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LIE DERIVATIONS OF DUAL EXTENSIONS OF ALGEBRAS

BY

YANBO LI (Qinhuangdao) and FENG WEI (Beijing)

Abstract. Let K be a field and Γ a finite quiver without oriented cycles. Let $\Lambda := K(\Gamma, \rho)$ be the quotient algebra of the path algebra $K\Gamma$ by the ideal generated by ρ, and let $\mathscr{D}(\Lambda)$ be the dual extension of Λ. We prove that each Lie derivation of $\mathscr{D}(\Lambda)$ is of the standard form.

1. Introduction. In the study of the representation theory of quasi-hereditary algebras, Xi [19] defined dual extensions of algebras without oriented cycles. Roughly speaking, these algebras A are constructed by adding to the ordinary quiver (without oriented cycles) of a given algebra B a reverse arrow for any original arrow, and extending the relations to this extended quiver in a suitable way. They are a class of finite-dimensional quasi-hereditary algebras, and they were investigated in detail by Deng and Xi [6, 8, 20]. A dual extension algebra is a BGG-algebra in the sense of R. Irving [11], that is, a quasi-hereditary algebra with a duality which fixes all simple modules. A much more general construction, the twisted doubles, were studied by Deng, Koenig and Xi [7, 12, 21].

Derivations and Lie derivations of associative algebras play significant roles in various mathematical areas, such as Lie theory, matrix theory, noncommutative algebras and operator algebras. Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ be a unital algebra over $\mathcal{R}$, and $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$. We write $[a, b] = ab - ba$ for all $a, b \in \mathcal{A}$. Let $\Theta\colon \mathcal{A} \to \mathcal{A}$ be a linear mapping. We call Θ an (*associative*) *derivation* if

$$\Theta(ab) = \Theta(a)b + a\Theta(b)$$

for all $a, b \in \mathcal{A}$. Further, Θ is called a *Lie derivation* if

$$\Theta([a, b]) = [\Theta(a), b] + [a, \Theta(b)]$$

for all $a, b \in \mathcal{A}$. It is clear that every associative derivation is a Lie derivation. But the converse statement is not true in general. Moreover, if $D\colon \mathcal{A} \to \mathcal{A}$ is an associative derivation and $\Delta\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is a linear mapping such that

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$\Delta([a,b]) = 0$ for all $a, b \in \mathcal{A}$, then the mapping

$$\Theta = D + \Delta \qquad (\spadesuit)$$

is a Lie derivation. Such a Lie derivation is said to be *of the standard form.*

A common and popular problem in the study of Lie derivations is whether they have the above mentioned standard form. Equivalently, how every Lie derivation is approximate to a derivation to the utmost extent. The first result in this area is due to Martindale [17], who proved that each Lie derivation of a prime ring satisfying some conditions is of the standard form. Alaminos et al. [1] showed that every Lie derivation on the full matrix algebra over a field of characteristic zero has the standard form. Cheung [5] considered Lie derivations of triangular algebras and gave a sufficient and necessary condition for every Lie derivation to be standard. Benkovič [4] studied the structure of Lie derivations from a triangular algebra into its bi-module. The description of the standard form of Lie triple derivations of triangular algebras was obtained by Xiao and Wei [23]. Recently, the current authors and Xiao investigated the associative-type, Lie-type and Jordan-type linear mappings of generalized matrix algebras. For details, we refer the reader to [13, 14, 16, 22].

The path algebras of quivers naturally appear in the study of tensor algebras of bimodules over semisimple algebras. It is well known that any finite-dimensional basic K-algebra is given by a quiver with relations when K is an algebraically closed field. In [10], Guo and Li studied the Lie algebra of differential operators on a path algebra $K\Gamma$, and related this Lie algebra to the algebraic and combinatorial properties of $K\Gamma$. In [14], the current authors studied Lie derivations of a class of path algebras of quivers without oriented cycles, which can be viewed as one-point extensions. It was proved that in this case each Lie derivation is of the standard form. Moreover, the standard decomposition is unique. On the other hand, we remark that the dual extension algebra of an arbitrary finite-dimensional algebra inherits many nice properties from the given algebra. Then for the path algebra of a finite quiver without oriented cycles, it is natural to ask whether all Lie derivations on the dual extension algebra are of the standard form. We will give a positive answer to this question. More precisely, our main result is

THEOREM. *Let K be a field of characteristic not 2. Let (Γ, ρ) be a finite quiver without oriented cycles. Then each Lie derivation on the dual extension of the algebra $K(\Gamma, \rho)$ is of the standard form ($\spadesuit$). Moreover, the standard decomposition is unique.*

Jordan derivations, another important class of linear mappings on dual extension algebras, have been characterized in [15], where we show that every Jordan derivation on a dual extension algebra is a derivation.

Note that each associative algebra with nontrivial idempotents is isomorphic to a generalized matrix algebra. Recently, Du and Wang [9] studied Lie derivations of generalized matrix algebras with bimodules M being faithful. Although the methods of matrix algebras are also employed in our current work, we do not assume faithfulness conditions. In Cheung's study [5] of Lie derivations of triangular algebras, the faithfulness assumption is not needed. In this sense, Section 3 of this paper is a natural generalization of Cheung's work. Simultaneously, our work is an attempt to deal with the path algebras of quivers with oriented cycles. So this article is also a continuation and development of [14].

The paper is organized as follows. After a rapid review of some necessary preliminaries in Section 2, we characterize Lie derivations of generalized matrix algebras in Section 3. We study Lie derivations of dual extensions in Section 4, where the main result of this paper is obtained. Throughout, we freely use the quiver representation terminology of [2] and [3], where the reader can find basic facts on path algebras and quiver K-linear representations.

2. Dual extension. Let us first recall the definition of dual extensions of path algebras which were introduced by Xi [19]. Moreover, in order to be able to use the methods of matrix algebras, we will also give some descriptions of dual extensions from the point of view of generalized matrix algebras. This kind of algebra was introduced by Morita [18], who studied Morita duality theory of modules and its applications to Artinian algebras. Let us begin with the definition of generalized matrix algebras.

