \#_T g_3 = \sum \varepsilon[(g_{2t})_1((g_{2t})_2 S^{-1}(g_1))_T] a_t b_T \#_T g_3 \\
&\quad \overset{(5)}{=} \sum \varepsilon[(g_{21})_t (g_{22} S^{-1}(g_1))_T] a_t b_T \#_T g_3 \\
&\quad = \sum (a_t \#_T 1_H)[\varepsilon(g_{2t}(g_3 S^{-1}(g_1))_T) b_T \#_T g_4] \\
&\quad = \sum \alpha((a_t \#_T 1_H) \# \varepsilon)[\varepsilon(g_{2t}(g_3 S^{-1}(g_1))_T) b_T \#_T g_4] \\
&\quad \overset{(6)}{=} \sum_{j=1}^{r} \alpha((a_j \#_T 1_H) \# \varepsilon)[\sum \varepsilon((f_j(g_2) g_3 S^{-1}(g_1))_T) b_T \#_T g_4] \\
&\quad = \sum_{j=1}^{r} \alpha((a_j \#_T 1_H) \# \varepsilon)[\sum \varepsilon(((g_2 \leftharpoonup f_j) S^{-1}(g_1))_T) b_T \#_T g_3] \\
&\quad = \sum_{j=1}^{r} \alpha((a_j \#_T 1_H) \# \varepsilon)[\sum \varepsilon((g_2 \leftharpoonup f_j)_t S^{-1}(g_1)_T) b_{Tt} \#_T g_3] \\
&\quad = \sum_{j=1}^{r} \alpha((a_j \#_T 1_H) \# \varepsilon)[\sum \varepsilon((g_2 \leftharpoonup f_j)_{1t})\varepsilon(S^{-1}(g_1)_T) b_{Tt} \#_T (g_2 \leftharpoonup f_j)_2] \\
&\quad = \sum_{j=1}^{r} \alpha((a_j \#_T 1_H) \# \varepsilon) \circ \nu[\sum \varepsilon(S^{-1}(g_1)_T) b_T \#_T (g_2 \leftharpoonup f_j)] \\
&\quad = \sum_{j=1}^{r} \alpha((a_j \#_T 1_H) \# \varepsilon) \circ \nu \circ (\mathrm{id}_A \otimes \varrho_{H,U}(f_j))[\sum \varepsilon(S^{-1}(g_1)_T) b_T \#_T g_2] \\
&\quad = \sum_{j=1}^{r} \alpha((a_j \#_T 1_H) \# \varepsilon) \circ \nu \circ (\mathrm{id}_A \otimes \varrho_{H,U}(f_j)) \circ \gamma(b \#_T g)
\end{aligned}
$$

as required.

(ii) Similarly to (i), we have

$$
\begin{aligned}
&\gamma \circ \alpha((a \#_T 1_H) \# \varepsilon) \circ \nu(b \#_T g) = \sum \gamma \circ \alpha((a \#_T 1_H) \# \varepsilon)\big(\varepsilon(g_{1T}) b_T \#_T g_2\big) \\
&\qquad = \sum \gamma[(a \#_T 1_H)(\varepsilon(g_{1T}) b_T \#_T g_2)] = \sum \gamma(\varepsilon(g_{1T}) a b_T \#_T g_2) \\
&\qquad = \sum \varepsilon(g_{1T} S^{-1}(g_2)_t)(ab_T)_t \#_T g_3 = \sum \varepsilon(g_{1T} S^{-1}(g_2)_{tt'}) a_t b_{Tt'} \#_T g_3 \\
&\qquad = \sum \varepsilon((S^{-1}(g_2)_t g_1)_T) a_t b_T \#_T g_3 \\
&\qquad = \sum \varepsilon((S^{-1}(g_2)_{t2} g_1)_T)\varepsilon(S^{-1}(g_2)_{t1}) a_t b_T \#_T g_3 \\
&\qquad \overset{(5)}{=} \sum \varepsilon((S^{-1}(g_2)_2 g_1)_T)\varepsilon((S^{-1}(g_2)_1)_t) a_t b_T \#_T g_3 \\
&\qquad = \sum \varepsilon((S^{-1}(g_2) g_1)_T)\varepsilon(S^{-1}(g_3)_t) a_t b_T \#_T g_4 = \sum \varepsilon(S^{-1}(g_1)_t) a_t b \#_T g_2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum \beta\big(\varepsilon(S^{-1}(g_1)_t)a_t \otimes (1_H \# \varepsilon)\big)(b \#_T g_2) \\
&= \sum_{j=1}^{r} \beta(a_j \otimes (1_H \# \varepsilon))\big(\sum b \#_T f_j(S^{-1}(g_1))g_2\big) \\
&= \sum_{j=1}^{r} \beta(a_j \otimes (1_H \# \varepsilon))(b \#_T (g \leftharpoonup S^{-1}(f_j))) \\
&= \sum_{j=1}^{r} \beta(a_j \otimes (1_H \# \varepsilon)) \circ \big(\mathrm{id}_A \otimes \varrho_{H,U}(S^{-1}(f_j))\big)(b \#_T g). \ \blacksquare
\end{aligned}
$$

Now, we are ready to give the main result of this section, that is, the duality theorem for twisted smash products.

