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SOME IDENTITIES INVOLVING DIFFERENCES OF PRODUCTS OF GENERALIZED FIBONACCI NUMBERS

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Abstract. Melham discovered the Fibonacci identity

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n.$$

He then considered the generalized sequence W_n where $W_0 = a$, $W_1 = b$, and $W_n = pW_{n-1} + qW_{n-2}$ and a, b, p and q are integers and $q \neq 0$. Letting $e = pab - qa^2 - b^2$, he proved the following identity:

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}(p^3 W_{n+2} - q^2 W_{n+1}).$$

There are similar differences of products of Fibonacci numbers, like this one discovered by Fairgrieve and Gould:

$$F_n F_{n+4} F_{n+5} - F_{n+3}^3 = (-1)^{n+1} F_{n+6}.$$

We prove similar identities. For example, a generalization of Fairgrieve and Gould's identity is

$$W_n W_{n+4} W_{n+5} - W_{n+3}^3 = eq^n (p^3 W_{n+4} - qW_{n+5}).$$

1. Introduction and results. Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Many authors have studied Fibonacci identities and generalized Fibonacci identities. For example, Fairgrieve and Gould [FG], Hoggatt and Bergum [HB], and Horadam [H] stated and proved Fibonacci identities involving differences of products of Fibonacci numbers. And Melham [M] found, proved, and generalized the following one:

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n.$$

We will attempt to prove some more such identities.

The following is the sequence Melham used to generalize his identity above.

Definition. Let W_n be defined by $W_0 = a$, $W_1 = b$, and $W_n = pW_{n-1} + qW_{n-2}$ for $n \geq 2$, where a, b, p and q are integers and $q \neq 0$. Let $e = pab - qa^2 - b^2$.

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Here is a list of some known and some new identities involving differences of products of generalized Fibonacci numbers:

1a	$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n$
1b	$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = eq^{n+1}(p^3W_{n+2} - q^2W_{n+1})$
2a	$F_nF_{n+4}F_{n+5} - F_{n+3}^3 = (-1)^{n+1}F_{n+6}$
2b	$W_nW_{n+4}W_{n+5} - W_{n+3}^3 = eq^n(p^3W_{n+4} - qW_{n+5})$
3a	$F_nF_{n+3}^2 - F_{n+2}^3 = (-1)^{n+1}F_{n+1}$
3b	$W_nW_{n+3}^2 - W_{n+2}^3 = eq^n(pW_{n+3} + qW_{n+2})$
4a	$F_n^2F_{n+3} - F_{n+1}^3 = (-1)^{n+1}F_{n+2}$
4b	$W_n^2W_{n+3} - W_{n+1}^3 = eq^n(pW_n + W_{n+1})$
5a	$F_nF_{n+5}F_{n+6} - F_{n+3}F_{n+4}^2 = (-1)^{n+1}L_{n+6}$
5b	$W_nW_{n+5}W_{n+6} - W_{n+3}W_{n+4}^2 = eq^n(pW_{n+8} + p^3qW_{n+4})$
6a	$F_nF_{n+4}^2 - F_{n+2}F_{n+3}^2 = (-1)^{n+1}L_{n+3}$
6b	$W_nW_{n+4}^2 - W_{n+2}W_{n+3}^2 = eq^n(p^2W_{n+4} + q^2W_{n+2})$
7a	$F_nF_{n+3}F_{n+5} - F_{n+2}^2F_{n+4} = (-1)^{n+1}L_{n+2}$
7b	$W_nW_{n+3}W_{n+5} - W_{n+2}^2W_{n+4} = eq^n(p^2W_{n+4} + q^3W_n)$
8a	$F_nF_{n+3}^2 - F_{n+1}^2F_{n+4} = (-1)^{n+1}L_{n+2}$
8b	$W_nW_{n+3}^2 - W_{n+1}^2W_{n+4} = eq^n(W_{n+4} - q^2W_n)$
9a	$F_nF_{n+2}F_{n+5} - F_{n+1}F_{n+3}^2 = (-1)^{n+1}L_{n+3}$
9b	$W_nW_{n+2}W_{n+5} - W_{n+1}W_{n+3}^2 = eq^n(W_{n+5} + p^2qW_{n+1})$
10	$F_nF_{n+2}F_{n+4}F_{n+6} - F_{n+3}^4 = (-1)^{n+1}L_{n+3}^2$
11	$F_nF_{n+4}^3 - F_{n+2}^3F_{n+6} = (-1)^{n+1}F_{n+3}L_{n+3}$
12	$F_n^2F_{n+5}^3 - F_{n+1}^3F_{n+6}^2 = (-1)^{n+1}L_{n+3}^3$