2.1. Generalized matrix algebras. The definition of generalized matrix algebras is via the notion of Morita context. Let $\mathcal{R}$ be a commutative ring with identity. A *Morita context* consists of two $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called *pairings* $\Phi_{MN} : M \underset{B}{\otimes} N \to A$ and $\Psi_{NM} : N \underset{A}{\otimes} M \to B$ making the following diagrams commutative:

$$\begin{CD}
M \underset{B}{\otimes} N \underset{A}{\otimes} M @>{\Phi_{MN}\otimes I_M}>> A \underset{A}{\otimes} M\\
@V{I_M\otimes\Psi_{NM}}VV @VV{\cong}V\\
M \underset{B}{\otimes} B @>{\cong}>> M
\end{CD}$$

and

$$\begin{CD}
N \underset{A}{\otimes} M \underset{B}{\otimes} N @>{\Psi_{NM}\otimes I_N}>> B \underset{B}{\otimes} N\\
@V{I_N\otimes\Phi_{MN}}VV @VV{\cong}V\\
N \underset{A}{\otimes} A @>{\cong}>> N
\end{CD}$$

Let us write this Morita context as $(A, B, {}_AM_B, {}_BN_A, \Phi_{MN}, \Psi_{NM})$. If $(A, B, {}_AM_B, {}_BN_A, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \,\middle|\, a \in A,\, m \in M,\, n \in N,\, b \in B \right\}$$

forms an $\mathcal{R}$-algebra under matrix-like addition and matrix-like multiplication. There is no constraint on the bimodules M and N (which may be zero). Such an $\mathcal{R}$-algebra is called a *generalized matrix algebra* of order 2 and is usually denoted by $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$. Its center is

$$\begin{aligned} &\mathcal{Z}(\mathcal{G}) \\ &= \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a \in \mathcal{Z}(A),\, b \in \mathcal{Z}(B),\, am = mb,\, na = bn,\ \forall m \in M,\, n \in N \right\}. \end{aligned}$$

Thus we have two natural $\mathcal{R}$-linear projections $\pi_A : \mathcal{G} \to A$ and $\pi_B : \mathcal{G} \to B$,

$$\pi_A : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto b.$$

Then $\pi_A(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(A)$, and $\pi_B(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(B)$. If M is faithful as a right B-module and as a left A-module, then for every element $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, there exists a unique $b \in \pi_B(\mathcal{Z}(\mathcal{G}))$, denoted by $\varphi(a)$, such that $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$. It is easy to verify that the mapping $\varphi : \pi_A(\mathcal{Z}(\mathcal{G})) \to \pi_B(\mathcal{Z}(\mathcal{G}))$ is an algebra isomorphism such that $am = m\varphi(a)$ and $na = \varphi(a)n$ for all $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, $m \in M$ and $n \in N$.

REMARK 2.1. Any unital $\mathcal{R}$-algebra $\mathcal{A}$ with nontrivial idempotents is isomorphic to a generalized matrix algebra. In fact, suppose that there exists a nontrivial idempotent $e \in \mathcal{A}$. We construct the following *natural generalized matrix algebra*:

$$\begin{aligned} \mathcal{G} &= \begin{bmatrix} e\mathcal{A}e & e\mathcal{A}(1-e) \\ (1-e)\mathcal{A}e & (1-e)\mathcal{A}(1-e) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} eae & ec(1-e) \\ (1-e)de & (1-e)b(1-e) \end{bmatrix} \,\middle|\, a, b, c, d \in \mathcal{A} \right\}. \end{aligned}$$

It is easy to check that the $\mathcal{R}$-linear mapping

$$\xi : \mathcal{A} \to \mathcal{G}, \qquad a \mapsto \begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix},$$

is an algebra isomorphism.

2.2. Dual extension of a path algebra. Recall that a *finite quiver* Γ is an oriented graph with the set Γ_0 of vertices and the set Γ_1 of arrows between vertices, both finite. For an arrow α, we write $s(\alpha) = i$ and $e(\alpha) = j$

if α goes from vertex i to vertex j. A *sink* is a vertex from which no arrows start, and a *source* is a vertex where no arrows end. A *nontrivial path* in Γ is a sequence $p = \alpha_n \cdots \alpha_1$ of arrows such that $e(\alpha_m) = s(\alpha_{m+1})$ for each $1 \leq m < n$. Define $s(p) = s(\alpha_1)$ and $e(p) = e(\alpha_n)$. A *trivial path* is the symbol e_i for each $i \in \Gamma_0$. In this case, we set $s(e_i) = e(e_i) = i$. A nontrivial path p is called an *oriented cycle* if $s(p) = e(p)$. Denote the set of all paths by $\mathscr{P}$.

Let K be a field and Γ be a quiver. Then the *path algebra* $K\Gamma$ is the K-algebra spanned by the paths in Γ, where the product of two paths $x = \alpha_n \cdots \alpha_1$ and $y = \beta_t \cdots \beta_1$ is defined by

$$xy = \begin{cases} \alpha_n \cdots \alpha_1 \beta_t \cdots \beta_1 & e(y) = s(x), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $K\Gamma$ is an associative algebra with the identity $1 = \sum_{i \in \Gamma_0} e_i$, where e_i $(i \in \Gamma_0)$ are pairwise orthogonal primitive idempotents of $K\Gamma$.

A *relation* σ on a quiver Γ over a field K is a K-linear combination of paths $\sigma = \sum_{i=1}^n k_i p_i$, where $k_i \in K$ and

$$e(p_1) = \cdots = e(p_n), \qquad s(p_1) = \cdots = s(p_n).$$

Moreover, the number of arrows in each path is assumed to be at least 2. Let ρ be a set of relations on Γ over K. The pair (Γ, ρ) is called a *quiver with relations* over K. Denote by $\langle \rho \rangle$ the ideal of $K\Gamma$ generated by the set of relations ρ. The K-algebra $K(\Gamma, \rho) = K\Gamma / \langle \rho \rangle$ is always associated with (Γ, ρ). For $x \in K\Gamma$, write $\overline{x}$ for the corresponding element in $K(\Gamma, \rho)$. We often write x instead of $\overline{x}$ if no confusion can arise. We refer the reader to [3] for the basic facts on path algebras.

Let $\Lambda = K(\Gamma, \rho)$, where Γ is a finite quiver. Let Γ^* be the quiver whose vertex set is Γ_0 and whose arrow set is

$$\Gamma_1^* = \{\alpha^* : i \to j \mid \alpha : j \to i \text{ is an arrow in } \Gamma_1\}.$$

In other words, Γ^* is the opposite quiver of Γ. Let $p = \alpha_n \cdots \alpha_1$ be a path in Γ. Denote the path $\alpha_1^* \cdots \alpha_n^*$ in Γ^* by p^*. Define $\mathscr{D}(\Lambda)$ to be the quotient algebra of the path algebra of the quiver $(\Gamma_0, \Gamma_1 \cup \Gamma_1^*)$ by the ideal generated by

$$\rho \cup \rho^* \cup \{\alpha\beta^* \mid \alpha, \beta \in \Gamma_1\}.$$

If Γ has no oriented cycles, then $\mathscr{D}(\Lambda)$ is called the *dual extension* of Λ. It is a BGG-algebra in the sense of [11]. Clearly, if $|\Gamma_0| = 1$, then the algebra is trivial. Let us assume that $|\Gamma_0| \geq 2$ from now on. Note that in this case, $\mathscr{D}(\Lambda)$ has nontrivial idempotents. In view of Remark 2.1, $\mathscr{D}(\Lambda)$ is isomorphic to a generalized matrix algebra $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$.