THEOREM 2.12. *Let H be a Hopf algebra with bijective antipode, and U a Hopf subalgebra of H^0 with bijective antipode. Assume that A is a U-locally finite algebra, and U satisfies the RL-condition with respect to H. Then*

$$(A \#_T H) \# U \cong A \otimes (H \# U).$$

Proof. Let $a \in A$, $h \in H$ and $f \in U$. We first show that

$$\nu \circ \beta(a \otimes (h \# f)) \circ \gamma \in \alpha((A \#_T H) \# U).$$

Since $a \otimes (h \# f) = (a \otimes (1_H \# \varepsilon))(1_A \otimes (h \# f))$, since α and β are algebra homomorphisms by Lemma 2.8, and since $\nu = \gamma^{-1}$ by Lemma 2.10, it suffices to show that $\nu \circ \beta(1_A \otimes (h \# f)) \circ \gamma$ and $\nu \circ \beta(a \otimes (1_H \# \varepsilon)) \circ \gamma$ both belong to $\alpha((A \#_T H) \# U)$. The first does by Lemma 2.11. By the RL-condition, there exists some $z \in H \# U$ such that $\mathrm{id}_A \otimes \varrho_{H,U}(f_j) = \beta(1_A \otimes z)$. Then by Lemma 2.11(i), $\nu \circ \beta(a \otimes (1_H \# \varepsilon)) \circ \gamma = \sum_{j=1}^{r} \alpha((a_j \#_T 1_H) \# \varepsilon) \circ \nu \circ \beta(1_A \otimes z) \circ \gamma$, hence by Lemma 2.10 we know that $\nu \circ \beta(a \otimes (1_H \# \varepsilon)) \circ \gamma$ belongs to $\alpha((A \#_T H) \# U)$.

One can similarly prove that

$$\gamma \circ \alpha((a \#_T h) \# f) \circ \nu \in \beta(A \otimes (H \# U)).$$

Since we know that $(a \#_T h) \# f = ((a \#_T 1_H) \# \varepsilon)((1_A \#_T h) \# f)$, it suffices to show that $\gamma \circ \alpha((a \#_T 1_H) \# \varepsilon) \circ \nu$ and $\gamma \circ \alpha((1_A \#_T h) \# f) \circ \nu$ both belong to $\beta(A \otimes (H \# U))$. That the second one does can be immediately seen by Lemma 2.10. Lemma 2.11(ii) and the RL-condition imply that the first one also does.

We have proved that

$$\gamma^{-1} \circ \beta(A \otimes (H \# U)) \circ \gamma = \alpha((A \#_T H) \# U).$$

Since α and β are injective homomorphisms by Lemma 2.7, our theorem is proved. $\blacksquare$

If H is a finite-dimensional Hopf algebra, then it is not difficult to see that A is an H^*-locally finite algebra and H^* satisfies the RL-condition with respect to H. Hence, by [5, Corollary 2.7], we get

Corollary 2.13. *Let H be a finite-dimensional Hopf algebra with $\dim(H) = n$, and $A\#_T H$ a twisted smash product such that T is H-colinear. Then*

$$(A \#_T H) \# H^* \cong A \otimes (H \# H^*) \cong \mathbb{M}_n(A).$$

3. Applications of the duality theorem. In this section, we give some applications of the duality theorem to the global dimensions for twisted smash products.

In what follows, we always suppose that H is a finite-dimensional Hopf algebra and $A \#_T H$ a twisted smash product. Assume further that T is H-colinear and satisfies the following condition as in [12]:

$$\sum aS(h_1) \otimes h_2 = \sum S(h_1)a_T \otimes h_{2T} \tag{7}$$

for all $a \in A$ and $h \in H$, where we denote $ah := a \#_T h$ and $ha := (1_A \#_T h)(a \#_T 1_H)$, respectively.

Lemma 3.1. *Let H be a finite-dimensional semisimple Hopf algebra, and P a left $A \#_T H$-module. Then P is a projective left $A \#_T H$-module if and only if P is a projective left A-module.*

Proof. Suppose that P is a projective left $A\#_T H$-module. Since $A\#_T H$ is a free left A-module, P is a projective left A-module.

Conversely, for any left $A \#_T H$-modules M and N, let $g : M \to N$ and $h : P \to N$ be $A \#_T H$-module homomorphisms such that g is onto. In order to prove that P is projective as a left $A \#_T H$-module, it is sufficient to find $\widetilde{f} \in \mathrm{Hom}_{A\#_T H}(P, M)$ satisfying $h = g \circ \widetilde{f}$.

