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ON SELFINJECTIVE ALGEBRAS OF TILTED TYPE

BY

ANDRZEJ SKOWROŃSKI (Toruń) and KUNIO YAMAGATA (Tokyo)

Dedicated to Piotr Dowbor on the occasion of his 60th birthday

Abstract. We provide a characterization of all finite-dimensional selfinjective algebras over a field K which are socle equivalent to a prominent class of selfinjective algebras of tilted type.

Introduction and the main results. Throughout the paper, by an *algebra* we mean a basic, indecomposable, finite-dimensional associative K-algebra with an identity over a (fixed) field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules, by D the standard duality $\operatorname{Hom}_K(-, K)$ on $\operatorname{mod} A$, and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules. Moreover, we denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^{-1} the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We do not distinguish between a module in $\operatorname{ind} A$ and the vertex of Γ_A corresponding to it. An algebra A is called *selfinjective* if A_A is an injective module, or equivalently, the projective modules in $\operatorname{mod} A$ are injective. For a selfinjective algebra A, we denote by Γ_A^s the stable Auslander–Reiten quiver of A, obtained from Γ_A by removing the projective modules and the arrows attached to them. If A is a selfinjective algebra, then the left socle of A and the right socle of A coincide, and we denote them by $\operatorname{soc} A$. Two selfinjective algebras A and Λ are said to be *socle equivalent* if the quotient algebras $A/\operatorname{soc} A$ and $\Lambda/\operatorname{soc} \Lambda$ are isomorphic. Moreover, two selfinjective algebras A and Λ are called *stably equivalent* if their stable module categories $\underline{\operatorname{mod}}\, A$ and $\underline{\operatorname{mod}}\, \Lambda$ are equivalent.

In the representation theory of selfinjective algebras an important role is played by the selfinjective algebras A which admit Galois coverings of the form $\widehat{B} \to \widehat{B}/G = A$, where $\widehat{B}$ is the repetitive category of an algebra B with acyclic Gabriel quiver and G is an admissible group of automorphisms of $\widehat{B}$. Namely, frequently interesting selfinjective algebras are socle equiv-

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alent to such orbit algebras $\widehat{B}/G$ and we may reduce their representation theory to that for the corresponding algebras of finite global dimension occurring in $\widehat{B}$. For example, for K algebraically closed, this is the case for selfinjective algebras of polynomial growth (see [34], [35]), the restricted enveloping algebras of restricted Lie algebras [11], or more generally the tame Hopf algebras of infinitesimal group schemes [12], in odd characteristic, as well as for special biserial algebras [25]. We also mention that for algebras B of finite global dimension the stable module category $\underline{\mathrm{mod}}\, \widehat{B}$ is equivalent (as a triangulated category) to the derived category $D^b(\mathrm{mod}\, B)$ of bounded complexes in $\mathrm{mod}\, B$ (see [14]).

Among the algebras of finite global dimension a prominent role is played by the tilted algebras of hereditary algebras, for which the representation theory is rather well understood (see [3], [7], [15], [19], [20], [21], [23], [26], [27], [29], [30], [31], [32], [33] for some basic results and characterizations). This made it possible to understand the representation theory of the orbit algebras $\widehat{B}/G$ of tilted algebras B (see [2], [4], [10], [16], [17], [18], [22], [35], [36], [38], [39], [43]), called *selfinjective algebras of tilted type*. In particular, it has been proved that every admissible group G of automorphisms of the repetitive category $\widehat{B}$ of a tilted algebra B is an infinite cyclic group generated by a strictly positive automorphism of $\widehat{B}$. It would be interesting to characterize the selfinjective algebras which are socle equivalent (respectively, stably equivalent) to selfinjective algebras of tilted type. In the series of papers [36], [37], [38], [40], [41], [42] we developed the theory of selfinjective algebras with deforming ideals and established necessary and sufficient conditions for a selfinjective algebra A to be socle equivalent to an orbit algebra $\widehat{B}/G$, for an algebra B and an infinite cyclic group G generated by a strictly positive automorphism of $\widehat{B}$ being the composition $\varphi\nu_{\widehat{B}}$ of the Nakayama automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ and a positive automorphism φ of $\widehat{B}$. The structure and stable equivalences of selfinjective algebras of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$, with B a tilted algebra and φ a positive automorphism of $\widehat{B}$, were investigated in [24], [37], [38], [40], [42]. We also refer to [5], [6] for some recent investigation of related selfinjective algebras of finite representation type.

The aim of this paper is to establish a characterization of the class of selfinjective algebras of tilted type by the existence of a double τ-rigid module. For an algebra A, a module M in $\mathrm{mod}\, A$ is called *τ_A-rigid* if $\mathrm{Hom}_A(M, \tau_A M) = 0$. It has been proved in [33] that the number of pairwise nonisomorphic indecomposable direct summands of a τ_A-rigid module M in $\mathrm{mod}\, A$ is less than or equal to the rank of the Grothendieck group $K_0(A)$ of A. We also refer to [1] for a theory of τ-rigid modules and its applications.

Let A be a selfinjective algebra. A full valued subquiver Δ of the Auslander–Reiten quiver Γ_A of A is said to be a *stable slice* if the following conditions are satisfied:

(1) Δ is connected, acyclic, and without projective modules.
(2) For any valued arrow $V \xrightarrow{(d,d')} U$ in Γ_A with U in Δ and V nonprojective, V belongs to Δ or to $\tau_A \Delta$.
(3) For any valued arrow $U \xrightarrow{(e,e')} V$ in Γ_A with U in Δ and V nonprojective, V belongs to Δ or to $\tau_A^{-1} \Delta$.

A stable slice Δ of Γ_A is said to be *regular* if Δ contains neither the socle factor $P/\operatorname{soc} P$ nor the radical $\operatorname{rad} P$ of an indecomposable projective module P in $\operatorname{mod} A$. Further, a stable slice Δ of Γ_A is said to be *semiregular* if Δ does not contain both the socle factor $Q/\operatorname{soc} Q$ of an indecomposable projective module Q and the radical $\operatorname{rad} P$ of an indecomposable projective module P in $\operatorname{mod} A$. Moreover, a stable slice Δ of Γ_A is said to be *double τ_A-rigid* if $\operatorname{Hom}_A(X, \tau_A Y) = 0$ and $\operatorname{Hom}_A(\tau_A^{-1} X, Y) = 0$ for all indecomposable modules X and Y from Δ. We note that Δ is then finite and hence the direct sum $M = M_\Delta$ of the indecomposable modules from Δ is a τ_A-rigid module, and $\tau_A^{-1} M$ is also a τ_A-rigid module. Moreover, if Δ is a stable slice in Γ_A, then Δ is a full valued subquiver of a connected component $\mathcal{C}$ of Γ_A^s intersecting every τ_A-orbit in $\mathcal{C}$ exactly once.

The following theorem is the main result of the paper.

THEOREM 1. *Let A be a basic, indecomposable, finite-dimensional selfinjective algebra over a field K. The following statements are equivalent:*

(i) *Γ_A admits a semiregular double τ_A-rigid stable slice.*
(ii) *A has one of the following forms:*
 (a) *A is isomorphic to the orbit algebra $\widehat{B}/(\varphi \nu_{\widehat{B}})$, where $B = \operatorname{End}_H(T)$ for a hereditary algebra H and a tilting module T in* $\operatorname{mod} H$ *either without nonzero projective direct summand or without nonzero injective direct summand, and φ is a strictly positive automorphism of $\widehat{B}$.*
 (b) *A is socle equivalent to the orbit algebra $\widehat{B}/(\varphi \nu_{\widehat{B}})$, where $B =$* $\operatorname{End}_H(T)$ *for a hereditary algebra H and a tilting module T in* $\operatorname{mod} H$ *without nonzero projective or injective direct summands, and φ is a rigid automorphism of $\widehat{B}$.*

Moreover, if K is an algebraically closed field, then we may replace in (ii)(b) *"socle equivalent" by "isomorphic".*

We would like to stress that in general we cannot replace in (ii) "socle equivalent" by "isomorphic" without assuming that φ is strictly positive (see [39, Proposition 4]).

It follows from the results in [2], [4], [10] (see also [38], [39]) that the repetitive category $\widehat{B}$ of a tilted algebra B not of Dynkin type is isomorphic to the repetitive category $\widehat{B}^*$ of a tilted algebra $B^* = \mathrm{End}_{H^*}(T^*)$, where H^* is a hereditary algebra not of Dynkin type and T^* is a tilting module in $\mathrm{mod}\, H^*$ without nonzero projective or injective direct summands.

Then we obtain the following consequence of Theorem 1.

Theorem 2. *Let A be a basic, indecomposable, finite-dimensional selfinjective algebra of infinite representation type over a field K. The following statements are equivalent:*

(i) *Γ_A admits a regular double τ_A-rigid stable slice.*
(ii) *A is socle equivalent to the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a tilted algebra not of Dynkin type and φ is a positive automorphism of $\widehat{B}$.*

Moreover, if K is an algebraically closed field, we may replace in (ii) *"socle equivalent" by "isomorphic".*

We will present in Section 4 examples of tilted algebras B of Dynkin type for which every section in Γ_B contains either an indecomposable projective or an indecomposable injective module, and even an indecomposable projective-injective module. It would be interesting to describe all tilted algebras of Dynkin type with these properties. In particular, we conclude that there are trivial extension algebras $\mathrm{T}(B) = \widehat{B}/(\nu_{\widehat{B}})$ of tilted algebras B of Dynkin type for which the Auslander–Reiten quiver $\Gamma_{\mathrm{T}(B)}$ does not admit a semiregular double $\tau_{\mathrm{T}(B)}$-rigid stable slice. Moreover, we will show that there are r-fold trivial extension algebras $\mathrm{T}(B)^{(r)}$ of tilted algebras B of Dynkin type, with $r \geq 2$, for which the Auslander–Reiten quiver $\Gamma_{\tau_{\mathrm{T}(B)^{(r)}}}$ admits a semiregular but nonregular double $\tau_{\mathrm{T}(B)^{(r)}}$-rigid stable slice. We also mention that all selfinjective orbit algebras $A = \widehat{B}/G$ of tilted algebras B of Dynkin type and admissible infinite cyclic automorphism groups G of $\widehat{B}$ having a maximal almost split sequence in $\mathrm{mod}\, A$ do have a regular double τ_A-rigid stable slice in Γ_A (see [6, Theorem 5.2]).