Take the nontrivial idempotent to be e_i, where i is a source of Γ. According to the definition of dual extension, it is easy to verify that $\Phi_{MN} = 0$ and

$\Psi_{NM} \neq 0$. If $M \neq 0$, then $N \neq 0$. Moreover, M need not be faithful as a left A-module or as a right B-module. Let us illustrate this by two examples.

EXAMPLE 2.2. Let Γ be the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}$$

and let $\Lambda = K\Gamma$. The dual extension $\mathscr{D}(\Lambda)$ has a basis

$$\{e_1, e_2, e_3, \alpha, \beta, \alpha^*, \beta^*, \alpha^*\alpha, \beta^*\alpha, \beta^*\beta, \alpha^*\beta\}.$$

If we take the nontrivial idempotent to be e_1, then $\mathscr{D}(\Lambda)$ is isomorphic to the generalized matrix algebra $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$, where A has a basis $\{e_2, e_3, \beta, \beta^*, \beta^*\beta\}$, B has a basis $\{e_1, \alpha^*\alpha\}$, M has a basis $\{\alpha, \beta^*\alpha\}$, and N has a basis $\{\alpha^*, \alpha^*\beta\}$. It follows from $\beta\alpha = 0$ and $\beta\beta^*\alpha = 0$ that $\beta \in \mathrm{anni}({}_AM)$, that is, M is not faithful as a left A-module. It is easy to check that $\alpha^*\alpha \in \mathrm{anni}(M_B)$. This implies that M is not faithful as a right B-module. Similarly, we obtain $\alpha^*\alpha \in \mathrm{anni}({}_BN)$ and $\beta^*\beta \in \mathrm{anni}(N_A)$. That is, N is neither a faithful left B-module nor a faithful right A-module.

EXAMPLE 2.3. Let Γ be the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[dr, "\alpha"'] \arrow[drr, bend left, "\gamma"] & & \\ & \underset{2}{\bullet} \arrow[r, "\beta"'] & \underset{3}{\bullet} \end{tikzcd}$$

and let $\Lambda = K\Gamma$. Taking the nontrivial idempotent to be e_1, the dual extension $\mathscr{D}(\Lambda)$ is isomorphic to $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$, where A has a basis $\{e_2, e_3, \beta, \beta^*, \beta^*\beta\}$, M has a basis $\{\alpha, \beta^*\beta\alpha, \beta^*\gamma, \beta\alpha, \gamma\}$, N has a basis $\{\alpha^*, \gamma^*\beta, \alpha^*\beta^*\beta, \alpha^*\beta^*, \gamma^*\}$, B has a basis $\{e_1, \alpha^*\alpha, \gamma^*\gamma, \alpha^*\beta^*\gamma, \gamma^*\beta\alpha, \alpha^*\beta^*\beta\alpha\}$. Clearly, $e_2\alpha \neq 0$, $e_3\beta\alpha \neq 0$. Then $e_2, e_3 \notin \mathrm{anni}(M)$. Similarly, $\beta\alpha \neq 0$ implies that $\beta \notin \mathrm{anni}(M)$, and $\beta^*\beta\alpha \neq 0$ implies that $\beta^* \notin \mathrm{anni}(M)$ and $\beta^*\beta \notin \mathrm{anni}(M)$. Hence M is faithful as a left A-module. On the other hand, it is easy to check that $\alpha^*\beta^*\beta\alpha \in \mathrm{anni}(M_B)$. Thus M is not faithful as a right B-module. Likewise, we find that N is faithful as a right A-module, while it is not faithful as a left B-module.

REMARK 2.4. In order to study the global dimension of dual extensions, one more general definition of dual extension algebras was proposed by Xi [20]. We omit the details here because it will not be used in our current work.

3. Lie derivations of generalized matrix algebras. In Section 2 we have pointed out that the dual extension of a path algebra can be viewed as a generalized matrix algebra. In order to study Lie derivations of dual extension algebras, it is necessary to provide some basic facts concerning Lie derivations of generalized matrix algebras. In this section, we will give a sufficient and necessary condition for every Lie derivation to be standard (♠).

From now on, we always assume that all algebras and bimodules are 2-torsion free. Note that the forms of derivations and Lie derivations of a generalized matrix have been described in [13].

Lemma 3.1 ([13, Proposition 4.1]). *Let Θ_{Lied} be a Lie derivation of a generalized matrix algebra $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$. Then*

$$\Theta_{\mathrm{Lied}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0 n + \delta_4(b) & am_0 - m_0 b + \tau_2(m) \\ n_0 a - bn_0 + \nu_3(n) & \mu_1(a) + n_0 m + nm_0 + \mu_4(b) \end{bmatrix}$$

for all $\left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right] \in \mathcal{G}$, where $m_0 \in M$, $n_0 \in N$ and

$$\delta_1 : A \to A, \quad \delta_4 : B \to \mathcal{Z}(A), \quad \tau_2 : M \to M,$$
$$\nu_3 : N \to N, \quad \mu_1 : A \to \mathcal{Z}(B), \quad \mu_4 : B \to B$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) *δ_1 is a Lie derivation of A and $\delta_1(mn) = \delta_4(nm) + \tau_2(m)n + m\nu_3(n)$;*
(2) *μ_4 is a Lie derivation of B and $\mu_4(nm) = \mu_1(mn) + n\tau_2(m) + \nu_3(n)m$;*
(3) *$\delta_4([b, b']) = 0$ for all $b, b' \in B$, and $\mu_1([a, a']) = 0$ for all $a, a' \in A$;*
(4) *$\tau_2(am) = a\tau_2(m) + \delta_1(a)m - m\mu_1(a)$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b) - \delta_4(b)m$;*
(5) *$\nu_3(na) = \nu_3(n)a + n\delta_1(a) - \mu_1(a)n$ and $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n - n\delta_4(b)$.*

Lemma 3.2 ([13, Proposition 4.2]). *An additive mapping $\Theta_{\mathrm{d}} : \mathcal{G} \to \mathcal{G}$ is a derivation if and only if*

$$\Theta_{\mathrm{d}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) - mn_0 - m_0 n & am_0 - m_0 b + \tau_2(m) \\ n_0 a - bn_0 + \nu_3(n) & n_0 m + nm_0 + \mu_4(b) \end{bmatrix}$$

for all $\left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right] \in \mathcal{G}$, where $m_0 \in M$, $n_0 \in N$ and

$$\delta_1 : A \to A, \quad \tau_2 : M \to M, \quad \tau_3 : N \to M,$$
$$\nu_2 : M \to N, \quad \nu_3 : N \to N, \quad \mu_4 : B \to B$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) *δ_1 is a derivation of A with $\delta_1(mn) = \tau_2(m)n + m\nu_3(n)$;*
(2) *μ_4 is a derivation of B with $\mu_4(nm) = n\tau_2(m) + \nu_3(n)m$;*
(3) *$\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ and $\tau_2(mb) = \tau_2(m)b + m\mu_4(b)$;*
(4) *$\nu_3(na) = \nu_3(n)a + n\delta_1(a)$ and $\nu_3(bn) = b\nu_3(n) + \mu_4(b)n$.*

In [5], Cheung gave a necessary and sufficient condition for each Lie derivation on a triangular algebra to have the standard form (♠). We extend this result to the generalized matrix algebras context.