Since A and H are subalgebras of $A \#_T H$, we know that M, N are left A-modules, and h, g are left A-module and H-module homomorphisms. Since P is projective as an A-module, there exists $f \in \mathrm{Hom}_A(P, M)$ such that $h = g \circ f$.

Define

$$\widetilde{f}(p) = \sum S(t_1) \cdot f(t_2 \cdot p)$$

for any $p \in P$, where $t \in \int^r$ is such that $\varepsilon(t) = 1$. Then $\widetilde{f}$ is an $A\#_T H$-module homomorphism.

As a matter of fact, for any $a \#_T h \in A \#_T H$ and $p \in P$, since it is well known that

$$\sum S(t_1) \otimes t_2 h = \sum hS(t_1) \otimes t_2, \tag{8}$$

we have

$$\begin{aligned}
\widetilde{f}((a \#_T h) \cdot p) &= \sum S(t_1) \cdot f(t_2(a \#_T h) \cdot p) = \sum S(t_1) \cdot f((a_T \#_T t_{2T} h) \cdot p) \\
&= \sum S(t_1)a_T \cdot f((1_A \#_T t_{2T} h) \cdot p) \\
&\overset{(7)}{=} \sum aS(t_1) \cdot f((1_A \#_T t_2 h) \cdot p) \\
&\overset{(8)}{=} \sum ahS(t_1) \cdot f((1_A \#_T t_2) \cdot p) = (a \#_T h) \cdot \widetilde{f}(p).
\end{aligned}$$

Moreover,

$$\begin{aligned} g \circ \widetilde{f}(p) &= g\big(\sum S(t_1) \cdot f(t_2 \cdot p)\big) = \sum S(t_1) \cdot gf(t_2 \cdot p) \\ &= \sum S(t_1) \cdot h(t_2 \cdot p) = \sum S(t_1)t_2 \cdot h(p) = \varepsilon(t)h(p) = h(p). \end{aligned}$$

Hence, P is a projective left A-module. ∎

By the proof of sufficiency in Lemma 3.1, we can get the following

REMARK 3.2. Let H be a finite-dimensional semisimple Hopf algebra, and Q a left $A \#_T H$-module. If Q is an injective left $A \#_T H$-module, then Q is an injective left A-module.

It is obvious that the k-space $H \otimes A$ (and $A\#_T H$) is both a left H-module via the left multiplication of H and a right A-module via the right multiplication of A (via the left and right multiplication of $A \#_T H$, respectively). This allows us to prove the following useful lemma.

LEMMA 3.3. *If $A \#_T H$ is strong, then T and T^{-1} are isomorphisms of left H-modules and right A-modules.*

Proof. For any $a, b \in A$ and $h, g \in H$,

$$\begin{aligned} T(g \cdot (h \otimes a)) &= T(gh \otimes a) = \sum a_{Tt} \#_T g_t h_T \\ &= (1_A \#_T g)(\sum a_T \#_T h_T) = g \cdot T(h \otimes a), \\ T((h \otimes a) \cdot b) &= T(h \otimes ab) = \sum a_T b_t \#_T h_{Tt} \\ &= (\sum a_T \#_T h_T)(b \#_T 1_H) = T(h \otimes a) \cdot b. \end{aligned}$$

Hence, T is both left H-linear and right A-linear. Moreover,

$$\begin{aligned} T^{-1}(g \cdot (a \#_T h)) &= T^{-1}(\sum a_T \#_T g_T h) = \sum g_{TT^{-1}} h_{t^{-1}} \otimes a_{TT^{-1}t^{-1}} \\ &= \sum g h_{t^{-1}} \otimes a_{t^{-1}} = g \cdot T^{-1}(a \#_T h), \\ T^{-1}((a \#_T h) \cdot b) &= T^{-1}(\sum ab_T \#_T h_T) = \sum h_{TT^{-1}t^{-1}} \otimes a_{t^{-1}} b_{TT^{-1}} \\ &= \sum h_{t^{-1}} \otimes a_{t^{-1}} b = T^{-1}(a \#_T h) \cdot b. \end{aligned}$$

Hence, T^{-1} is also both left H-linear and right A-linear. ∎

LEMMA 3.4. *Let H be a finite-dimensional Hopf algebra with $z \in \int^l$ such that H^* is unimodular. Suppose that Q is a left $A \#_T H$-module, and $\sum \varepsilon(z_T)c_T = 1_A$ for some $c \in Z(A)$, the center of A. If Q is an injective left A-module, then Q is an injective left $A \#_T H$-module.*

Proof. By the assumptions and [17], one can see that $S^2(z) = z$ and $t = S(z) \in \int^r$. For any left $A \#_T H$-modules M, N, let $g : M \to N$ and $h : M \to Q$ be $A\#_T H$-module homomorphisms such that g is monomorphic. In order to prove that Q is an injective left $A \#_T H$-module, it is sufficient to find $\widetilde{f} \in \mathrm{Hom}_{A\#_T H}(N, Q)$ such that $\widetilde{f} \circ g = h$.