The paper is organized as follows. In Section 1 we recall the background on orbit algebras of repetitive categories of algebras. Section 2 is devoted to presenting the theory of selfinjective algebras with deforming ideals, playing a prominent role in the proof of our main result. In Section 3 we prove Theorem 1. In Section 4 we present some examples illustrating Theorem 1.

For basic background on the relevant representation theory we refer to [3], [29], [30], [43], [44].

1. Orbit algebras of repetitive categories. Let B be an algebra and $1_B = e_1 + \cdots + e_n$ a decomposition of the identity of B into a sum of pairwise orthogonal primitive idempotents. We associate to B a selfinjective locally bounded K-category $\widehat{B}$, called the *repetitive category* of B (see [17]). The objects of $\widehat{B}$ are $e_{m,i}$, $m \in \mathbb{Z}$, $i \in \{1,\dots,n\}$, and the morphism spaces are defined as follows:

$$\widehat{B}(e_{m,i}, e_{r,j}) = \begin{cases} e_jBe_i, & r = m, \\ D(e_iBe_j), & r = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $e_jBe_i = \mathrm{Hom}_B(e_iB, e_jB)$, $D(e_iBe_j) = e_jD(B)e_i$ and

$$\bigoplus_{(m,i)\in\mathbb{Z}\times\{1,\dots,n\}} \widehat{B}(e_{m,i}, e_{r,j}) = e_jB \oplus D(Be_j),$$

for any $r \in \mathbb{Z}$ and $j \in \{1,\dots,n\}$. We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ defined by

$$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i} \quad \text{for all } (m,i) \in \mathbb{Z}\times\{1,\dots,n\}.$$

An automorphism φ of the K-category $\widehat{B}$ is said to be:

- *positive* if, for each pair $(m,i) \in \mathbb{Z}\times\{1,\dots,n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and some $j \in \{1,\dots,n\}$;
- *rigid* if, for each pair $(m,i) \in \mathbb{Z}\times\{1,\dots,n\}$, there exists $j \in \{1,\dots,n\}$ such that $\varphi(e_{m,i}) = e_{m,j}$;
- *strictly positive* if it is positive but not rigid.

The automorphisms $\nu_{\widehat{B}}^r$, $r \geq 1$, are strictly positive automorphisms of $\widehat{B}$.

A group G of automorphisms of $\widehat{B}$ is said to be *admissible* if G acts freely on the set of objects of $\widehat{B}$ and has finitely many orbits. Following P. Gabriel [13], we may then consider the orbit category $\widehat{B}/G$ of $\widehat{B}$ with respect to G whose objects are the G-orbits of objects in $\widehat{B}$, and the morphism spaces are given by

$$(\widehat{B}/G)(a,b) = \Big\{ (f_{y,x}) \in \prod_{(x,y)\in a\times b} \widehat{B}(x,y) \;\Big|\; gf_{y,x} = f_{gy,gx}, \forall_{g\in G,(x,y)\in a\times b} \Big\}$$

for all objects a, b of $\widehat{B}/G$. Since $\widehat{B}/G$ has finitely many objects and the morphism spaces in $\widehat{B}/G$ are finite-dimensional, we have the associated finite-dimensional selfinjective K-algebra $\bigoplus(\widehat{B}/G)$ which is the direct sum of all morphism spaces in $\widehat{B}/G$, called the *orbit algebra* of $\widehat{B}$ with respect to G. We will identify $\widehat{B}/G$ with $\bigoplus(\widehat{B}/G)$. For example, for each positive integer r, the infinite cyclic group $(\nu_{\widehat{B}}^r)$ generated by the rth power $\nu_{\widehat{B}}^r$ of $\nu_{\widehat{B}}$ is an admissible group of automorphisms of $\widehat{B}$, and we have the associated selfinjective orbit algebra

$$T(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^r) = \left\{ \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ f_2 & b_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & f_3 & b_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & f_{r-1} & b_{r-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & f_1 & b_1 \end{bmatrix} \\ b_1, \dots, b_{r-1} \in B,\ f_1, \dots, f_{r-1} \in D(B) \right\},$$

called the *r-fold trivial extension algebra* of B. In particular, $T(B)^{(1)} \cong T(B) = B \ltimes D(B)$ is the *trivial extension algebra* of B by the injective cogenerator $D(B)$.

Let B be an algebra. By a *finite-dimensional* $\widehat{B}$*-module* we mean a contravariant functor M from $\widehat{B}$ to the category of K-vector spaces such that $\sum_{x \in \mathrm{ob}\,\widehat{B}} \dim_K M(x)$ is finite. We denote by $\mathrm{mod}\,\widehat{B}$ the category of all finite-dimensional $\widehat{B}$-modules. For a module M in $\mathrm{mod}\,\widehat{B}$, we denote by $\mathrm{supp}(M)$ the full subcategory of $\widehat{B}$ formed by all objects x with $M(x) \neq 0$, and call it the *support* of M. Following [8], the category $\widehat{B}$ is said to be *locally support-finite* if for any object x of $\widehat{B}$ the full subcategory $\widehat{B}_x$ of $\widehat{B}$ formed by the supports of all indecomposable modules M in $\mathrm{mod}\,\widehat{B}$ with $M(x) \neq 0$ is finite. We also recall that for a group G of automorphisms of $\widehat{B}$ we have the induced action of G on $\mathrm{mod}\,\widehat{B}$ given by ${}^gM = M \circ g^{-1}$ for any module M in $\mathrm{mod}\,\widehat{B}$ and element g of G. Then we denote by $F_\lambda : \mathrm{mod}\,\widehat{B} \to \mathrm{mod}\,\widehat{B}/G$ the *push-down functor* associated to the Galois covering $F : \widehat{B} \to \widehat{B}/G$ (see [13]).

The following theorem is a consequence of results established in [2], [4], [10], [16], [17].

THEOREM 1.1. *Let B be a tilted algebra. Then $\widehat{B}$ is locally support finite.*

Then we obtain the following consequence of [8, Theorem] (or [9, Proposition 2.5]) (the density part) and [13, Theorem 3.6].

THEOREM 1.2. *Let B be a tilted algebra, G an admissible infinite cyclic group of automorphisms of $\widehat{B}$ and $A = \widehat{B}/G$ the associated orbit algebra. Then:*

(i) *The push-down functor $F_\lambda : \mathrm{mod}\,\widehat{B} \to \mathrm{mod}\,A$ associated to the Galois covering $F : \widehat{B} \to \widehat{B}/G = A$ is dense and preserves indecomposable modules and almost split sequences.*

(ii) *The Auslander–Reiten quiver Γ_A is the orbit quiver $\Gamma_{\widehat{B}}/G$ with respect to the induced action of G on the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$.*

2. Selfinjective algebras with deforming ideals. In this section we present criteria for selfinjective algebras to be socle equivalent to orbit algebras of the repetitive categories of algebras with respect to infinite cyclic automorphism groups, playing a fundamental role in our proof of Theorem 1.

Let A be a selfinjective algebra. For a subset X of A, we may consider its left annihilator $l_A(X) = \{a \in A \mid aX = 0\}$ and right annihilator $r_A(X) = \{a \in A \mid Xa = 0\}$. Then by a theorem due to T. Nakayama (see [44, Theorem IV.6.10]) the annihilator operation l_A induces a Galois correspondence from the lattice of right ideals of A to the lattice of left ideals of A, and r_A is the inverse Galois correspondence to l_A. Let I be an ideal of A, $B = A/I$, and e an idempotent of A such that $e + I$ is the identity of B. We may assume that $1_A = e_1 + \cdots + e_r$ with $e_1, \ldots, e_r$ pairwise orthogonal primitive idempotents of A, $e = e_1 + \cdots + e_n$ for some $n \leq r$, and $\{e_i \mid 1 \leq i \leq n\}$ is the set of all idempotents in $\{e_i \mid 1 \leq i \leq r\}$ which are not in I. Such an idempotent e is uniquely determined by I up to an inner automorphism of A, and is called a *residual identity* of $B = A/I$. Observe also that $B \cong eAe/eIe$.

We have the following lemma from [41, Lemma 5.1].

LEMMA 2.1. *Let A be a selfinjective algebra, I an ideal of A, and e an idempotent of A such that $l_A(I) = Ie$ or $r_A(I) = eI$. Then e is a residual identity of A/I.*

We also recall the following proposition proved in [36, Proposition 2.3].

PROPOSITION 2.2. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$, e a residual identity of B, and assume that $IeI = 0$. The following conditions are equivalent:*

(i) *Ie is an injective cogenerator in* $\operatorname{mod} B$.
(ii) *eI is an injective cogenerator in* $\operatorname{mod} B^{\mathrm{op}}$.
(iii) *$l_A(I) = Ie$.*
(iv) *$r_A(I) = eI$.*

Moreover, under these equivalent conditions, we have $\operatorname{soc} A \subseteq I$ *and* $l_{eAe}(I) = eIe = r_{eAe}(I)$.

The following theorem, proved in [38, Theorem 3.8] (sufficiency part) and [41, Theorem 5.3] (necessity part), will be fundamental for our considerations.

THEOREM 2.3. *Let A be a selfinjective algebra. The following conditions are equivalent:*

(i) *A is isomorphic to the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is an algebra and φ is a positive automorphism of $\widehat{B}$.*
(ii) *There is an ideal I of A and an idempotent e of A such that*
   (1) *$r_A(I) = eI$;*
   (2) *the canonical algebra epimorphism $eAe \to eAe/eIe$ is a retraction.*

Moreover, in this case, B is isomorphic to A/I.

Let A be a selfinjective algebra, I an ideal of A, and e a residual identity of A/I. Following [36], I is said to be a *deforming ideal* of A if:

(D1) $l_{eAe}(I) = eIe = r_{eAe}(I)$;
(D2) the valued quiver $Q_{A/I}$ of A/I is acyclic.