THEOREM 3.3. *Let $\Theta_{\rm Lied}$ be a Lie derivation of a generalized matrix algebra $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$. Then $\Theta_{\rm Lied}$ is of the standard form (♠) if and only if there exist linear mappings $l_A : A \to \mathcal{Z}(A)$ and $l_B : B \to \mathcal{Z}(B)$ satisfying*

(1) *$p_A = \delta_1 - l_A$ is a derivation on A, $l_A([a, a']) = 0$, $l_A(mn) = \delta_4(nm)$, $l_A(a)m = m\mu_1(a)$ and $nl_A(a) = \mu_1(a)n$.*
(2) *$p_B = \mu_4 - l_B$ is a derivation on B, $l_B([b, b']) = 0$, $l_B(nm) = \mu_1(mn)$, $l_B(b)n = n\delta_4(b)$ and $ml_B(b) = \delta_4(b)m$.*

Proof. For necessity, suppose that $\Theta_{\rm Lied} = \delta + h$, where δ is a derivation and h maps $\mathcal{G}$ into $\mathcal{Z}(\mathcal{G})$. Then by Lemma 3.2, there exist linear mappings $l_A : A \to A$ and $l_B : B \to B$ such that $p_A = \delta_1 - l_A$ is a derivation of A and $p_B = \mu_4 - l_B$ is a derivation of B. This gives

$$h\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} l_A(a) + \delta_4(b) & 0 \\ 0 & \mu_1(a) + l_B(b) \end{bmatrix} \in Z(\mathcal{G}), \quad \forall \begin{bmatrix} a & m \\ n & b \end{bmatrix} \in \mathcal{G}.$$

By Lemma 3.1 we know that $\delta_4(b) \in \mathcal{Z}(A)$ and $\mu_1(a) \in \mathcal{Z}(B)$ for all $b \in B$ and $a \in A$. Then the structure of $\mathcal{Z}(\mathcal{G})$ implies that l_A maps into $\mathcal{Z}(A)$ and l_B maps into $\mathcal{Z}(B)$. Furthermore, $l_A(a)m = m\mu_1(a)$, $nl_A(a) = \mu_1(a)n$, $l_B(b)n = m\delta_4(b)$ and $ml_B(b) = \delta_4(b)m$.

Note that h is also a Lie derivation of $\mathcal{G}$. In view of Lemma 3.1 we have $l_A(mn) = \delta_4(nm)$ and $l_B(nm) = \mu_1(mn)$ for all $m \in M$ and $n \in N$. Let us substitute $G_1 = \left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $G_2 = \left[\begin{smallmatrix} a' & 0 \\ 0 & 0 \end{smallmatrix}\right]$ into

$$h([G_1, G_2]) = [h(G_1), G_2] + [G_1, h(G_2)]. \tag{3.1}$$

It is not difficult to calculate that

$$h([G_1, G_2]) = \begin{bmatrix} l_A([a, a']) & 0 \\ 0 & \mu_1([a, a']) \end{bmatrix}, \tag{3.2}$$

$$[h(G_1), G_2] + [G_1, h(G_2)] = \begin{bmatrix} [l_A(a), a'] + [a, l_A(a')] & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.3}$$

It follows from $l_A(a), l_A(a') \in \mathcal{Z}(A)$ that

$$[l_A(a), a'] + [a, l_A(a')] = 0.$$

Combining (3.2) with (3.3) yields $l_A([a, a']) = 0$. Similarly, if we take $G_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right]$ and $G_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & b' \end{smallmatrix}\right]$ in (3.1), we get $l_B([b, b']) = 0$.

For sufficiency, set

$$\delta\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} p_A(a) - mn_0 - m_0n & am_0 - m_0b + \tau_2(m) \\ n_0a - bn_0 + \nu_3(n) & n_0m + nm_0 + p_B(b) \end{bmatrix},$$

$$h\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} l_A(a) + \delta_4(b) & 0 \\ 0 & \mu_1(a) + l_B(b) \end{bmatrix}$$

for all $\left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right] \in \mathcal{G}$. It is easy to verify that δ is a derivation of $\mathcal{G}$ and h maps into $\mathcal{Z}(\mathcal{G})$. For all $G = \left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right] \in \mathcal{G}$ and $G' = \left[\begin{smallmatrix} a' & m' \\ n' & b' \end{smallmatrix}\right] \in \mathcal{G}$, we have

$$h([G, G']) = \begin{bmatrix} l_A(x+u) + \delta_4(v+y) & 0 \\ 0 & \mu_1(x+u) + l_B(v+y) \end{bmatrix},$$

where $x = [a, a']$, $y = [b, b']$, $u = mn' - m'n$ and $v = nm' - n'm$. Note that (1) implies $l_A(x+u)+\delta_4(v+y) = 0$, and (2) implies $\mu_1(x+u)+l_B(v+y) = 0$. Therefore $h([G, G']) = 0$. ∎

The following corollary provides a sufficient condition for each Lie derivation of $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ to be standard, where M is faithful as a left A-module and as a right B-module.

Corollary 3.4. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra. Suppose that M is faithful as a left A-module and as a right B-module. If $\mathcal{Z}(A) = \pi_A(\mathcal{Z}(\mathcal{G}))$ and $\mathcal{Z}(B) = \pi_B(\mathcal{Z}(\mathcal{G}))$, then every Lie derivation of $\mathcal{G}$ has the standard form (♠).*

Proof. Let Θ be a Lie derivation of $\mathcal{G}$ of the form described in Lemma 3.1. Then it follows from $\mathcal{Z}(A) = \pi_A(\mathcal{Z}(\mathcal{G}))$ and $\mathcal{Z}(B) = \pi_B(\mathcal{Z}(\mathcal{G}))$ and Lemma 3.1 that

$$h\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \varphi^{-1}(\mu_1(a)) + \delta_4(b) & 0 \\ 0 & \mu_1(a) + \varphi(\delta_4(b)) \end{bmatrix} \in Z(\mathcal{G}).$$

On the other hand, a direct computation shows that $\Theta - h$ is a derivation of $\mathcal{G}$. This completes the proof. ∎

Remark 3.5. Du and Wang [9] obtained a much more general version of Corollary 3.4. We omit the details.