Since Q is injective as an A-module, there exists an A-module homomorphism $f : N \to Q$ such that $f \circ g = h$. We now define

$$\widetilde{f} : N \to Q, \qquad \widetilde{f}(n) = \sum S(t_1)c \cdot f(t_2 \cdot n).$$

Then $\widetilde{f}$ is a left $A \#_T H$-module morphism. Indeed, for any $a \in A$, $h \in H$ and $n \in N$,

$$\begin{aligned}
\widetilde{f}((a \#_T h) \cdot n) &= \sum S(t_1)c \cdot f(t_2(a \#_T h) \cdot n) = \sum S(t_1)c \cdot f((a_T \#_T t_{2T}h) \cdot n) \\
&= \sum S(t_1)ca_T \cdot f((1_A \#_T t_{2T}h) \cdot n) = \sum S(t_1)a_T c \cdot f((1_A \#_T t_{2T}h) \cdot n) \\
&\overset{(7)}{=} \sum aS(t_1)c \cdot f((1_A \#_T t_2h) \cdot n) \overset{(8)}{=} \sum ahS(t_1)c \cdot f((1_A \#_T t_2) \cdot n) \\
&= (a \#_T h) \cdot \widetilde{f}(n).
\end{aligned}$$

Moreover, for any $m \in M$, since g and h are H-linear, and

$$S(t) \otimes 1_H = \sum S(t_2) \otimes S(t_1)t_3, \tag{9}$$

by [25], we obtain

$$\begin{aligned}
&\widetilde{f} \circ g(m) = \sum (c_T \#_T S(t_1)_T) \cdot f(t_2 \cdot g(m)) \\
&= \sum \big(c_T\varepsilon(S(t_1)_{T1}) \#_T S(t_1)_{T2}\big) \cdot f(g(t_2 \cdot m)) \\
&\overset{(5)}{=} \sum \big(c_T\varepsilon(S(t_2)_T) \#_T S(t_1)\big) \cdot h(t_3 \cdot m) = \sum \big(c_R\varepsilon(S(t_2)_T) \#_T S(t_1)t_3\big) \cdot h(m) \\
&\overset{(9)}{=} \sum (c_T\varepsilon(S(t)_T) \#_T 1_H) \cdot h(m) = \sum (c_T\varepsilon(z_T) \#_T 1_H) \cdot h(m) \\
&= (1_A \#_T 1_H) \cdot h(m) = h(m).
\end{aligned}$$

Hence, Q is an injective left $A \#_T H$-module. ∎

LEMMA 3.5. *Let H be a finite-dimensional Hopf algebra with $z \in \int^l$ and M a right $A \#_T H$-module. Suppose that T is invertible. If either*

(i) *H is semisimple, or*
(ii) *H^* is unimodular and there exists $c \in Z(A)$ with $\sum \varepsilon(z_T)c_T = 1_A$, then M is a flat right A-module if and only if M is a flat right $A \#_T H$-module.*

Proof. Any free right $A \#_T H$-module is free as a right A-module, since $A \#_T H$ is free as a right A-module by Lemma 3.3. Let M be a flat right $A \#_T H$-module, and consider an exact sequence of right $A \#_T H$-modules

$$0 \to K \to F \to M \to 0,$$

where F is a free right $A \#_T H$-module. Then F is free as a right A-module. Now, one can easily deduce that M is a flat right A-module from [18, Theorem 3.62].

Conversely, let M be a flat right A-module. We regard $\mathbb{Q}$ (the field of rational numbers) and $\mathbb{Z}$ (the ring of integers) as $\mathbb{Z}$-modules. Then $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a left $A \#_T H$-module and hence a left A-module defined in the natural way $[(a \#_T h)\cdot f](m) = f(m\cdot a)\varepsilon(h)$. Thus, M^* is an injective left A-module. Now, if (i) or (ii) holds, M^* is an injective left $A \#_T H$-module by Remark 3.2 or Lemma 3.4, and it follows that M is a flat right $A \#_T H$-module. ∎

Let us recall from [4] that the *finitistic dimension* of an algebra A is defined by

$$\text{fin.dim}(A) = \sup\{\text{proj.dim}(A) < \infty \mid M \text{ is an } A\text{-module and } \text{proj.dim}(M) < \infty\}.$$

PROPOSITION 3.6. *Assume that H and H^* are semisimple. Then:*

(i) $\text{gl.dim}(A \#_T H) = \text{gl.dim}(A)$. *Hence $A \#_T H$ is semisimple (resp. hereditary) if and only if A is semisimple (resp. hereditary).*
(ii) $\text{fin.dim}(A \#_T H) = \text{fin.dim}(A)$.
(iii) *If T is bijective, then* $\text{w.dim}(A \#_T H) = \text{w.dim}(A)$. *Hence $A \#_T H$ is von Neumann if and only if A is von Neumann.*

Proof. (i) It is harmless to assume that $\text{gl.dim}(A) = n < \infty$. For any left $A \#_T H$-module N, consider any one of its projective resolutions

$$P_N:\ \cdots P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \xrightarrow{d_0} N \to 0.$$

By Lemma 3.1, P_N is also a projective resolution for N as an A-module and hence a projective resolution as an $A \#_T H$-module. This implies that $\text{proj.dim}(N) \le n$.