Assume I is a deforming ideal of A. Then we have a canonical isomorphism of algebras $eAe/eIe \to A/I$ and I can be considered as an (eAe/eIe)-(eAe/eIe)-bimodule. Denote by $A[I]$ the direct sum of K-vector spaces $(eAe/eIe) \oplus I$ with the multiplication

$$(b, x) \cdot (c, y) = (bc, by + xc + xy)$$

for $b, c \in eAe/eIe$ and $x, y \in I$. Then $A[I]$ is a K-algebra with the identity $(e + eIe, 1_A - e)$, and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider I to be ideal of $A[I]$. Observe that $e = (e + eIe, 0)$ is a residual identity of $A[I]/I = eAe/eIe \cong A/I$, $eA[I]e = (eAe/eIe) \oplus eIe$, and the canonical algebra epimorphism $eA[I]e \to eA[I]e/eIe$ is a retraction.

The following properties of the algebra $A[I]$ were established in [36, Theorem 4.1], [37, Theorem 3] and [42, Lemma 3.1].

THEOREM 2.4. *Let A be a selfinjective algebra and I a deforming ideal of A. Then:*

(i) *$A[I]$ is a selfinjective algebra with the same Nakayama permutation as A and I is a deforming ideal of $A[I]$.*
(ii) *A and $A[I]$ are socle equivalent.*
(iii) *A and $A[I]$ are stably equivalent.*
(iv) *$A[I]$ is a symmetric algebra if A is a symmetric algebra.*

We note that if A is a selfinjective algebra, I an ideal of A, $B = A/I$, e an idempotent of A such that $r_A(I) = eI$, and the valued quiver Q_B of B is acyclic, then by Lemma 2.1 and Proposition 2.2, I is a deforming ideal of A and e is a residual identity of B.

The following theorem proved in [38, Theorem 4.1] shows the importance of the algebras $A[I]$.

THEOREM 2.5. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$ and e an idempotent of A. Assume that $r_A(I) = eI$ and Q_B is acyclic. Then*

$A[I]$ is isomorphic to the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.

We point out that there are selfinjective algebras A with deforming ideals I such that the algebras A and $A[I]$ are not isomorphic (see [38, Example 4.2]), and A is not isomorphic to the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is an algebra and φ is a positive automorphism of $\widehat{B}$ (see [39, Proposition 4]).

The following result proved in [40, Proposition 3.2] describes a situation when the algebras A and $A[I]$ are isomorphic.

THEOREM 2.6. *Let A be a selfinjective algebra with a deforming ideal I, $B = A/I$, e be a residual identity of B, and ν the Nakayama permutation of A. Assume that $IeI = 0$ and $e_i \neq e_{\nu(i)}$ for any primitive summand e_i of e. Then the algebras A and $A[I]$ are isomorphic. In particular, A is isomorphic to the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$.*

Moreover, we have the following consequence of [36, Theorem 3.2].

THEOREM 2.7. *Let A be a selfinjective algebra over an algebraically closed field K and I a deforming ideal of A. Then the algebras A and $A[I]$ are isomorphic.*

3. Proof of Theorem 1. We first prove that (ii) implies (i).

Let B be the tilted algebra $\mathrm{End}_H(T)$, where H is a hereditary algebra and T is a tilting module in $\mathrm{mod}\, H$. Recall that $\mathrm{Ext}^1_H(T,T) = 0$ and T is a direct sum of n pairwise nonisomorphic indecomposable modules in $\mathrm{mod}\, H$, where n is the rank of the Grothendieck group $K_0(H)$ of H (see [7], [15]). Let $I_1, \ldots, I_n$ be a complete family of pairwise nonisomorphic indecomposable injective modules in $\mathrm{mod}\, H$. Then, by general theory, the images $\mathrm{Hom}_H(T, I_1), \ldots, \mathrm{Hom}_H(T, I_n)$ of these modules via the functor $\mathrm{Hom}_H(T, -) : \mathrm{mod}\, H \to \mathrm{mod}\, B$ form a complete section Δ_T of a connected component $\mathcal{C}_T$ of Γ_B, called the *connecting component* of Γ_B *determined by* T, which connects the torsion-free part $\mathcal{Y}(T) = \{Y \in \mathrm{mod}\, B \mid \mathrm{Tor}^B_1(Y,T) = 0\}$ to the torsion part $\mathcal{X}(T) = \{X \in \mathrm{mod}\, B \mid X \otimes_B T = 0\}$ of $\mathrm{mod}\, B$ (see [3], [15]). Moreover, Δ_T is isomorphic to the opposite quiver Q_H^{op} of Q_H, and hence Δ_T is a connected acyclic valued quiver. Recall also that the section Δ_T is a convex subquiver of $\mathcal{C}_T$ intersecting every τ_B-orbit of $\mathcal{C}_T$ exactly once. Since H is a hereditary algebra, the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\mathrm{mod}\, B$ is splitting, that is, every indecomposable module in $\mathrm{mod}\, B$ belongs to $\mathcal{X}(T)$ or to $\mathcal{Y}(T)$.

PROPOSITION 3.1. *Let $\Lambda = \widehat{B}/(\varphi\nu_{\widehat{B}})$, where $B = \mathrm{End}_H(T)$ for a hereditary algebra H and a tilting module T in $\mathrm{mod}\, H$, and φ is a positive automorphism of $\widehat{B}$. Moreover, let $F_\lambda : \mathrm{mod}\, \widehat{B} \to \mathrm{mod}\, \Lambda$ be the push-down* *functor associated to the Galois covering* $F : \widehat{B} \to \widehat{B}/(\varphi\nu_{\widehat{B}}) = \Lambda$. *Then:*

(i) $F_\lambda(\Delta_T)$ *is a stable slice of* Γ_Λ.

(ii) $F_\lambda(\Delta_T)$ *contains the radical* $\operatorname{rad} P$ *of an indecomposable projective module* P *in* $\operatorname{mod} \Lambda$ *if and only if* T *admits an indecomposable projective direct summand in* $\operatorname{mod} H$.

(iii) $F_\lambda(\Delta_T)$ *contains the socle factor* $Q/\operatorname{soc} Q$ *of an indecomposable projective module* Q *in* $\operatorname{mod} \Lambda$ *if and only if* T *admits an indecomposable injective direct summand in* $\operatorname{mod} H$.

Proof. (i) It follows from the results in [2], [10], [16], [17] that there exists a connected acyclic component $\mathcal{C}$ of $\Gamma_{\widehat{B}}$ such that Δ_T is a connected, convex, full valued subquiver of $\mathcal{C}$ which intersects every $\tau_{\widehat{B}}$-orbit of the stable part $\mathcal{C}^s$ of $\mathcal{C}$ exactly once. Since the push-down functor F_λ induces an isomorphism of translation quivers $\Gamma_{\widehat{B}}/G \to \Gamma_\Lambda$, we conclude that $F_\lambda(\Delta_T)$ is a connected, full valued subquiver of the connected component $F_\lambda(\mathcal{C})$ of Γ_Λ intersecting every τ_Λ-orbit of the stable part $F_\lambda(\mathcal{C})^s$ of $F_\lambda(\mathcal{C})$ exactly once. In particular, $F_\lambda(\Delta_T)$ is a stable slice of Γ_Λ. Moreover the valued quivers Δ_T and $F_\lambda(\Delta_T)$ are isomorphic, because Λ is the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ with φ a positive automorphism of $\widehat{B}$.

(ii) Observe that $F_\lambda(\Delta_T)$ contains the radical $\operatorname{rad} P$ of an indecomposable projective module P in $\operatorname{mod} \Lambda$ if and only if Δ_T contains $\operatorname{rad} P^*$ for an indecomposable projective module P^* in $\operatorname{mod} \widehat{B}$ such that $P = F_\lambda(P^*)$. Further, by the results in [2], [10], [16], [17], this is equivalent to the fact that Δ_T contains an injective module R from $\operatorname{mod} B$ which has no proper injective predecessor on Δ_T (and then $R = \operatorname{rad} P^*$ for an indecomposable projective module P^* in $\operatorname{mod} \widehat{B}$). Since Δ_T is a finite acyclic quiver, this is equivalent to the fact that Δ_T contains an indecomposable injective module from $\operatorname{mod} B$. Finally, it follows from the connecting lemma [3, Lemma VI.4.9] (see also [3, Proposition VI.5.8]) that, for an indecomposable injective module I in $\operatorname{mod} H$, the right B-module $\operatorname{Hom}_H(T, I)$ is injective in $\operatorname{mod} B$ if and only if the indecomposable projective module P_I in $\operatorname{mod} H$ with $\operatorname{top} P_I = \operatorname{soc} I$ is a direct summand of T. This completes the proof of (ii).

(iii) Observe that $F_\lambda(\Delta_T)$ contains the socle factor $Q/\operatorname{soc} Q$ of an indecomposable projective module Q in $\operatorname{mod} \Lambda$ if and only if Δ_T contains $Q^*/\operatorname{soc} Q^*$ for an indecomposable projective module Q^* in $\operatorname{mod} \widehat{B}$ such that $Q = F_\lambda(Q^*)$. Further, by the results of [2], [10], [16], [17], this is equivalent to the fact that Δ_T contains a projective module R from $\operatorname{mod} B$ which has no proper projective successor on Δ_T (and then $R = Q^*/\operatorname{soc} Q^*$ for an indecomposable projective module Q^* in $\operatorname{mod} \widehat{B}$). Since Δ_T is a finite acyclic quiver, this in turn is equivalent to the fact that Δ_T contains an indecomposable projective module from $\operatorname{mod} B$. Finally, for an indecomposable injective module I in $\operatorname{mod} H$, the right B-module $\operatorname{Hom}_H(T, I)$ is projective in $\operatorname{mod} B$

if and only if I is a direct summand of T (see [3, Lemma VI.3.1]). This completes the proof of (iii). ∎

PROPOSITION 3.2. *Let Λ be an orbit algebra of one of the forms:*

(a) $\widehat{B}/(\varphi\nu_{\widehat{B}})$, *where $B = \mathrm{End}_H(T)$ for a hereditary algebra H and a tilting module T in* $\mathrm{mod}\, H$, *and φ is a strictly positive automorphism of $\widehat{B}$.*

(b) $\widehat{B}/(\varphi\nu_{\widehat{B}})$, *where $B = \mathrm{End}_H(T)$ for a hereditary algebra H and a tilting module T in* $\mathrm{mod}\, H$ *without nonzero projective or injective direct summands, and φ is a rigid automorphism of $\widehat{B}$.*

Then the push-down $F_\lambda(\Delta_T)$ of the section Δ_T of the connecting component $\mathcal{C}_T$ of Γ_B determined by T via the push-down functor $F_\lambda : \mathrm{mod}\, \widehat{B} \to \mathrm{mod}\, \Lambda$ associated to the Galois covering $F : \widehat{B} \to \widehat{B}/(\varphi\nu_{\widehat{B}}) = \Lambda$ is a double τ_Λ-rigid stable slice of Γ_Λ.