Corollary 3.6. *Let $\mathcal{U} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra. Suppose that M is faithful as a left A-module and as a right B-module. If $\mathcal{Z}(A) = \pi_A(\mathcal{Z}(\mathcal{U}))$ and $\mathcal{Z}(B) = \pi_B(\mathcal{Z}(\mathcal{U}))$, then every Lie derivation of $\mathcal{U}$ is of the standard form (♠).*

Let us now extend Theorem 11 of [5] to the case of generalized matrix algebras. As in [5], we need two preliminary lemmas.

Lemma 3.7. *Let δ_1 be a Lie derivation of A, and let $\mu_1 : A \to \mathcal{Z}(B)$, $\tau_2 : M \to M$ and $\nu_3 : N \to N$ be linear mappings satisfying*

$$\tau_2(am) = a\tau_2(m) + \delta_1(a)m - m\mu_1(a),$$
$$\nu_3(na) = \nu_3(n)a + n\delta_1(a) - \mu_1(a)n.$$

Define $\Gamma : A \times A \to A$ by

$$\Gamma(x, y) = \delta_1(xy) - x\delta_1(y) - \delta_1(x)y.$$

Then:

(1) $\Gamma(x, y) = \Gamma(y, x)$;
(2) $\Gamma(x, y)m = m\mu_1(xy) - xm\mu_1(y) - ym\mu_1(x)$ *and* $n\Gamma(x, y) = \mu_1(xy)n - \mu_1(y)nx - \mu_1(x)ny$.
(3) *Let $f(t) = \sum_{j=0}^{k} r_j t^j \in K[t]$ and $x \in A$. Then there exists $a_x \in A$ such that*

$$a_x m = m\mu_1(f(x)) - f'(x)m\mu_1(x) \quad \text{for all } m \in M,$$
$$na_x = \mu_1(f(x))n - \mu_1(x)nf'(x) \quad \text{for all } n \in N,$$

where $f'(t) = \sum_{j=1}^{k} jr_j t^{j-1}$.

Moreover, if δ_1 is of the standard form, that is, $\delta_1 = p_A + l_A$, where p_A is a derivation of A and l_A maps into $\mathcal{Z}(A)$, then $a_x = l_A(f(x)) - f'(x)l(x))$.

Proof. The equality (1) and the first one of (2) can be obtained from [5, Lemma 9(i), (ii)]. It suffices to prove the second equality of (2). In fact, the equality $\nu_3(na) = \nu_3(n)a + n\delta_1(a) - \mu_1(a)n$ implies that

$$\begin{aligned} \nu_3(n(xy)) &= \nu_3(n)xy + n\delta_1(xy) - \mu_1(xy)n, \\ \nu_3((nx)y) &= \nu_3(nx)y + nx\delta_1(y) - \mu_1(y)nx \\ &= \nu_3(n)xy + n\delta_1(x)y - \mu_1(x)ny. \end{aligned}$$

Comparing the above two equalities gives the required result.

In order to prove (3), it is enough to consider $f(t) = t^k$, where $k = 0, 1, \ldots$. For $k = 0$, we can take $a_x = \delta_1(1)$, by conditions (4) and (5) of Lemma 3.1. For $k > 0$, take

$$a_x = \sum_{j=1}^{k-1} x^{k-1-j} G(x^j, x).$$

Then the two equalities of (2) imply the conclusion. ∎

Lemma 3.8. *Assume that $\delta_1 = p_A + l_A$, where p_A is a derivation and $l_A(a) \in \mathcal{Z}(A)$, $l_A([a, a']) = 0$ for all $a, a' \in A$. Let*

$$V_A = \{a \in A \mid l_A(a)m = m\mu_1(a),\ nl_A(a) = \mu_1(a)n\ \forall m \in M,\ \forall n \in N\}.$$

Then V_A is a subalgebra of A satisfying the following conditions:

(1) $[x,y] \in V_A$ *for all* $x,y \in A$.
(2) *Let* $f(t) \in K[t]$ *and* $x \in A$. *If* $f'(x) = 0$, *then* $f(x) \in V_A$.
(3) V_A *contains all the idempotents of* A.

Proof. For all $x,y \in V_A$, from Lemma 3.7(2) we have

$$\begin{aligned} m\mu_1(xy) &= \Gamma(x,y)m + xm\mu_1(y) + ym\mu_1(x) \\ &= \delta_1(xy)m - x\delta_1(y)m - \delta_1(x)ym + xl_A(y)m + yl_A(x)m \\ &= p_A(xy)m + l_A(xy)m - xp_A(y)m \\ &\quad - xl_A(y)m - p_A(x)ym - l_A(x)ym + xl_A(y)m + yl_A(x)m. \end{aligned}$$

Note that p_A is a derivation and $l_A(x) \in \mathcal{Z}(A)$. Hence

$$m\mu_1(xy) = l_A(xy)m. \tag{3.4}$$

On the other hand,

$$\begin{aligned} \mu_1(xy)n &= n\Gamma(x,y) + \mu_1(y)nx + \mu_1(x)ny \\ &= n\delta_1(xy) - nx\delta_1(y) - n\delta_1(x)y + nl_A(y)x + nl_A(x)y \\ &= np_A(xy) + nl_A(xy) - nxp_A(y) \\ &\quad - nxl_A(y) - np_A(x)y - nl_A(x)y + nl_A(y)x + nl_A(x)y \\ &= nl_A(xy). \end{aligned} \tag{3.5}$$

Combining (3.4) with (3.5) shows that V_A is a subalgebra of A.

We now prove (1)–(3). Clearly, (1) follows from the fact that μ_1 annihilates all commutators.

To prove (2), take $x \in A$ with $f'(x) = 0$ and $f(t) \in \mathcal{R}[t]$. In view of Lemma 3.7(3), there exists $a_x \in A$ such that $a_x m = m\mu_1(f(x))$ and $na_x = \mu_1(f(x))n$. Since δ_1 is of the standard form, we have $a_x m = l_A(f(x))m$ and $na_x = nl_A(f(x))$ by Lemma 3.7. Therefore

$$l_A(f(x))m = m\mu_1(f(x)) \quad \text{and} \quad nl_A(f(x)) = \mu_1(f(x))n.$$

That is, $f(x) \in V_A$.

The proof of (3) is the same as that of [5, Lemma 10(4)]: for any idempotent $e \in A$, let $f(t) = 3t^2 - 2t^3$. Then clearly $f'(e) = 0$. Hence $e = f(e) \in V_A$. ∎

As in [5], we define $W(X)$ to be the smallest subalgebra of an algebra X satisfying conditions (1)–(3) of Lemma 3.8. Then the following is a direct consequence of Theorem 3.3.

Corollary 3.9. *Let* $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ *be a generalized matrix algebra with zero bilinear pairings. If*

(1) $W(A) = A$ *and every Lie derivation of* A *is of the standard form* (♠),
(2) $W(B) = B$ *and every Lie derivation of* B *is of the standard form* (♠),

then each Lie derivation of $\mathcal{G}$ *is of the standard form* (♠).