Since $\text{gl.dim}(B \# H) \le \text{gl.dim}(B)$ for an H-module algebra B by [22, proof of Theorem 2.2], we can obtain

$$\text{gl.dim}((A \#_T H) \# H^*) \le \text{gl.dim}(A \#_T H).$$

Since $\mathbb{M}_n(A)$ is Morita equivalent to A, and by Corollary 2.12

$$(A \#_T H) \# H^* \cong \mathbb{M}_n(A),$$

we get

$$\text{gl.dim}(A) = \text{gl.dim}((A \#_T H) \# H^*) \le \text{gl.dim}(A \#_T H) \le \text{gl.dim}(A).$$

This shows that $\text{gl.dim}(A \#_T H) = \text{gl.dim}(A)$.

(ii) If the finitistic dimension of A is infinite, the result is obviously true. Assume that $\text{fin.dim}(A) < \infty$. For any $A \#_T H$-module P with finite projective dimension, we have

$$\text{proj.dim}({}_{A\#_T H}P) = \text{proj.dim}({}_A P)$$

by Lemma 3.1. This implies $\text{fin.dim}(A \#_T H) \le \text{fin.dim}(A)$.

Next, similarly to the proof of (i), we have

$$\text{fin.dim}(A) = \text{fin.dim}((A \#_T H) \# H^*) \le \text{fin.dim}(A \#_T H) \le \text{fin.dim}(A).$$

This shows that $\text{fin.dim}(A \#_T H) = \text{fin.dim}(A)$.

(iii) Suppose that T is invertible. Since H is semisimple, there exists $z \in \int^l$ such that $\varepsilon(z) = 1$. Then $\sum \varepsilon(z_T)(1_A)_T = \varepsilon(z)1_A = 1_A$. Hence by Lemma 3.5, we know that $\mathrm{w.dim}(A \#_T H) \le \mathrm{w.dim}(A)$. On the other hand, we also have $\mathrm{w.dim}((A \#_T H) \# H^*) \le \mathrm{w.dim}(A \#_T H)$ by [25, Lemma 2]. Then

$$\mathrm{w.dim}(A) = \mathrm{w.dim}((A \#_T H) \# H^* \le \mathrm{w.dim}(A \#_T H) \le \mathrm{w.dim}(A),$$

which implies that $\mathrm{w.dim}(A \#_T H) = \mathrm{w.dim}(A)$. ■

4. Examples of twisted smash products. In this section, we give some examples of twisted smash products.

EXAMPLE 4.1. For a given field k of characteristic $\neq 2$, let H_4 denote the four-dimensional Sweedler Hopf algebra over k (see [14]). It is described as follows:

$$H_4 = k\langle 1, g, x, gx \mid g^2 = 1,\ x^2 = 0,\ xg = -gx\rangle$$

with coalgebra structure

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

and antipode

$$S(g) = g, \quad S(x) = -gx.$$

Let $A = k\langle 1, x\rangle$. Define

$$T : H_4 \otimes A \to A \otimes H_4$$

by

$$T(1, 1_A) = 1_A \otimes 1, \quad T(1, x) = x \otimes 1, \quad T(g, 1_A) = 1_A \otimes g,$$
$$T(gx, 1_A) = 1_A \otimes gx, \quad T(g, x) = x \otimes g, \quad T(x, x) = 0, \quad T(gx, x) = 0.$$

Then $A \#_T H_4$ is a twisted smash product.

Indeed, it is easy to check that $T(h, x^2) = 0 = (m \otimes \mathrm{id})T_{23}T_{12}(h \otimes x \otimes x)$ for any $h \in H$, that is, condition (4) holds.

In what follows, we prove that $T(hh', x) = (\mathrm{id} \otimes m)T_{12}T_{23}(h \otimes h' \otimes x)$; for any $h, h' \in H_4$: if $h' = g$, we have

$$T(gg, x) = T(1, x) = x \otimes 1 = (\mathrm{id} \otimes m)T_{12}T_{23}(g \otimes g \otimes x) = x \otimes 1,$$
$$T(xg, x) = 0 = (\mathrm{id} \otimes m)T_{12}T_{23}(x \otimes g \otimes x),$$
$$T((gx)g, x) = 0 = (\mathrm{id} \otimes m)T_{12}T_{23}(gx \otimes g \otimes x).$$

The rest of the proof is straightforward, so condition (3) holds. Hence, by Lemma 2.2, $A \#_T H_4$ is a twisted smash product.