Proof. We abbreviate $g = \varphi\nu_{\widehat{B}}$ and $G = (g)$. Consider the canonical Galois covering functor $F : \widehat{B} \to \widehat{B}/G = \Lambda$ and the associated push-down functor $F_\lambda : \mathrm{mod}\, \widehat{B} \to \mathrm{mod}\, \Lambda$. Then, applying Theorems 1.1 and 1.2, we conclude that F_λ is a dense functor, preserves indecomposable modules and almost split sequences, and the Auslander–Reiten quiver Γ_Λ is the orbit quiver $\Gamma_{\widehat{B}}/G$ with respect to the induced action of G on $\Gamma_{\widehat{B}}$. Moreover, for any indecomposable modules X and Y in $\mathrm{mod}\, \widehat{B}$, the functor F_λ induces isomorphisms of K-vector spaces

$$\bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\widehat{B}}(X, {}^{g^r}Y) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(F_\lambda(X), F_\lambda(Y)),$$
$$\bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\widehat{B}}({}^{g^r}X, Y) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(F_\lambda(X), F_\lambda(Y)).$$

Let $e_1, \dots, e_n$ be a set of pairwise orthogonal primitive idempotents of B whose sum is the identity of B. Then $\widehat{B}$ is the category with the objects $e_{m,i}$, $m \in \mathbb{Z}$, $i \in \{1, \dots, n\}$. We identify the algebra B with the full subcategory of $\widehat{B}$ given by the objects $e_{0,i}$, $i \in \{1, \dots, n\}$. It follows from the results in [2], [4], [10], [16], [17] that there exists a connected acyclic component $\mathcal{C}$ in $\Gamma_{\widehat{B}}$ such that:

- Δ_T is a connected, convex, full valued subquiver of $\mathcal{C}$ and intersects every $\tau_{\widehat{B}}$-orbit of the stable part of $\mathcal{C}^s$ of $\mathcal{C}$ exactly once.
- $\mathcal{C}$ is a generalized standard component of $\Gamma_{\widehat{B}}$, that is, $\mathrm{rad}^\infty_{\widehat{B}}(X, Y) = 0$ for all X and Y in $\mathcal{C}$ (see [32]).
- $\Gamma_{\widehat{B}}$ has a disjoint decomposition $\Gamma_{\widehat{B}} = \mathcal{P} \vee \mathcal{C} \vee \mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ are families of connected components of $\Gamma_{\widehat{B}}$ such that $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}, \mathcal{P}) = 0$, $\mathrm{Hom}_{\widehat{B}}(\mathcal{Q}, \mathcal{C}) = 0$, and $\mathrm{Hom}_{\widehat{B}}(\mathcal{Q}, \mathcal{P}) = 0$.

We note that $\mathcal{C} = \Gamma_{\widehat{B}}$, hence $\mathcal{P}$ and $\mathcal{Q}$ are empty, if B is a tilted algebra of Dynkin type.

It follows from Proposition 3.1 that the push-down functor $F_\lambda(\Delta_T)$ of Δ_T is a stable slice of Γ_Λ. We claim that $F_\lambda(\Delta_T)$ is a double τ_Λ-rigid stable slice of Γ_Λ. Denote by M_T the direct sum of all indecomposable modules in $\operatorname{mod}\widehat{B}$ lying on Δ_T. Then $F_\lambda(M_T)$ is the direct sum of all indecomposable modules in $\operatorname{mod}\Lambda$ lying on $F_\lambda(\Delta_T)$. We will show that

$$\operatorname{Hom}_\Lambda\big(F_\lambda(M_T), \tau_\Lambda F_\lambda(M_T)\big) = 0, \qquad \operatorname{Hom}_\Lambda\big(\tau_\Lambda^{-1} F_\lambda(M_T), F_\lambda(M_T)\big) = 0.$$

We know that $\tau_\Lambda F_\lambda(M_T) = F_\lambda(\tau_{\widehat{B}} M_T)$ and $\tau_\Lambda^{-1} F_\lambda(M_T) = F_\lambda(\tau_{\widehat{B}}^{-1} M_T)$. Moreover, since F_λ is a Galois covering of module categories, it induces isomorphisms of K-vector spaces

$$\bigoplus_{r\in\mathbb{Z}} \operatorname{Hom}_{\widehat{B}}({}^{g^r}M_T, \tau_{\widehat{B}} M_T) \xrightarrow{\sim} \operatorname{Hom}_\Lambda\big(F_\lambda(M_T), F_\lambda(\tau_{\widehat{B}} M_T)\big),$$
$$\bigoplus_{r\in\mathbb{Z}} \operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^{-1} M_T, {}^{g^r}M_T) \xrightarrow{\sim} \operatorname{Hom}_\Lambda\big(F_\lambda(\tau_{\widehat{B}}^{-1} M_T), F_\lambda(M_T)\big).$$

Observe that $\operatorname{Hom}_{\widehat{B}}(M_T, \tau_{\widehat{B}} M_T) = 0$ and $\operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^{-1} M_T, M_T) = 0$, because Δ_T is contained in the generalized standard acyclic component $\mathcal{C}$ of $\Gamma_{\widehat{B}}$ and is a stable slice of $\Gamma_{\widehat{B}}$. We claim that $\operatorname{Hom}_{\widehat{B}}({}^{g^r}M_T, \tau_{\widehat{B}} M_T) = 0$ and $\operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^{-1} M_T, {}^{g^r}M_T) = 0$ for any $r \in \mathbb{Z}\setminus\{0\}$. We have two cases to consider.

Assume first that Λ is of the form (a), so φ is a strictly positive automorphism of $\widehat{B}$. Then it follows from [2], [4], [10], [16], [17] that for any $r \in \mathbb{Z}\setminus\{0\}$ the supports of ${}^{g^r}M_T$ and $\tau_{\widehat{B}} M_T$ (respectively, ${}^{g^r}M_T$ and $\tau_{\widehat{B}}^{-1} M_T$) have no common objects, and hence the claim follows.

Assume now that Λ is of the form (b). Then it follows from general theory (see [3, Lemma VI.3.1, Proposition VI.5.8] that the section Δ_T of $\mathcal{C}_T$ does not contain an indecomposable projective or indecomposable injective module. Applying again the results of [2], [10], [16], [17], we conclude that $\tau_{\widehat{B}} M_T = \tau_B M_T$ and $\tau_{\widehat{B}}^{-1} M_T = \tau_B^{-1} M_T$, so $\tau_{\widehat{B}} M_T$ and $\tau_{\widehat{B}}^{-1} M_T$ have supports contained in B. On the other hand, for $g = \varphi\nu_{\widehat{B}}$ with φ a rigid automorphism of $\widehat{B}$, the support of ${}^{g^r}M_T$ is the Nakayama shift $\nu_{\widehat{B}}^r(B)$ of the support B of M_T. Then, for $r \in \mathbb{Z}\setminus\{0\}$, the supports of ${}^{g^r}M_T$ and $\tau_{\widehat{B}} M_T = \tau_B M_T$ (respectively, ${}^{g^r}M_T$ and $\tau_{\widehat{B}}^{-1} M_T = \tau_B^{-1} M_T$) have no common objects, and hence the claim follows.

Summing up, we obtain the equalities $\operatorname{Hom}_\Lambda(F_\lambda(M_T), \tau_\Lambda F_\lambda(M_T)) = 0$ and $\operatorname{Hom}_\Lambda(\tau_\Lambda^{-1} F_\lambda(M_T), F_\lambda(M_T)) = 0$. Therefore, $F_\lambda(\Delta_T)$ is a double τ_Λ-rigid stable slice of Γ_Λ. ■

The following lemma completes the proof of the implication (ii)⇒(i) in Theorem 1.

Lemma 3.3. *Let A and Λ be socle equivalent selfinjective algebras, and assume that Γ_Λ admits a double τ_Λ-rigid stable slice Δ. Then Γ_A admits a double τ_A-rigid stable slice Δ^*. Moreover, if Δ is regular* (*respectively, semiregular*) *then Δ^* is regular* (*respectively, semiregular*).