Proof. Since the bilinear pairings are both zero, (1) implies condition (1) of Theorem 3.3, and (2) implies condition (2) of Theorem 3.3. ∎

COROLLARY 3.10 ([5, Theorem 11]). *Every Lie derivation of a triangular algebra* $\mathcal{U} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ *is of the standard form if:*

(1) $W(A) = A$ *and every Lie derivation of* A *is of the standard form;*
(2) $W(B) = B$ *and every Lie derivation of* B *is of the standard form.*

To end this section, let us give an application of Corollary 3.9. We will construct a class of algebras with bilinear pairings being both zero, called *generalized one-point extension algebras.* Note that they are not triangular algebras. We will show that each Lie derivation of a generalized one-point extension algebra is of the standard form (♠). Moreover, the standard decomposition is unique.

DEFINITION 3.11. Let (Γ_0, Γ_1) be a finite quiver without oriented cycles and $|\Gamma_0| \geq 2$. Let Γ^* be a quiver whose vertex set is Γ_0 and whose arrow set is

$$\Gamma_1^* = \{\alpha^* : i \to j \mid \alpha : j \to i \text{ is an arrow in } \Gamma_1\}.$$

For a path $p = \alpha_n \cdots \alpha_1$ in Γ, write the path $\alpha_1^* \cdots \alpha_n^*$ in Γ^* by p^*. Given a set ρ of relations, denote $\Lambda = K(\Gamma, \rho)$. Define the *generalized one-point extension algebra* $E(\Lambda)$ to be the quotient algebra of the path algebra of the quiver $(\Gamma_0, \Gamma_1 \cup \Gamma_1^*)$ by the ideal generated by

$$\rho \cup \rho^* \cup \{\alpha\beta^* \mid \alpha, \beta \in \Gamma_1\} \cup \{\alpha^*\beta \mid \alpha, \beta \in \Gamma_1\}.$$

In order to study the Lie derivations of $E(\Lambda)$, we need the following lemmas.

LEMMA 3.12. $W(E(\Lambda)) = E(\Lambda)$.

Proof. According to the definition, $W(E(\Lambda))$ contains all idempotents e_i. Furthermore, for every arrow α with $e(\alpha) = j$, the fact that $\alpha = [e_j, \alpha]$ implies $\alpha \in W(E(\Lambda))$. For the same reason, $\alpha^* \in W(E(\Lambda))$ and then $W(E(\Lambda)) = E(\Lambda)$. ∎

Since Γ is a quiver without oriented cycles, we can take a source i in Γ. Let e_i be the corresponding idempotent in $E(\Lambda)$. Then $E(\Lambda)$ is isomorphic to a generalized matrix algebra $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ with $A \simeq E(\Lambda')$, where the quiver (Γ', ρ') of Λ' is obtained from Γ by removing vertex i and the relations starting at i. Moreover, in view of the construction of $E(\Lambda)$ the bilinear pairings are both zero.

LEMMA 3.13. *An additive mapping* $\Theta_{\rm d}$ *is a derivation of* $E(\Lambda)$ *if and only if*

$$\Theta_{\rm d}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \delta_1(a) & am_0 - m_0 b + \tau_2(m) \\ n_0 a - bn_0 + \nu_3(n) & 0 \end{bmatrix}$$

for all $\left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right] \in \mathcal{G}$, *where* $m_0 \in M$, $n_0 \in N$ *and*

$$\delta_1 : A \to A, \qquad \tau_2 : M \to M, \qquad \nu_3 : N \to N$$

are all $\mathcal{R}$*-linear mappings satisfying the following conditions:*

(1) δ_1 *is a derivation of* A*;*
(2) $\tau_2(am) = a\tau_2(m) + \delta_1(a)m$ *and* $\tau_2(mb) = \tau_2(m)b$*;*
(3) $\nu_3(na) = \nu_3(n)a + n\delta_1(a)$ *and* $\nu_3(bn) = b\nu_3(n)$*.*

Proof. Since the bilinear pairings are both zero, by Lemma 3.2 we only need to show that $\mu_4 = 0$. But this is clear by condition (2) of Lemma 3.2. ■

Lemma 3.14. *Every derivation* Θ *of* $E(\Lambda)$ *with* $\mathrm{Im}(\Theta) \in \mathcal{Z}(E(\Lambda))$ *is zero.*

Proof. If there exists a nonzero such derivation, then Lemma 3.13 yields $\delta_1 \neq 0$ and $\mathrm{Im}(\delta_1) \in \mathcal{Z}(A)$. Repeating this process, we eventually get a nonzero derivation f of K. However, this is impossible. ■

Proposition 3.15. *Every Lie derivation of* $E(\Lambda)$ *is of the standard form* (♠)*. Moreover, the standard decomposition is unique.*

Proof. Obviously, $W(K) = K$. If $|\Gamma_0'| = 1$, then $A \simeq K$ and hence $W(A) = A$. If $|\Gamma_0'| > 1$, then A can be viewed as a generalized one-point extension too. Thus $W(A) = A$. By Corollary 3.9, if each Lie derivation of A has the standard form (♠), then so does $E(\Lambda)$. Note that Γ is a finite quiver. Repeating the above process finitely many times, we arrive at the algebra K. Of course, each Lie derivation of K is of the standard form. This implies that each Lie derivation of $E(\Lambda)$ is standard. The uniqueness of the standard decomposition is due to Lemma 3.14. ■

4. Lie derivations of dual extensions

Lemma 4.1. *Let* Γ *be a finite quiver without oriented cycles,* $\Lambda = K(\Gamma, \rho)$ *and* $\mathscr{D}(\Lambda)$ *the dual extension of* Λ*. Then* $\mathscr{D}(\Lambda) = W(\mathscr{D}(\Lambda))$*.*

Proof. If Γ only contains one vertex, then the algebra $\mathscr{D}(\Lambda)$ is trivial, that is, $\mathscr{D}(\Lambda) \simeq K$. In this case $W(\mathscr{D}(\Lambda)) = \mathscr{D}(\Lambda)$.

Now suppose that the number of vertices in Γ is at least 2. It follows from condition (3) in the definition of $W(\mathscr{D}(\Lambda))$ that all the trivial paths are contained in $W(\mathscr{D}(\Lambda))$. On the other hand, Γ is a quiver without oriented cycles. Thus for every arrow $\alpha \in \Gamma$, we have $\alpha = [\alpha, s(\alpha)]$, since $\alpha s(\alpha) = \alpha$ and $s(\alpha)\alpha = 0$. Condition (1) of the definition of $W(\mathscr{D}(\Lambda))$ shows that α is in $W(\mathscr{D}(\Lambda))$. Analogously, $\alpha^* \in W(\mathscr{D}(\Lambda))$. Therefore all paths are contained in $W(\mathscr{D}(\Lambda))$, and hence $\mathscr{D}(\Lambda) = W(\mathscr{D}(\Lambda))$. ■

Lemma 4.2. *Let $\mathscr{D}(\Lambda)$ be the dual extension of $\Lambda = K(\Gamma, \rho)$. Let i be a source in Γ and*

$$\mathcal{P}_i = \{p \in \mathscr{P} \mid s(p) = e(p) = i,\, p^2 = 0\}.$$

Denote by V the vector space spanned by all paths of $\mathcal{P}_i$. Assume that Θ_{Lied} is a Lie derivation on $\mathscr{D}(\Lambda)$. Then $\Theta_{\mathrm{Lied}}(v)$ is in the center of $\mathscr{D}(\Lambda)$ for all $v \in V$.