Identities 1a and 1b were discovered and proved by Melham [M]. Identity 2a was discovered and proved by Fairgrieve and Gould [FG]. Identities 3a, 4a, and 8a were discovered and proved by Hoggatt and Bergum [HB]. As far as we know, the other identities in the table are new. In the next section, we will prove some of the new identities. The proofs of all the generalized identities are similar to the proof of 1b by Melham [M].

2. Some proofs

Proof of 2b. We require the identity

$$W_nW_{n+2} - W_{n+1}^2 = eq^n,$$

proved by Horadam [H, p. 171, eq. (4.3)]. We also need

$$
\begin{aligned}
W_{n+2} &= pW_{n+1} - qW_n,\\
W_{n+3} &= (p^2 - q)W_{n+1} - pqW_n,\\
W_{n+4} &= (p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n,\\
W_{n+5} &= (p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n.
\end{aligned}
$$

These identities are obtained by the use of the recurrence for W_n. To prove identity 2b, we write its LHS and RHS in terms of W_n, W_{n+1}, p and q. The RHS of 2b is

$$
\begin{aligned}
&eq^n(p^3W_{n+4} - qW_{n+5})\\
&\quad = (W_nW_{n+2} - W_{n+1}^2)(p^3W_{n+4} - qW_{n+5})\\
&\quad = \big(W_n(pW_{n+1} - qW_n) - W_{n+1}^2\big)\\
&\qquad \times \big(p^3\big((p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n\big)\\
&\qquad\qquad - q\big((p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n\big)\big)\\
&\quad = (p^7 - 2p^5q + p^3q^2 + pq^3)W_{n+1}^2W_n + (-2p^6q + 5p^4q^2 - 5p^2q^3 + q^4)W_{n+1}W_n^2\\
&\qquad + (-p^6 + 3p^4q - 3p^2q^2 + q^3)W_{n+1}^3 + (p^5q^2 - 2p^3q^3 + 2pq^4)W_n^3.
\end{aligned}
$$

The LHS of 2b is

$$
\begin{aligned}
&W_nW_{n+4}W_{n+5} - W_{n+3}^2\\
&\quad = W_n\big((p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n\big)\\
&\qquad \times \big((p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n\big)\\
&\qquad - \big((p^2 - q)W_{n+1} - pqW_n\big)^3\\
&\quad = (p^7 - 2p^5q + p^3q^2 + pq^3)W_{n+1}^2W_n + (-2p^6q + 5p^4q^2 - 5p^2q^3 + q^4)W_{n+1}W_n^2\\
&\qquad + (-p^6 + 3p^4q - 3p^2q^2 + q^3)W_{n+1}^3 + (p^5q^2 - 2p^3q^3 + 2pq^4)W_n^3.
\end{aligned}
$$

Since the LHS and RHS are equal, the identity is proved. ∎

Proof of 5b. We again require the Horadam identity

$$W_nW_{n+2} - W_{n+1}^2 = eq^n.$$

We also need the identities

$$
\begin{aligned}
W_{n+2} &= pW_{n+1} - qW_n,\\
W_{n+3} &= (p^2 - q)W_{n+1} - pqW_n,\\
W_{n+4} &= (p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n,\\
W_{n+5} &= (p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n,\\
W_{n+6} &= (p^5 - 4p^3q + 3pq^2)W_{n+1} - (p^4q - 3p^2q^2 + q^3)W_n,\\
W_{n+8} &= (p^7 - 6p^5q + 10p^3q^2 - 4pq^3)W_{n+1} - (p^6q - 5p^4q^2 + 6p^2q^3 - q^4)W_n,
\end{aligned}
$$

obtained by the use of the recurrence for W_n. We write the LHS and RHS of identity 5b in terms of W_n, W_{n+1}, p and q. The RHS of 5b is