Proof. Let $\varphi : \Lambda/\operatorname{soc}\Lambda \to A/\operatorname{soc} A$ be an isomorphism of algebras. Then φ induces an isomorphism of module categories $\phi : \operatorname{mod}(\Lambda/\operatorname{soc}\Lambda) \to \operatorname{mod}(A/\operatorname{soc} A)$. Clearly, ϕ induces an isomorphism of Auslander–Reiten quivers $\Gamma_{\Lambda/\operatorname{soc}\Lambda} \to \Gamma_{A/\operatorname{soc} A}$. Let M_Δ be the direct sum of all indecomposable modules in $\operatorname{mod}\Lambda$ lying on Δ. Since Δ contains no projective module, we conclude that M_Δ is a module in $\operatorname{mod}(\Lambda/\operatorname{soc}\Lambda)$. Thus we may consider the module $\phi(M_\Delta)$ in $\operatorname{mod}(A/\operatorname{soc} A)$, and hence in $\operatorname{mod} A$. Observe that $\phi(M_\Delta)$ is the direct sum of all indecomposable modules in $\operatorname{mod} A$ lying on the valued quiver $\Delta^* = \phi(\Delta)$. Moreover, Δ^* is a stable slice of Γ_A, because ϕ induces an isomorphism $\Gamma_{\Lambda/\operatorname{soc}\Lambda} \xrightarrow{\sim} \Gamma_{A/\operatorname{soc} A}$ of translation quivers. In particular,

$$\begin{aligned}
\tau_A\phi(M_\Delta) &= \tau_{A/\operatorname{soc} A}\phi(M_\Delta) = \phi(\tau_{\Lambda/\operatorname{soc}\Lambda}M_\Delta) = \phi(\tau_\Lambda M_\Delta),\\
\tau_A^{-1}\phi(M_\Delta) &= \tau^{-1}_{A/\operatorname{soc} A}\phi(M_\Delta) = \phi(\tau^{-1}_{\Lambda/\operatorname{soc}\Lambda}M_\Delta) = \phi(\tau_\Lambda^{-1}M_\Delta).
\end{aligned}$$

Hence, we obtain isomorphisms of K-vector spaces

$$\begin{aligned}
\operatorname{Hom}_A(\phi(M_\Delta), \tau_A\phi(M_\Delta)) &= \operatorname{Hom}_{A/\operatorname{soc} A}(\phi(M_\Delta), \tau_{A/\operatorname{soc} A}\phi(M_\Delta))\\
&\cong \operatorname{Hom}_{\Lambda/\operatorname{soc}\Lambda}(M_\Delta, \tau_{\Lambda/\operatorname{soc}\Lambda}M_\Delta)\\
&= \operatorname{Hom}_\Lambda(M_\Delta, \tau_\Lambda M_\Delta) = 0,\\
\operatorname{Hom}_A(\tau_A^{-1}\phi(M_\Delta), \phi(M_\Delta)) &= \operatorname{Hom}_{A/\operatorname{soc} A}(\tau^{-1}_{A/\operatorname{soc} A}\phi(M_\Delta), \phi(M_\Delta))\\
&\cong \operatorname{Hom}_{\Lambda/\operatorname{soc}\Lambda}(\tau^{-1}_{\Lambda/\operatorname{soc}\Lambda}M_\Delta, M_\Delta)\\
&= \operatorname{Hom}_\Lambda(\tau_\Lambda^{-1}M_\Delta, M_\Delta) = 0.
\end{aligned}$$

This shows that $\Delta^* = \phi(M_\Delta)$ is a double τ_A-rigid stable slice in Γ_A. We also note that for an indecomposable projective module P in $\operatorname{mod} A$ there is an indecomposable projective module P^* in $\operatorname{mod}\Lambda$ such that $\phi(P/\operatorname{soc} P) = P^*/\operatorname{soc} P^*$ and $\phi(\operatorname{rad} P) = \operatorname{rad} P^*$. Hence, the remaining statements follow. ∎

We will prove now that (i) implies (ii) in Theorem 1.

Let A be a basic, indecomposable, finite-dimensional selfinjective algebra over a field K. Assume that Γ_A admits a semiregular double τ_A-rigid stable slice Δ. Let M be the direct sum of all indecomposable modules in $\operatorname{mod} A$ lying on Δ, $I = r_A(M)$, and $B = A/I$.

Lemma 3.4. *The following statements hold:*

(i) *Let P be an indecomposable projective module in* $\operatorname{mod} A$ *which is a direct predecessor of a module from Δ in Γ_A. Then* $\operatorname{Hom}_A(M, P) = 0$, *and hence the socle of P is not a simple right B-module.*

(ii) *Let P be an indecomposable projective module in* $\operatorname{mod} A$ *which is a direct successor of a module from Δ in Γ_A. Then* $\operatorname{Hom}_A(P, M) = 0$, *and hence the top of P is not a simple right B-module.*

Proof. (i) Suppose that $\operatorname{Hom}_A(M, P) \neq 0$. Since Δ does not contain a projective module, we infer that $\operatorname{Hom}_A(M, \operatorname{rad} P) \neq 0$. On the other hand, $P/\operatorname{soc} P$ is a unique direct successor of the projective module P in Γ_A, so $P/\operatorname{soc} P$ belongs to Δ. But then $\operatorname{rad} P = \tau_A(P/\operatorname{soc} P)$ is a direct summand of $\tau_A M$. Therefore $\operatorname{Hom}_A(M, \tau_A M) \neq 0$, contrary to assumption.

The proof of (ii) is similar. ∎

We have the following known facts (see [3, Lemma VIII.5.2] and its dual).

LEMMA 3.5. *The following the statements hold:*

(i) *$\tau_B M$ is the largest right B-submodule of $\tau_A M$.*
(ii) *$\tau_B^{-1} M$ is the largest quotient right B-module of $\tau_A^{-1} M$.*

Then we have following direct consequence of the double τ_A-rigidity of the stable slice Δ.

COROLLARY 3.6. $\operatorname{Hom}_B(M, \tau_B M) = 0$ *and* $\operatorname{Hom}_B(\tau_B^{-1} M, M) = 0$.

The following lemma will be essential for further considerations.

LEMMA 3.7. *Let X be an indecomposable module lying on Δ and Y an indecomposable module in* $\operatorname{mod} B$ *not lying on Δ. Then:*

(i) *Every homomorphism from Y to X in* $\operatorname{mod} B$ *factors through the module $(\tau_B M)^s$ for some positive integer s.*
(ii) *Every homomorphism from X to Y in* $\operatorname{mod} B$ *factors through the module $(\tau_B^{-1} M)^t$ for some positive integer t.*

Proof. (i) Let $f : Y \to X$ be a nonzero homomorphism in $\operatorname{mod} B$. It follows from Lemma 3.4(i) that Y is not isomorphic to the radical of an indecomposable projective module P in $\operatorname{mod} A$ with $P/\operatorname{soc} P$ lying on Δ. Then there are a positive integer s and homomorphisms $g : Y \to (\tau_A M)^s$ and $h : (\tau_A M)^s \to X$ in $\operatorname{mod} A$ such that $f = hg$, by [3, Lemma VIII.5.4(a)]. Then it follows from Lemma 3.5(i) that the image of g is contained in $(\tau_B M)^s$, and hence f factors through $(\tau_B M)^s$.

The proof of (ii) is similar and applies [3, Lemma VIII.5.4(b)] and Lemmas 3.4(ii) and 3.5(ii). ∎

PROPOSITION 3.8. *The following statements hold:*

(i) *M is a tilting module in* $\operatorname{mod} B$.
(ii) *$H = \operatorname{End}_B(M)$ is a hereditary algebra.*
(iii) *$T = D(M)$ is a tilting module in* $\operatorname{mod} H$.
(iv) *$B = \operatorname{End}_H(T)$.*

(v) *Δ is the section Δ_T of the connecting component $\mathcal{C}_T$ determined by T.*

Proof. Corollary 3.6 yields $\mathrm{Hom}_B(M,\tau_B M)=0$ and $\mathrm{Hom}_B(\tau_B^{-1}M,M)=0$. Since M is a faithful right B-module, applying [3, Lemma VIII.5.1], we conclude that $\mathrm{pd}_B M\le 1$ and $\mathrm{id}_B M\le 1$. Moreover, we have $\mathrm{Ext}_B^1(M,M)\cong D\,\mathrm{Hom}_B(M,\tau_B M)=0$, by [3, Corollary IV.2.14].

We will now show that M is a tilting module in $\mathrm{mod}\,B$. Let $f_1,\dots,f_d$ be a basis of the K-vector space $\mathrm{Hom}_B(B,M)$. Then we have a monomorphism $f:B\to M^d$ in $\mathrm{mod}\,B$, induced by $f_1,\dots,f_d$, and a short exact sequence

$$0\to B\xrightarrow{f} M^d\xrightarrow{g} N\to 0$$

in $\mathrm{mod}\,B$, where $N=\mathrm{Coker}\,f$ and g is a canonical epimorphism.

We now give the standard arguments showing that $M\oplus N$ is a tilting module in $\mathrm{mod}\,B$. Since B is a projective module in $\mathrm{mod}\,B$, we have $\mathrm{Ext}_B^2(N,-)\cong\mathrm{Ext}_B^2(M^d,-)$, and so $\mathrm{pd}_B N\le 1$, because $\mathrm{pd}_B M\le 1$. Hence, $\mathrm{pd}_B(M\oplus N)\le 1$. Applying $\mathrm{Hom}_B(-,M)$ to the above short exact sequence, we obtain a short exact sequence in $\mathrm{mod}\,K$ of the form

$$\mathrm{Hom}_B(M^d,M)\xrightarrow{\mathrm{Hom}_B(f,M)}\mathrm{Hom}_B(B,M)\to\mathrm{Ext}_B^1(N,M)\to\mathrm{Ext}_B^1(M^d,M),$$

where $\mathrm{Ext}_B^1(M^d,M)=0$ and $\mathrm{Hom}_B(f,M)$ is an epimorphism by the choice of f, and so $\mathrm{Ext}_B^1(N,M)=0$. Applying now $\mathrm{Hom}_B(N,-)$, we obtain an epimorphism $\mathrm{Ext}_B^1(N,g):\mathrm{Ext}_B^1(N,M^d)\to\mathrm{Ext}_B^1(N,N)$, because $\mathrm{pd}_B N\le 1$ implies $\mathrm{Ext}_B^2(N,B)=0$, and consequently $\mathrm{Ext}_B^1(N,N)=0$. Finally, applying $\mathrm{Hom}_B(M,-)$, we obtain an epimorphism $\mathrm{Ext}_B^1(M,g):\mathrm{Ext}_B^1(M,M^d)\to\mathrm{Ext}_B^1(M,N)$, because $\mathrm{pd}_B M\le 1$ implies $\mathrm{Ext}_B^2(M,B)=0$, and hence $\mathrm{Ext}_B^1(M,N)=0$. Summing up, we have $\mathrm{pd}_B(M\oplus N)\le 1$ and $\mathrm{Ext}_B^1(M\oplus N,M\oplus N)=0$, and so $M\oplus N$ is a tilting module in $\mathrm{mod}\,B$.

We will now show that N belongs to the additive category $\mathrm{add}\,M$ of M. Assume to the contrary that there exists an indecomposable direct summand W of N which is not in $\mathrm{add}\,M$, or equivalently W does not lie on Δ. Clearly, $\mathrm{Hom}_B(M,W)\ne 0$ because N is a quotient module of M^d. Hence, applying Lemma 3.7, we conclude that $\mathrm{Hom}_B(\tau_B^{-1}M,W)\ne 0$. Since $\mathrm{id}_B M\le 1$, applying [3, Corollary IV.2.14], we find that $\mathrm{Ext}_B^1(W,M)\cong D\,\mathrm{Hom}_B(\tau_B^{-1}M,W)\ne 0$, which contradicts $\mathrm{Ext}_B^1(N,M)=0$. Therefore, M is a tilting module in $\mathrm{mod}\,B$. We also conclude that the rank of $K_0(B)$ coincides with the number of indecomposable modules lying on Δ.