However, A has no non-trivial left H_4-module algebra for any module action.

Indeed, assume that A is a left H_4-module algebra for some module action "$\cdot$". Then, from $g \cdot x = g \cdot (x + x^2) = g \cdot x + (g \cdot x)^2$, we see that $(g \cdot x)^2 = 0$. In addition, $x = 1 \cdot x = g \cdot (g \cdot x)$, so $g \cdot x \neq 0$, hence $g \cdot x = \pm x$. Here, we only consider the case $g \cdot x = x$.

Let $x \cdot x = a1_A + bx$ with $a, b \in k$. Then $g \cdot (x \cdot x) = a(g \cdot 1_A) + b(g \cdot x) = a1_A + bx = x \cdot x$. Since $g \cdot (x \cdot x) = (gx) \cdot x = (-x) \cdot (g \cdot x) = -x \cdot x$, we have $x \cdot x = g \cdot (x \cdot x) = -x \cdot x$, and hence $x \cdot x = 0$.

Consequently, $(gx) \cdot x = 0$. In this case, A has only a trivial left H_4-module algebra.

Example 4.2. Let H be a bialgebra and $\sigma : H \otimes H \to k$ a linear map. If for any $h, g, x \in H$,

(P1) $\sigma(1, x) = \varepsilon(x)$,
(P2) $\sigma(h, 1) = \varepsilon(h)$,
(P3) $\sigma(hg, x) = \sum \sigma(h, x_2)\sigma(g, x_1)$,
(P4) $\sigma(h, xy) = \sum \sigma(h_1, x)\sigma(h_2, y)$,

then (H, σ) is called a *skew paired bialgebra* (see [23]).

Define

$$T : H \otimes H \to H \otimes H, \qquad h \otimes g \mapsto \sum \sigma(h_1, g_1) g_2 \otimes h_2.$$

Then $H \#_T H$ is a twisted smash product. If σ is invertible, then T is a bijection, that is, $H \#_T H$ is a strong twisted smash product. Indeed, for any $h, g, x, y \in H$, we have

$$\begin{aligned}
T(1, x) &\overset{\text{(P1)}}{=} \sum \sigma(1, x_1)x_2 \otimes 1 = x \otimes 1, \\
T(h, 1) &\overset{\text{(P2)}}{=} \sum \sigma(h_1, 1)1 \otimes h_2 = 1 \otimes h, \\
T(hg, x) &= \sum \sigma(h_1g_1, x_1)x_2 \otimes h_2g_2 \overset{\text{(P3)}}{=} \sum \sigma(h_1, x_2)\sigma(g_1, x_1)x_3 \otimes h_2g_2 \\
&= (\mathrm{id} \otimes m)T_{12}T_{23}(h \otimes g \otimes x), \\
T(h, xy) &= \sum \sigma(h_1, x_1y_1)x_2y_2 \otimes h_2 \overset{\text{(P4)}}{=} \sum \sigma(h_1, x_1)\sigma(h_2, y_1)x_2y_2 \otimes h_3 \\
&= (m \otimes \mathrm{id})T_{23}T_{12}(h \otimes x \otimes y),
\end{aligned}$$

so, by Lemma 2.2, $H \#_T H$ is a twisted smash product.

If σ is invertible with inverse σ^{-1}, then it is easy to see that T is a bijection with inverse

$$T^{-1} : H \otimes H \to H \otimes H, \qquad h \otimes g \mapsto \sum \sigma^{-1}(g_1, h_1) g_2 \otimes h_2.$$

Moreover, T is colinear by a direct computation.

In particular, the Long bialgebra (H, σ) with antipode S (see [13] and [23]) and the coquasi-triangular Hopf algebra (H, σ) of [14] are skew paired bialgebras with bijection σ.

Example 4.3. Let H be a Hopf algebra with bijective antipode S, and A an H-bimodule algebra. The diagonal crossed product $A \bowtie H$ (see [7], [10] and [15]) is the k-space $A \otimes H$ with multiplication given by

$$(a \bowtie h)(b \bowtie g) = \sum a(h_1 \rightharpoonup b \leftharpoonup S^{-1}(h_3)) \bowtie h_2 g$$

for all $a, b \in A$ and $h, g \in H$.

Then $A \bowtie H$ is a strong twisted smash product, where

$$T : H \otimes A \to A \otimes H, \qquad T(h \otimes a) = \sum h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_3) \otimes h_2,$$

with inverse

$$T^{-1} : A \otimes H \to H \otimes A, \qquad T^{-1}(a \otimes h) = \sum h_2 \otimes S^{-1}(h_1) \rightharpoonup a \leftharpoonup h_3,$$

for all $a \in A$ and $h \in H$.