$$\begin{aligned}
&eq^n(pW_{n+8} + p^3qW_{n+4})\\
&= (W_nW_{n+2} - W_{n+1}^2)(pW_{n+8} + p^3qW_{n+5})\\
&= (W_n(pW_{n+1} - qW_n) - W_{n+1}^2)\\
&\quad \times \big(p\big((p^7 - 6p^5q + 10p^3q^2 - 4pq^3)W_{n+1} - (p^6q - 5p^4q^2 + 6p^2q^3 - q^4)W_n\big)\\
&\qquad\qquad + p^3q\big((p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n\big)\big)\\
&= W_n^3(-pq^5 + 5p^3q^4 - 4p^5q^3 + p^7q^2)\\
&\quad + W_n^2W_{n+1}(5p^2q^4 - 13p^4q^3 + 9p^6q^2 - 2p^8q)\\
&\quad + W_nW_{n+1}^2(-pq^4 + p^3q^3 + 4p^5q^2 - 4p^7q + p^9)\\
&\quad + W_{n+1}^3(4p^2q^3 - 8p^4q^2 + 5p^6q - p^8).
\end{aligned}$$

The LHS of 5b is

$$\begin{aligned}
&W_nW_{n+5}W_{n+6} - W_{n+3}W_{n+4}^2\\
&\qquad = W_n\big((p^4 - 3p^2q + q^2)W_{n+1} - (p^3q - 2pq^2)W_n\big)\\
&\qquad\quad \times \big((p^5 - 4p^3q + 3pq^2)W_{n+1} - (p^4q - 3p^2q^2 + q^3)W_n\big)\\
&\qquad\quad - \big((p^2 - q)W_{n+1} - pqW_n\big)\big((p^3 - 2pq)W_{n+1} - (p^2q - q^2)W_n\big)^2\\
&\qquad = W_n^3(-pq^5 + 5p^3q^4 - 4p^5q^3 + p^7q^2)\\
&\qquad\quad + W_n^2W_{n+1}(5p^2q^4 - 13p^4q^3 + 9p^6q^2 - 2p^8q)\\
&\qquad\quad + W_nW_{n+1}^2(-pq^4 + p^3q^3 + 4p^5q^2 - 4p^7q + p^9)\\
&\qquad\quad + W_{n+1}^3(4p^2q^3 - 8p^4q^2 + 5p^6q - p^8).
\end{aligned}$$

Since the LHS and RHS are equal, the identity is proved. ■

Proof of 10. We require Cassini's identity

$$F_nF_{n+2} - F_{n+1}^2 = (-1)^{n+1}.$$

We also need

$$\begin{aligned}
F_{n+2} &= F_{n+1} + F_n,\\
F_{n+3} &= 2F_{n+1} + F_n,\\
F_{n+4} &= 3F_{n+1} + 2F_n,\\
F_{n+6} &= 8F_{n+1} + 5F_n,\\
L_{n+3} &= 4F_{n+1} + 3F_n,
\end{aligned}$$

which are obtained by the use of the recurrence for F_n and the fact that $L_{n+3} = F_{n+4} + F_{n+2}$. To prove 10, we write its LHS and RHS in terms of F_n and F_{n+1}. The RHS of 10 is

$$\begin{aligned}(-1)^{n+1}L_{n+3}^2 &= (F_{n+2}F_n - F_{n+1}^2)L_{n+3}^2\\ &= (F_n^2 + F_nF_{n+1} - F_{n+1}^2)(4F_{n+1} + 3F_n)^2\\ &= 9F_n^4 + 33F_n^3F_{n+1} + 31F_n^2F_{n+1}^2 - 8F_nF_{n+1}^3 - 16F_{n+1}^4.\end{aligned}$$

The LHS of 10 is

$$\begin{aligned}&F_nF_{n+2}F_{n+4}F_{n+6} - F_{n+3}^4\\ &\quad = F_n(F_{n+1} + F_n)(3F_{n+1} + 2F_n)(8F_{n+1} + 5F_n) - (2F_{n+1} + F_n)^4\\ &\quad = 9F_n^4 + 33F_n^3F_{n+1} + 31F_n^2F_{n+1}^2 - 8F_nF_{n+1}^3 - 16F_{n+1}^4.\end{aligned}$$

Since the LHS and RHS are equal, the identity is proved. ∎

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