(ii) Let Q be an indecomposable projective module in $\mathrm{mod}\,H$, R an indecomposable right H-submodule of Q, and $f:R\to Q$ the inclusion homomorphism. We claim that R is a projective module. The tilting module M induces the torsion pair $(\mathcal{T}(M),\mathcal{F}(M))$ in $\mathrm{mod}\,B$ with $\mathcal{T}(M)=\{U\in\mathrm{mod}\,B\mid\mathrm{Ext}_B^1(M,U)=0\}$ and $\mathcal{F}(M)=\{W\in\mathrm{mod}\,B\mid\mathrm{Hom}_B(M,W)=0\}$, and the torsion pair $(\mathcal{X}(M), \mathcal{Y}(M))$ in $\operatorname{mod} H$ with $\mathcal{X}(M) = \{X \in \operatorname{mod} H \mid X \otimes_H M = 0\}$ and $\mathcal{Y}(M) = \{Y \in \operatorname{mod} H \mid \operatorname{Tor}_1^H(Y, M) = 0\}$. Since Q belongs to $\mathcal{Y}(M)$ and the torsion-free class $\mathcal{Y}(M)$ is closed under submodules, we conclude that R belongs to $\mathcal{Y}(M)$. Moreover, the functor $\operatorname{Hom}_B(M, -) : \operatorname{mod} B \to \operatorname{mod} H$ induces an equivalence of categories $\mathcal{T}(M) \xrightarrow{\sim} \mathcal{Y}(M)$. Hence there exists a homomorphism $g : V \to U$ in $\operatorname{mod} B$ with V, U indecomposable modules from $\mathcal{T}(M)$, U from Δ, such that $\operatorname{Hom}_B(M, V) = R$, $\operatorname{Hom}_B(M, U) = Q$, and $\operatorname{Hom}_B(M, g) = f$.

Take now a nonzero homomorphism $h : Q' \to R$ in $\operatorname{mod} H$ with Q' an indecomposable projective module. Then there exists a nonzero homomorphism $u : V' \to V$ in $\operatorname{mod} B$ such that V' is in Δ, $\operatorname{Hom}_B(M, V') = Q'$, and $\operatorname{Hom}_B(M, u) = h$. Since f is a monomorphism, we conclude that $fh \neq 0$, and hence $gu \neq 0$. We claim that V lies on Δ. Suppose V is not on Δ. Applying Lemma 3.7, we conclude that there exist homomorphisms $p : V \to W$ and $q : W \to U$ in $\operatorname{mod} B$, with W being a direct sum of modules from $\tau_B \Delta$, such that $g = qp$. But then $qpu = gu \neq 0$ implies $pu \neq 0$, and hence $\operatorname{Hom}_B(M, \tau_B M) \neq 0$, contrary to Corollary 3.6. Thus V belongs to Δ, and consequently $R = \operatorname{Hom}_B(M, V)$ is a projective module in $\operatorname{mod} H$. This shows that every right H-submodule of Q is projective. Therefore, H is a hereditary algebra whose quiver Q_H is the opposite quiver Δ^{op} of Δ.

(iii)–(v). It follows from the Brenner–Butler tilting theorem [3, Theorem VI.3.8] that $T = D(M)$ is a tilting module in $\operatorname{mod} H$ and there is a canonical K-algebra isomorphism $B \xrightarrow{\sim} \operatorname{End}_H(T)$. In particular, B is a tilted algebra of type Δ^{op}. Moreover, Δ is the section Δ_T of Γ_B given by the images $\operatorname{Hom}_H(T, I_1), \ldots, \operatorname{Hom}_H(T, I_n)$ of a complete family $I_1, \ldots, I_n$ of pairwise nonisomorphic indecomposable injective modules in $\operatorname{mod} H$. Indeed, the direct sum of these modules is isomorphic to $D(H)$, and we have isomorphisms of right B-modules

$$\operatorname{Hom}_H(T, D(H)) = \operatorname{Hom}_H(D(M), D(H)) \cong \operatorname{Hom}_{H^{\mathrm{op}}}(H, M) \cong M,$$

since M is also a right H^{op}-module (left H-module). ■

A crucial step for proving the implication (i)⇒(ii) in Theorem 1 is the following theorem.

THEOREM 3.9. *The ideal I is a deforming ideal of A with $r_A(I) = eI$ for an idempotent e of A.*

We will prove the above theorem in several steps. Let $e_1, \ldots, e_r$ be a set of pairwise orthogonal primitive idempotents of A such that $1_A = e_1 + \cdots + e_r$, and $e = e_1 + \cdots + e_n$, for some $n \leq r$, is a residual identity of $B = A/I$. We denote by J the *trace ideal* of M in A, that is, the ideal of A generated by the images of all homomorphisms from M to A, and by J' the trace ideal of the left A-module $D(M)$ in A. Observe that $I = l_A(D(M))$. Then we have the following lemma.

LEMMA 3.10. *We have $J \subseteq I$ and $J' \subseteq I$.*

Proof. First we show that $J \subseteq I$. By definition, there exists an epimorphism $\varphi : M^s \to J$ for some integer $s \geq 1$. Suppose that J is not contained in I. Then there exists a homomorphism $f : A \to M$ in $\operatorname{mod} A$ such that $f(J) \neq 0$. We have in $\operatorname{mod} A$ a decomposition $A = P' \oplus P \oplus P''$, where P' is a maximal direct summand of A such that $P'/\operatorname{soc} P'$ belongs to $\operatorname{add} M$ and P'' is a maximal direct summand of A such that $\operatorname{rad} P''$ belongs to $\operatorname{add} M$. It follows from Lemma 3.4 that $\operatorname{Hom}_A(M, P') = 0$ and $\operatorname{Hom}_A(P'', M) = 0$. Then $J \subseteq P \oplus P''$ and $f(P'') = 0$. Hence, there are homomorphisms $u : J \to P$ and $v : P \to M$ such that $vu \neq 0$. Applying now [3, Lemma VIII.5.4(a)], we conclude that there are a positive integer t and homomorphisms $g : P \to (\tau_A M)^t$, $h : (\tau_A M)^t \to M$ such that $v = hg$. But then $hgu\varphi = vu\varphi \neq 0$, because $J = \operatorname{Im} \varphi$, and hence $gu\varphi \neq 0$. This implies that $\operatorname{Hom}_B(M, \tau_B M) \neq 0$, contradicting Corollary 3.6. Therefore, $J \subseteq I$.

Suppose now that J' is not contained in I. Then there is a homomorphism $f' : A \to D(M)$ in $\operatorname{mod} A^{\mathrm{op}}$ such that $f'(J') \neq 0$. Moreover, we have in $\operatorname{mod} A^{\mathrm{op}}$ an epimorphism $\varphi' : D(M)^m \to J'$ for some integer $m \geq 1$. Then $f'w'\varphi' \neq 0$ for $w' : J' \to A$ the inclusion homomorphism in $\operatorname{mod} A^{\mathrm{op}}$. Applying the duality functor $D : \operatorname{mod} A^{\mathrm{op}} \to \operatorname{mod} A$ we obtain homomorphisms

$$D(D(M)) \xrightarrow{D(f')} D(A) \xrightarrow{D(w')} D(J') \xrightarrow{D(\varphi')} D(D(M)^m)$$

in $\operatorname{mod} A$, where $D(D(M)) \cong M$, $D(D(M)^m) \cong M^m$, $D(A) \cong A$, and $D(\varphi')D(w')D(f') = D(f'w'\varphi') \neq 0$. Then, as in the first part of the proof, we conclude that $\operatorname{Hom}_A(M, \tau_A M) \neq 0$, a contradiction. Hence $J' \subseteq I$. ∎

LEMMA 3.11. *We have $l_A(I) = J$, $r_A(I) = J'$ and $I = r_A(J) = l_A(J')$.*

Proof. We prove that $l_A(I) = J$ and $I = r_A(J)$. Since J is a right B-module, we have $JI = 0$, and hence $I \subseteq r_A(J)$. In order to show the converse inclusion, take a monomorphism $u : M \to A_A^t$ for some integer $t \geq 1$, and let $u_i : M \to A$ be the composite of u with the projection of A_A^t on the ith component. Then there is a monomorphism $v : M \to \bigoplus_{i=1}^t \operatorname{Im} u_i$ induced by u. Further, by definition of J, $\bigoplus_{i=1}^t \operatorname{Im} u_i$ is contained in $\bigoplus_{i=1}^t J$. This leads to the inclusions

$$r_A(J) = r_A\Big(\bigoplus_{i=1}^t J\Big) \subseteq r_A(M) = I.$$

Therefore, $I = r_A(J)$. Moreover, applying a theorem by T. Nakayama (see [44, Corollary IV.6.11]), we obtain $l_A(I) = l_A(r_A(J)) = J$.

Similar arguments yield the equalities $I = l_A(J')$ and $r_A(I) = r_A(l_A(J')) = J'$. ∎

LEMMA 3.12. *We have $eIe = eJe$. In particular, $(eIe)^2 = 0$.*

Proof. Since e is a residual identity of $B = A/I$, we have $B \cong eAe/eIe$. In particular, M is a module in $\operatorname{mod} eAe$ with $r_{eAe}(M) = eIe$. Observe also that eJe is the trace ideal of M in eAe, generated by the images of all homomorphisms from M to eAe in $\operatorname{mod} eAe$. It follows from Lemma 3.10 that $eJe = eJ$ is an ideal of eAe with $eJe \subseteq eIe \subseteq \operatorname{rad} eAe$. Let $\Lambda = eAe/eJe$. Then M is a sincere module in $\operatorname{mod} \Lambda$. We will prove that M is a faithful module in $\operatorname{mod} \Lambda$. Observe that then $eIe/eJe = r_\Lambda(M) = 0$, and consequently $eIe = eJe$. Clerly then $(eIe)^2 = (eJe)(eIe) = 0$, because $JI = 0$.