As a matter of fact, the multiplication of $A \bowtie H$ is exactly that of the twisted smash product $A \#_T H$. For any $a, b \in A$ and $h, g \in H$, it is easy to see that $T(h \otimes 1_A) = 1_A \otimes h$ and $T(1_H \otimes a) = a \otimes 1_H$, and

$$\begin{aligned}
T(hg \otimes a) &= \sum h_1 g_1 \rightharpoonup a \leftharpoonup S^{-1}(h_3 g_3) \otimes h_2 g_2 \\
&= \sum h_1 \rightharpoonup (g_1 \rightharpoonup a \leftharpoonup S^{-1}(g_3)) \leftharpoonup S^{-1}(h_3) \otimes h_2 g_2 \\
&= \sum \big(h_1 \rightharpoonup a_T \leftharpoonup S^{-1}(h_3) \otimes h_2\big)(1_A \otimes g_T) \\
&= \sum T(h \otimes a_T)(1_A \otimes g_T) = \sum a_{Tt} \otimes h_t g_T, \\
T(h \otimes ab) &= \sum h_1 \rightharpoonup (ab) \leftharpoonup S^{-1}(h_3) \otimes h_2 \\
&= \sum (h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_5))(h_2 \rightharpoonup b \leftharpoonup S^{-1}(h_4)) \otimes h_3 \\
&= \sum \big(h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_3) \otimes 1_H\big)(b_t \otimes h_{2t}) \\
&= \sum \big(h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_3) \otimes h_{2t}\big)(b_t \otimes 1_H) \\
&= \sum (a_T \otimes h_{Tt})(b_t \otimes 1_H) = \sum a_T b_t \otimes h_{Tt},
\end{aligned}$$

so $A \bowtie H$ is a twisted smash product.

Moreover,

$$\begin{aligned}
TT^{-1}(a \otimes h) &= \sum T\big(h_2 \otimes S^{-1}(h_1) \rightharpoonup a \leftharpoonup h_3\big) \\
&= \sum h_2 \rightharpoonup (S^{-1}(h_1) \rightharpoonup a \leftharpoonup h_5) \leftharpoonup S^{-1}(h_4) \otimes h_3 \\
&= \sum h_2 S^{-1}(h_1) \rightharpoonup a \leftharpoonup h_5 S^{-1}(h_4) \otimes h_3 \\
&= a \otimes h, \\
T^{-1}T(h \otimes a) &= \sum T^{-1}\big(h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_3) \otimes h_2\big) \\
&= \sum h_3 \otimes S^{-1}(h_2) \rightharpoonup (h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_5)) \leftharpoonup h_4 \\
&= \sum h_3 \otimes S^{-1}(h_2) h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_5) h_4 \\
&= h \otimes a,
\end{aligned}$$

so T is invertible, and hence $A \bowtie H$ is a strong twisted smash product.

By Lemma 2.3, $A \bowtie H$ is a right H-comodule algebra if and only if (5) holds, that is,

$$\sum h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_4) \otimes h_2 \otimes h_3 = \sum h_1 \rightharpoonup a \leftharpoonup S^{-1}(h_3) \otimes h_2 \otimes h_4. \tag{10}$$

It is easy to see that (10) holds if and only if for any $a \in A$ and $h \in H$,

$$\sum a \leftharpoonup h_1 \otimes h_2 = \sum a \leftharpoonup h_2 \otimes h_1. \tag{11}$$

In fact, if (10) holds, then

$$\begin{aligned}\sum a \leftharpoonup S^{-1}(h_2) \otimes h_1 &= \sum S(h_1) \rightharpoonup (h_2 \rightharpoonup a \leftharpoonup S^{-1}(h_4)) \otimes h_3 \\ &= \sum S(h_1) \rightharpoonup (h_2 \rightharpoonup a \leftharpoonup S^{-1}(h_3)) \otimes h_4 \\ &= \sum a \leftharpoonup S^{-1}(h_1) \otimes h_2,\end{aligned}$$

so (11) holds. Conversely, if (11) holds, it is obvious that (10) holds.

So, $A \bowtie H$ is a right H-comodule algebra if and only if (11) holds, if and only if T is H-colinear.

In particular, if A is a left H-module algebra with the trivial right action, then (11) holds, so $T : H \otimes A \to A \otimes H,\ h \otimes a \mapsto \sum h_1 \cdot a \otimes h_2$, is H-colinear. In this case, the diagonal crossed product $A \bowtie H$ is exactly the usual smash product $A \# H$. So $A \# H$ is a strong twisted smash product such that T is H-colinear.