Proof. It is easy to see that

$$\mathscr{D}(\Lambda) \simeq \begin{bmatrix} (1-e_i)\mathscr{D}(\Lambda)(1-e_i) & (1-e_i)\mathscr{D}(\Lambda)e_i \\ e_i\mathscr{D}(\Lambda)(1-e_i) & e_i\mathscr{D}(\Lambda)e_i \end{bmatrix}.$$

Thus Θ_{Lied} has the form described in Lemma 3.1. Condition (1) of Lemma 3.1 implies that $\delta_4(p^*p) = 0$. Thus $\Theta_{\mathrm{Lied}}(p^*p) = \mu_4(p^*p)$. It follows from (2) of Lemma 3.1 that $\mu_4(p^*p) = p^*q + q'p$. This shows that $0m = m\mu_4(p^*p) = 0$ and $n0 = \mu_4(p^*p)n = 0$ for all $m \in M$ and $n \in N$. Note that B is commutative. Hence $\Theta_{\mathrm{Lied}}(p^*p) \in \mathcal{Z}(\mathscr{D}(\Lambda))$. ∎

Now we are in a position to prove the main result of this paper.

Theorem 4.3. *Let Γ be a quiver without oriented cycles, $\Lambda = K(\Gamma, \rho)$ and $\mathscr{D}(\Lambda)$ be the dual extension of Λ. Then each Lie derivation on $\mathscr{D}(\Lambda)$ is of the standard form (♠).*

Proof. Let Θ_{Lied} be a Lie derivation of $\mathscr{D}(\Lambda)$. Suppose that $\mathscr{D}(\Lambda)$ has a vector space decomposition $\mathscr{D}(\Lambda) = V \oplus W$. Define a linear mapping Θ' by $\Theta'|V = 0$, $\Theta'|W = \Theta_{\mathrm{Lied}}|W$, and define a linear mapping Δ' by $\Delta'(V) = \Theta_{\mathrm{Lied}}(V)$ and $\Delta'(W) = 0$. Clearly, $\Theta_{\mathrm{Lied}} = \Theta' + \Delta'$. Furthermore, it follows from Lemma 4.2 that $\mathrm{Im}(\Delta') \subset \mathcal{Z}(\mathscr{D}(\Lambda))$. This shows that Θ' is also a Lie derivation on $\mathscr{D}(\Lambda)$. Clearly, if each Lie derivation of type Θ' is of the standard form (♠), then every Lie derivation of $\mathscr{D}(\Lambda)$ has the standard form (♠).

Assume i is a source in Γ, and e_i the corresponding idempotent in $\mathscr{D}(\Lambda)$. Then

$$\mathscr{D}(\Lambda) \simeq \begin{bmatrix} (1-e_i)\mathscr{D}(\Lambda)(1-e_i) & (1-e_i)\mathscr{D}(\Lambda)e_i \\ e_i\mathscr{D}(\Lambda)(1-e_i) & e_i\mathscr{D}(\Lambda)e_i \end{bmatrix}.$$

Let Θ_{Lied} be a Lie derivation of $\mathscr{D}(\Lambda)$ satisfying $\Theta_{\mathrm{Lied}}(V) = 0$. Let us first prove that for $\mathscr{D}(\Lambda)$, condition (2) of Theorem 3.3 is satisfied. From the construction of $\mathscr{D}(\Lambda)$ we know that $e_i\mathscr{D}(\Lambda)e_i$ is an algebra with a basis $\{p^*p \mid s(p) = i\}$. Furthermore, if p, q are nontrivial, then $(p^*p)(q^*q) = 0$. Thus the algebra $e_i\mathscr{D}(\Lambda)e_i$ is commutative. Let $l_B = \mu_4$. Then $l_B([b, b']) = 0$ for all $b, b' \in e_i\mathscr{D}(\Lambda)e_i$. Note that $(1-e_i)\mathscr{D}(\Lambda)e_i\mathscr{D}(\Lambda)(1-e_i) = 0$. That is, $\Phi_{MN} = 0$. We conclude that $\mu_1(mn) = 0$ for all $m \in M$ and $n \in N$.

On the other hand, since $\Theta(p^*p) = 0$ for all nontrivial paths p, we arrive at $\mu_4(p^*p) = 0$ and hence $\mu_4(nm) = 0$. Therefore $l_B(nm) = \mu_1(mn)$.

Let $b = ke_i + v$, where $v \in V$. It follows from the structure of M and $\mathscr{D}(\Lambda)$ that

$$\tau_2(mb) = k\tau_2(me_i) = k\tau_2(m) = \tau_2(m)ke_i = \tau_2(m)(ke_i + v) = \tau_2(m)b.$$

Similarly, we can obtain $\nu_3(bn) = b\nu_3(n)$. Then it follows from conditions (4) and (5) of Lemma 3.1 that $l_B(b)n = n\delta_4(b)$ and $ml_B(b) = \delta_4(b)m$.

By the definition of dual extension, it is easy to check that

$$(1 - e_i)\mathscr{D}(\Lambda)(1 - e_i) \simeq \mathscr{D}(\Lambda'),$$

where $\Lambda' = K(\Gamma', \rho')$, (Γ', ρ') being the quiver obtained from Γ by removing vertex i and the relations starting at i. Clearly, Γ' has no oriented cycles. Then Lemma 4.1 implies that $\mathscr{D}(\Lambda') = W(\mathscr{D}(\Lambda'))$. Thus Θ_{Lied} is of the standard form (♠) if each Lie derivation on $\mathscr{D}(\Lambda')$ is standard.