We shall first show that $\operatorname{id}_\Lambda M \le 1$. Consider the exact sequence

$$0 \to eJe \xrightarrow{u} eAe \xrightarrow{v} \Lambda \to 0$$

in $\operatorname{mod} \Lambda$, where u is the inclusion homomorphism and v is the canonical epimorphism. Applying the functor $\operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, -) : \operatorname{mod} eAe \to \operatorname{mod} K$ to this sequence, we get the exact sequence in $\operatorname{mod} K$ of the form

$$\operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, eJe) \xrightarrow{\alpha} \operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, eAe)$$
$$\xrightarrow{\beta} \operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, \Lambda) \xrightarrow{\gamma} \operatorname{Ext}^1_{eAe}(\tau_{eAe}^{-1} M, eJe),$$

where $\alpha = \operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, u)$, $\beta = \operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, v)$, and γ is the connecting homomorphism. Observe that there is an epimorphism $M^t \to \tau_{eAe}^{-1} M$ in $\operatorname{mod} eAe$ for some positive integer t. Indeed, we first note that $\tau_{eAe}^{-1} M$ has no indecomposable projective direct summand in $\operatorname{mod} eAe$. Then a projective cover $Q \to \tau_{eAe}^{-1} M$ of $\tau_{eAe}^{-1} M$ in $\operatorname{mod} eAe$ factors through a module of the form M^t, and the claim follows. Observe that then the image of every homomorphism $g : \tau_{eAe}^{-1} M \to eAe$ in $\operatorname{mod} eAe$ is contained in eJe, and hence α is an isomorphism. This implies that γ is a monomorphism. Further, applying [3, Lemma VIII.5.4(b)], we conclude that every homomorphism $f : M \to eAe$ in $\operatorname{mod} eAe$ factors through a module of the form $(\tau_{eAe}^{-1} M)^s$ for some positive integer s. Hence there is an epimorphism $(\tau_{eAe}^{-1} M)^m \to eJe$ in $\operatorname{mod} eAe$ for some positive integer m. Then it follows from Lemma 3.5(ii) that there is an epimorphism $(\tau_B^{-1} M)^m \to eJe$ in $\operatorname{mod} eAe$. But then $\operatorname{Hom}_{eAe}(eJe, M) = 0$, because $\operatorname{Hom}_B(\tau_B^{-1} M, M) = 0$. Then we obtain $\operatorname{Ext}^1_{eAe}(\tau_{eAe}^{-1} M, eJe) \cong D\overline{\operatorname{Hom}}_{eAe}(eJe, M) = 0$. Summing up, we conclude that $\operatorname{Hom}_\Lambda(\tau_\Lambda^{-1} M, \Lambda) = \operatorname{Hom}_{eAe}(\tau_{eAe}^{-1} M, \Lambda) = 0$, or equivalently, $\operatorname{id}_\Lambda M \le 1$.

Clearly, $\mathrm{Ext}^1_\Lambda(M,M) = D\,\overline{\mathrm{Hom}}_\Lambda(M,\tau_\Lambda M) = D\,\overline{\mathrm{Hom}}_{eAe}(M,\tau_{eAe}M) = 0$, because $\tau_B M$ is the largest right B-submodule of $\tau_{eAe}M$ and $\mathrm{Hom}_B(M,\tau_B M) = 0$. Since the rank of $K_0(\Lambda)$ equals the rank of $K_0(B)$, we conclude that M is a cotilting module in $\mathrm{mod}\,\Lambda$, and hence $D(M)$ is a tilting module in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$. In particular, $D(M)$ is a faithful module in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$. Then we obtain the required fact $r_\Lambda(M) = l_{\Lambda^{\mathrm{op}}}(D(M)) = 0$. ■

We note that so far the semiregularity of Δ has not been used. It will be essential in the proofs of the next results.

LEMMA 3.13. *Assume that the stable slice Δ of Γ_A does not contain the radical of any indecomposable projective module in* $\mathrm{mod}\,A$. *Let f be a primitive idempotent in I such that $fJ \neq fAe$. Then $L = fAeAf + fJ + fAeAfAe + eAf + eIe$ is an ideal of $F = (e+f)A(e+f)$, and $N = fAe/fLe$ is a module in* $\mathrm{mod}\,B$ *such that* $\mathrm{Hom}_B(N,M) = 0$ *and* $\mathrm{Hom}_B(M,N) \neq 0$.

Proof. It follows from Lemma 3.12 that $fAeIe \subseteq fJ$. Then the fact that L is an ideal of F is a direct consequence of $fJ \subseteq fAe$. Observe also that $fLe = fJ + fAeAfAe$, $fLf \subseteq \mathrm{rad}(fAf)$, $eLe = eIe$, and $eLf = eAf$. We have $N \neq 0$. Indeed, if $fAe = fLe$ then, since $eAfAe \subseteq \mathrm{rad}(eAe)$, we obtain $fAe = fJ + fAe(\mathrm{rad}(eAe))$, and so $fAe = fJ$, by the Nakayama lemma [44, Lemma I.3.3], which contradicts our assumption. Further, $B = eAe/eIe$ and $(fAe)(eIe) = fAeJ \subseteq fJ \subseteq fLe$, and hence N is a right B-module. Moreover, N is also a left module over $S = fAf/fLf$ and F/L is isomorphic to the triangular matrix algebra

$$\Lambda = \begin{pmatrix} S & N \\ 0 & B \end{pmatrix}.$$

Since the module M has no indecomposable direct summand isomorphic to the radical of an indecomposable projective module in $\mathrm{mod}\,A$, it follows from definition of stable slice that $\tau_A^{-1}M = \tau_B^{-1}M$. Hence, for any indecomposable module X on Δ we have an almost split sequence in $\mathrm{mod}\,B$,

$$0 \to X \to Y \to Z \to 0,$$

which is also an almost split sequence in $\mathrm{mod}\,A$. Applying now [36, Lemma 5.6] (or [30, Theorem XV.1.6]) we conclude that $\mathrm{Hom}_A(N,X) = 0$. Hence $\mathrm{Hom}_B(N,M) = 0$. Moreover, every indecomposable direct summand of N is either generated or cogenerated by M. Therefore, $\mathrm{Hom}_B(M,N) \neq 0$. ■

PROPOSITION 3.14. *Assume that the stable slice Δ of Γ_A does not contain the radical of any indecomposable projective module in* $\mathrm{mod}\,A$. *Then $Ie = J$ and $eI = J'$.*

Proof. This follows exactly as [36, Proposition 5.9] by applying Lemmas 3.10–3.13. ■

PROPOSITION 3.15. *Assume that the stable slice Δ of Γ_A does not contain the socle factor of any indecomposable projective module in* $\operatorname{mod} A$. *Then $Ie = J$ and $eI = J'$.*

Proof. The opposite algebra A^{op} is a basic, indecomposable, finite-dimensional selfinjective algebra over K whose Auslander–Reiten quiver $\Gamma_{A^{\mathrm{op}}}$ admits the double $\tau_{A^{\mathrm{op}}}$-rigid stable slice $D(\Delta)$ which does not contain the radical of any indecomposable projective module in $\operatorname{mod} A^{\mathrm{op}}$. Moreover, $D(M)$ is the direct sum of all indecomposable modules in $\operatorname{mod} A^{\mathrm{op}}$ lying on $D(\Delta)$ and $r_{A^{\mathrm{op}}}(D(M)) = l_A(D(M)) = r_A(M) = I$. Then the claim follows from Proposition 3.14 ■

Proof of Theorem 3.9. It follows from Lemma 3.11 and Propositions 3.14 and 3.15 that $r_A(I) = J' = eI$ and $l_A(I) = J = Ie$. In particular, we have $IeI = 0$, because $JI = 0$. Then, applying Proposition 2.2, we conclude that $\operatorname{soc} A \subseteq I$ and $l_{eAe}(I) = eIe = r_{eAe}(I)$. Moreover, the valued quiver $Q_{A/I}$ of $A/I = B$ is acyclic, because B is a tilted algebra. Therefore, I is a deforming ideal of A with $r_A(I) = eI$. ■

We now complete the proof of the implication (i)⇒(ii) of Theorem 1. It follows from Theorems 2.5 and 3.9 that the algebra $A[I]$ associated to I is isomorphic to the orbit algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$. Moreover, applying Theorem 2.4, we conclude that A is socle equivalent to $A[I]$, and consequently A is socle equivalent to $\widehat{B}/(\varphi\nu_{\widehat{B}})$. Further, if φ is strictly positive, we have $e_i \neq e_{\nu(i)}$ for any primitive summand e_i of e, and so the algebras A and $A[I]$ are isomorphic, by Theorem 2.6. It follows from Proposition 3.8 that $B = \operatorname{End}_H(T)$ for the hereditary algebra $H = \operatorname{End}_B(M)$ and the tilting module $T = D(M)$ in $\operatorname{mod} H$, and the canonical section Δ_T of the connecting component $\mathcal{C}_T$ of Γ_B determined by T is the double τ_A-rigid stable slice Δ of Γ_A.

Let $\varphi : A/\operatorname{soc} A \to A[I]/\operatorname{soc} A[I]$ be an isomorphism of algebras and $\phi : \operatorname{mod}(A/\operatorname{soc} A) \to \operatorname{mod}(A[I]/\operatorname{soc} A[I])$ the induced isomorphism of module categories. Then $\phi(\Delta)$ is a double $\tau_{A[I]}$-rigid stable slice of $\Gamma_{A[I]}$, by Lemma 3.3. Moreover, $\phi(\Delta) = F_\lambda(\Delta_T)$ for the push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A[I]$ associated to the Galois covering functor $F : \widehat{B} \to \widehat{B}/(\varphi\nu_{\widehat{B}}) = A[I]$, under the usual identification of B with the corresponding full subcategory of $\widehat{B}$. Since Δ is a semiregular stable slice of Γ_A, we conclude from Lemma 3.3 that $\phi(\Delta) = F_\lambda(\Delta_T)$ is a semiregular stable slice of $\Gamma_{A[I]}$. Then it follows from Proposition 3.1 that the tilting module T is either without nonzero projective direct summand or without nonzero injective direct summand.