Example 4.4. Let H be a finite-dimensional Hopf algebra with bijective antipode S. Define the following actions: for all $h \in H$ and $f \in H^*$,

$$f \rightharpoonup h = \sum \langle f, h_2 \rangle h_1, \qquad h \leftharpoonup f = \sum \langle f, h_1 \rangle h_2.$$

Then, by [24], $(H, \rightharpoonup, \leftharpoonup)$ is an H^*-bimodule algebra. Hence, we have the diagonal crossed product $H \bowtie H^*$ with multiplication

$$(x \bowtie f)(y \bowtie g) = \sum x(f_1 \rightharpoonup y \leftharpoonup S^{-1}(f_3)) \bowtie f_2 g$$

for all $x, y \in H$ and $f, g \in H^*$. So, by the above example, $H \bowtie H^*$ is a strong twisted smash product.

Example 4.5. Let (H, σ) be a finite-dimensional coquasi-triangular Hopf algebra. Define two actions on H:

$$x \rightharpoonup h = \sum \sigma(x, h_1) h_2, \qquad h \leftharpoonup x = \sum \sigma(h_2, S(x)) h_1,$$

for all $x, y, h, g \in H$. Then, by [24], $(H, \rightharpoonup, \leftharpoonup)$ is an H-bimodule algebra. Hence, we have the diagonal crossed product $H \bowtie H$ with multiplication

$$\begin{aligned}(h \bowtie x)(g \bowtie y) &= \sum h(x_1 \rightharpoonup g \leftharpoonup S^{-1}(x_3)) \bowtie x_2 y \\ &= \sum h(\sigma(x_1, g_1) g_2 \leftharpoonup S^{-1}(x_3)) \bowtie x_2 y \\ &= \sum h\sigma(x_1, g_1) g_2 \sigma(g_3, x_3) \bowtie x_2 y.\end{aligned}$$

The diagonal crossed product $H \bowtie H$ is a strong twisted smash product.

Moreover, $h \leftharpoonup S^{-1}(x_2) \otimes x_1 = \sum \sigma(h_2, x_2) h_1 \otimes x_1$ and $h \leftharpoonup S^{-1}(x_1) \otimes x_2 = \sum \sigma(h_2, x_1) h_1 \otimes x_2$, so (11) holds if and only if for any $h, x \in H$,

$$\sum \sigma(h, x_1) x_2 = \sum \sigma(h, x_2) x_1. \tag{12}$$

Hence, by the above examples, we obtain the following duality theorems and Maschke theorems of diagonal crossed products and Long bialgebras.

PROPOSITION 4.6. *Let H be a finite-dimensional Hopf algebra, and $A \bowtie H$ the diagonal crossed product such that* (11) *holds. Then:*

(i) *There is an isomorphism of algebras*

$$(A \bowtie H) \# H^* \cong A \otimes (H \# H^*) \cong \mathbb{M}_n(A),$$

where $\dim(H) = n$.

(ii) *Assume that H and H^* are semisimple. Then $A \bowtie H$ is semisimple if and only if A is semisimple.*

Proof. (i) By Theorem 2.12 and Corollary 2.13.

(ii) It is easy to see that (7) holds for $A \bowtie H$ if (11) holds, so, by Proposition 3.6 and the well-known Maschke theorem for the smash product in [8], conclusion (ii) holds. ∎

PROPOSITION 4.7. *Let H be a finite-dimensional Hopf algebra with* $\dim(H) = n$.

(i) *Assume that (H, σ) is a skew paired bialgebra. Then there is an isomorphism of algebras*

$$(H \#_T H) \# H^* \cong H \otimes (H \# H^*) \cong \mathbb{M}_n(A).$$

(ii) *Assume that (H, σ) is a Long bialgebra. If H^* is semisimple, then $H \#_T H$ is semisimple if and only if H is semisimple.*

Proof. (i) By Corollary 2.13 and Example 4.2.

(ii) Since (H, σ) is a Long bialgebra, $\sum \sigma(h_1, x)h_2 = \sum \sigma(h_2, x)h_1$ for any $h, x \in H$. Hence

$$\begin{aligned}
\sum S(x_1)h_T \otimes x_{2T} &= \sum (1_H \#_T S(x_1))(h_T \#_T 1_H) \otimes x_{2T} \\
&= \sum (1_H \#_T S(x_1))(h_2 \#_T 1_H) \otimes \sigma(h_1, x_2)x_3 \\
&= \sum h_{2T} \#_T S(x_1)_T \otimes \sigma(h_1, x_2)x_3 \\
&= \sum h_3 \#_T \sigma(h_2, S(x_2))S(x_1) \otimes \sigma(h_1, x_3)x_4 \\
&= \sum h_3 \#_T \sigma(h_1, S(x_2))S(x_1) \otimes \sigma(h_2, x_3)x_4 \\
&= \sum h_3 \#_T \sigma(h_1, S(x_2))\sigma(h_2, x_3)S(x_1) \otimes x_4 \\
&= \sum h \#_T S(x_1) \otimes x_2,
\end{aligned}$$

so (7) holds. By Proposition 3.6 and the above duality theorem for $H \#_T H$, we see that conclusion (ii) holds. ∎

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