Note that Γ is a finite quiver. Repeating this process finitely many times, we arrive at the algebra K. That is, if each Lie derivation of K is standard, then so is the case for $\mathscr{D}(\Lambda)$. ∎

Corollary 4.4. *Let Θ_{Lied} be a Lie derivation of $\mathscr{D}(\Lambda)$. Then there exists a derivation D of $\mathscr{D}(\Lambda)$ with $\Theta_{\text{Lied}}(x) = D(x)$ for all $x = \sum_i k_i p_i \in \Lambda$, where p_i are nontrivial paths.*

Proof. By Theorem 4.3, Θ_{Lied} is of the standard form (♠), so $\Theta_{\text{Lied}} = D + \Delta$, where D is a derivation of $\mathscr{D}(\Lambda)$ and $\Delta(x) \in \mathcal{Z}(\mathscr{D}(\Lambda))$ for all $x \in \mathscr{D}(\Lambda)$. Note that Δ is also a Lie derivation of $\mathscr{D}(\Lambda)$. Thus for a path p with $s(p) \neq e(p)$, the fact that $p = [p, s(p)]$ gives

$$\Delta(p) = [\Delta(p), s(p)] + [p, \Delta(s(p))].$$

It follows from the image of Δ being in $\mathcal{Z}(\mathscr{D}(\Lambda))$ that $\Delta(p) = 0$. Moreover, let p be a nontrivial path with $s(p) = e(p)$. By the construction of $\mathscr{D}(\Lambda)$, p is of the form x^*x, where x is a nontrivial path in Γ. Therefore

$$\Delta(p) = \Delta(x^*x) = \Delta([x^*, x]) = [\Delta(x^*), x] + [x^*, \Delta(x)] = 0.$$

Thus for all $x = \sum_i k_i \bar{p}_i \in \Lambda$, where the p_i are nontrivial paths, we have $\Theta_{\text{Lied}}(x) = D(x)$. ∎

Let us address the problem of whether the standard decomposition of each Lie derivation of $\mathscr{D}(\Lambda)$ is unique. It turns out that the answer is positive. To see this, we first characterize the center of $\mathscr{D}(\Lambda)$.

Lemma 4.5. *Let Γ be a connected quiver with $|\Gamma_0| \geq 2$. Then the elements in $\mathcal{Z}(\mathscr{D}(\Lambda))$ are all of the form*

$$k + \sum_{e(p)=s(p),\, p^2=0} k_p p.$$

Proof. Assume that

$$x=\sum_{i\in\Gamma_0}k_ie_i+\sum_{s(p)\neq e(p)}k_pp+\sum_{s(p)=e(p),\,p^2=0}k_pp\in\mathcal{Z}(\mathscr{D}(\Lambda)).$$

Applying the fact that $e_tx=xe_t$ yields

$$\sum_{t=s(p)\neq e(p)}k_pp=\sum_{t=e(p)\neq s(p)}k_pp.$$

This implies that for all paths p with $s(p)\neq e(p)$, we have $k_p=0$ if $s(p)=t$ or $e(p)=t$. Since t is arbitrary, the coefficients of all paths p with $s(p)\neq e(p)$ are all zero.

Let α be an arrow in Γ_1 with $e(\alpha)=j$ and $s(\alpha)=t$. In view of $\alpha x=x\alpha$, we know that $k_j=k_t$. Note that Γ is a connected quiver. Thus $k_i=k$ for all $i\in\Gamma_0$, where $k\in K$. ∎

Lemma 4.6. *Let D be a derivation of $\mathscr{D}(\Lambda)$ with* $\mathrm{Im}(D)\subset\mathcal{Z}(\mathscr{D}(\Lambda))$. *Then $D=0$.*

Proof. Clearly, D is also a Lie derivation of $\mathscr{D}(\Lambda)$. By the proof of Corollary 4.4, we have $D(p)=0$ for all nontrivial paths p. We now prove $D(e_i)=0$ for all $i\in\Gamma_0$. According to Lemma 4.5, we can assume that $D(e_i)=k_i+\sum_{e(p)=s(p),\,p^2=0}k_p^ip$. Note that e_i is an idempotent. By the definition of derivation, it is easy to verify that $k_i=0$ and $k_p^i=0$ for paths p with $s(p)=i$. Suppose there exists some p with nonzero coefficient in $D(e_i)$. Let $s(p)=j\neq i$. Then $D(e_ie_j)=0$. On the other hand, $D(e_ie_j)=D(e_i)e_j+e_iD(e_j)\neq 0$, a contradiction. ∎

As a direct consequence of Lemma 4.6 we immediately get

Proposition 4.7. *Let Θ_{Lied} be a Lie derivation of a dual extension algebra $\mathscr{D}(\Lambda)$. Then the standard decomposition of Θ_{Lied} is unique.*

Remark 4.8. On the one hand, a Lie derivation of a dual extension algebra can be uniquely expressed as the sum of a derivation and a linear mapping annihilating all commutators with images in the center of the algebra. On the other hand, the sum of a derivation and such a linear mapping is clearly a Lie derivation. In this sense, the Lie derivations on dual extensions are totally characterized.

Now let us give an example of a Lie derivation which is not a derivation.

Example 4.9. Let Γ be the quiver

$$\overset{1}{\bullet}\xrightarrow{\quad\alpha\quad}\overset{2}{\bullet}\xrightarrow{\quad\beta\quad}\overset{3}{\bullet}$$

with relation $\beta\alpha$, and $\Lambda=K(\Gamma,\rho)$. Let $\mathscr{D}(\Lambda)$ be the dual extension of the algebra Λ. Define a linear mapping Θ_{Lied} on $\mathscr{D}(\Lambda)$ by

$$\Theta_{\mathrm{Lied}}(e_1) = k_1 + \alpha^*\alpha, \qquad \Theta_{\mathrm{Lied}}(e_2) = k_2 + \beta^*\beta, \qquad \Theta_{\mathrm{Lied}}(e_3) = k_3,$$
$$\Theta_{\mathrm{Lied}}(\alpha) = \alpha, \qquad \Theta_{\mathrm{Lied}}(\alpha^*) = \alpha^*, \qquad \Theta_{\mathrm{Lied}}(\beta) = \beta^*,$$
$$\Theta_{\mathrm{Lied}}(\beta^*) = \beta^*, \qquad \Theta_{\mathrm{Lied}}(\alpha^*\alpha) = 2\alpha^*\alpha, \qquad \Theta_{\mathrm{Lied}}(\beta^*\beta) = 2\beta^*\beta.$$

Then a direct computation shows that Θ_{Lied} is a Lie derivation of $\mathscr{D}(\Lambda)$ but not a derivation.

Moreover, we give the standard decomposition of Θ_{Lied}. Define a linear mapping Δ on $\mathscr{D}(\Lambda)$ by

$$\Delta(e_1) = k_1 + \alpha^*\alpha, \quad \Delta(e_2) = k_2 + \beta^*\beta, \quad \Delta(e_3) = k_3,$$

and let $D = \Theta_{\mathrm{Lied}} - \Delta$. Then $\Theta_{\mathrm{Lied}} = D + \Delta$ is the standard decomposition of Θ.

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Yanbo Li
School of Mathematics and Statistics
Northeastern University at Qinhuangdao
Qinhuangdao, 066004, P.R. China
E-mail: liyanbo707@163.com

Feng Wei
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing, 100081, P.R. China
E-mail: daoshuo@bit.edu.cn
daoshuo@hotmail.com

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