Assume now that φ is a rigid automorphism of $\widehat{B}$. We claim that T has no nonzero projective or injective direct summands. We abbreviate $g = \varphi\nu_{\widehat{B}}$. Suppose that T admits an indecomposable projective direct summand in $\operatorname{mod} H$. Then it follows from Proposition 3.1 that the stable slice $F_\lambda(\Delta_T)$ of $\Gamma_{A[I]}$ contains the radical $\operatorname{rad} P$ of an indecomposable projective module P in $\operatorname{mod} A[I]$, and consequently Δ_T contains the radical $\operatorname{rad} P^*$ of an indecomposable projective module P^* in $\operatorname{mod} \widehat{B}$. Observe also that $P^*/\operatorname{soc} P^* = \tau_{\widehat{B}}^{-1} \operatorname{rad} P^*$. Since φ is a rigid automorphism of $\widehat{B}$, we conclude that ${}^gP^*$ is an indecomposable projective module in $\operatorname{mod} \widehat{B}$ whose radical $\operatorname{rad} {}^gP^* = {}^g\operatorname{rad} P^*$ lies on the shift ${}^g\Delta_T$ of Δ_T, which is the canonical section of the connecting component ${}^g\mathcal{C}_T$ of the tilted algebra $g(B) = \nu_{\widehat{B}}(B)$, under the usual identification of B with the corresponding full subcategory of $\widehat{B}$. We also note that $\operatorname{top} P^* = \operatorname{soc} {}^gP^*$, and hence we have $\operatorname{Hom}_{\widehat{B}}(P^*/\operatorname{soc} P^*, \operatorname{rad} {}^gP^*) \neq 0$. Thus $\operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^{-1} M, {}^gM) \neq 0$. But this implies that $\operatorname{Hom}_{A[I]}(\tau_{A[I]}^{-1} M, M) \neq 0$, because the push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A[I]$, associated to the Galois covering $F : \widehat{B} \to \widehat{B}/(g) = A[I]$, induces an isomorphism of K-vector spaces

$$\bigoplus_{r\in\mathbb{Z}} \operatorname{Hom}_{\widehat{B}}(\tau_{\widehat{B}}^{-1} M, {}^{g^r}M) \xrightarrow{\sim} \operatorname{Hom}_{A[I]}(\tau_{A[I]}^{-1} M, M).$$

This contradicts the double $\tau_{A[I]}$-rigidity of $\phi(\Delta)$. We prove similarly that if T admits an indecomposable injective direct summand in $\operatorname{mod} H$, then $\operatorname{Hom}_{A[I]}(M, \tau_{A[I]} M) \neq 0$, again contradicting the double $\tau_{A[I]}$-rigidity of $\phi(\Delta)$. Therefore, the required claim follows. Finally, we note that if K is algebraically closed, then A is isomorphic to $A[I]$, by Theorem 2.7.

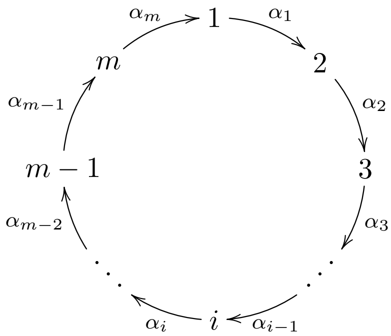
This finishes the proof of (i)$\Rightarrow$(ii), and hence the proof of Theorem 1.

4. Examples. In this section we present examples illustrating the main theorem of the paper.

EXAMPLE 4.1. Let $n \geq 2$ be an integer, $Q(n)$ be the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n,$$

and $B(n) = KQ(n)$ the path algebra of $Q(n)$ over a field K. Hence $B(n)$ is a tilted algebra of Dynkin type $\mathbb{A}_n$. Then every orbit algebra of the form $\widehat{B(n)}/(\varphi\nu_{\widehat{B(n)}})$, with φ a positive automorphism of $\widehat{B(n)}$, is isomorphic to a bound quiver algebra $A(m,n) = K\Omega(m)/J(m,n)$, where $\Omega(m)$ is the quiver



and $J(m,n)$ is the ideal in the path algebra $K\Omega(m)$ generated by all paths in $\Omega(m)$ of length $n+1$ (see [35, (2.7)]), for some integer $m \geq n$. For each $i \in \{1,\dots,m\}$, we denote by P_i the indecomposable projective module in $\operatorname{mod} A$ whose top is the simple module S_i at the vertex i. Observe that $\operatorname{soc} P_i = S_{i+n}$, where we identify $i+n$ with its remainder modulo m. Then $\operatorname{rad} P_i = P_{i+1}/S_{i+1+n}$ for all $i \in \{1,\dots,m\}$. In particular, we have $\tau_A \operatorname{rad} P_i = \operatorname{rad} P_{i+1}$ for all $i \in \{1,\dots,m\}$.

The stable Auslander–Reiten quiver $\Gamma^s_{A(m,n)}$ of $A(m,n)$ is isomorphic to the translation quiver $\mathbb{Z}Q(n)/(\tau^m)$. We observe that every stable slice in Γ_A admits an indecomposable module of the form $\operatorname{rad} P_i$ for some $i \in \{1,\dots,m\}$. On the other hand, for any $i \in \{1,\dots,m\}$, we have

$$\operatorname{Hom}_A(\operatorname{rad} P_i, \tau_A \operatorname{rad} P_i) = \operatorname{Hom}_A(P_{i+1}/S_{i+1+n}, \operatorname{rad} P_{i+1}) \neq 0$$

if and only if $m = n$. Similarly, for any $i \in \{1,\dots,m\}$,

$$\operatorname{Hom}_A(\tau_A^{-1} \operatorname{rad} P_i, \operatorname{rad} P_i) = \operatorname{Hom}_A(P_i/S_{i+n}, \operatorname{rad} P_i) \neq 0$$

if and only if $m = n$. This shows that $\Gamma_{A(m,n)}$ admits a double $\tau_{A(m,n)}$-rigid stable slice if and only if $m > n$. We note that the algebra $A(n,n)$ is isomorphic to the trivial extension algebra $\mathrm{T}(B(n)) = B(n) \ltimes D(B(n))$. On the other hand, for all $m \geq n \geq 2$, every stable slice in $\Gamma_{A(m,n)}$ contains an indecomposable module which is simultaneously the radical of an indecomposable projective module and the socle factor of an indecomposable projective module in $\operatorname{mod} A(m,n)$. Therefore, $\Gamma_{A(m,n)}$ does not admit a semiregular stable slice.

Example 4.2. Let B be the matrix algebra

$$B = \begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \;\middle|\; \begin{matrix} a \in \mathbb{R} \\ b, c \in \mathbb{C} \end{matrix} \right\},$$

where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively. Then B is a 5-dimensional hereditary $\mathbb{R}$-algebra whose valued Gabriel quiver Q_B is the quiver

$$1 \xleftarrow{(1,2)} 2$$

of Dynkin type $\mathbb{B}_2$. Moreover, the Auslander–Reiten quiver Γ_B is of the form

$$\begin{tikzcd}
 & P_2 \arrow[dr, "(2,1)"] & & \tau_B^{-1}P_2 = I_2 = S_2 \\
P_1 = S_1 \arrow[ur, "(1,2)"] & & \tau_B^{-1}P_1 = I_1 \arrow[ur, "(1,2)"] &
\end{tikzcd}$$

where P_i, I_i and S_i, for $i \in \{1,2\}$, denote the indecomposable projective, indecomposable injective and simple module in $\operatorname{mod} B$ at the vertex i. Observe that every section in Γ_B contains either a projective module or an injective module. Consider the trivial extension algebra $A = \mathrm{T}(B) = B \ltimes D(B)$. Then the Auslander–Reiten quiver Γ_A is of the form

$$\begin{tikzcd}
P(2) \arrow[dr] \arrow[dd, dash, dashed] & & & & & & P(2) \arrow[dd, dash, dashed] \\
 & P_2 \arrow[dr, "(2,1)"] & & S_2 \arrow[dr, "(2,1)"] & & \operatorname{rad} P(2) \arrow[ur] \arrow[dr, "(2,1)"] & \\
S_1 \arrow[ur, "(1,2)"] & & I_1 \arrow[ur, "(1,2)"] \arrow[dr] & & P(1)/S_1 \arrow[ur, "(1,2)"] & & S_1 \\
 & & & P(1) \arrow[ur] & & &
\end{tikzcd}$$

where $P(1)$ and $P(2)$ are the projective covers of S_1 and S_2 in $\operatorname{mod} A$, respectively (see [28], [45]).

Observe that every stable slice in Γ_A contains an indecomposable module which is either a direct predecessor or a direct successor of an indecomposable projective module in $\operatorname{mod} A$, and so Γ_A does not admit a regular stable slice. On the other hand, Γ_A admits four semiregular stable slices

$$S_1 \xrightarrow{(1,2)} P_2, \qquad I_1 \xrightarrow{(1,2)} S_2, \qquad S_2 \xrightarrow{(2,1)} P(1)/S_1, \qquad \operatorname{rad} P(2) \xrightarrow{(2,1)} S_1.$$

Moreover, $\operatorname{Hom}_A(P(i)/\operatorname{soc} P(i), \operatorname{rad} P(i)) \neq 0$ for $i \in \{1,2\}$. Therefore, Γ_A does not admit a stable slice which is double τ_A-rigid. We also note that, for $r \geq 2$, the Auslander–Reiten quivers $\Gamma_{\mathrm{T}(B)^{(r)}}$ of the r-fold trivial extension algebras $\mathrm{T}(B)^{(r)} = \widehat{B}/(\nu_{\widehat{B}}^r)$ admit semiregular double $\tau_{\mathrm{T}(B)^{(r)}}$-rigid stable slices, for example, the stable slices given by the four sections of Γ_B presented above.

Example 4.3. Let B be the matrix algebra

$$B = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q}(\sqrt[3]{2}) & \mathbb{Q}(\sqrt[3]{2}) \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \;\middle|\; \begin{matrix} a \in \mathbb{Q} \\ b, c \in \mathbb{Q}(\sqrt[3]{2}) \end{matrix} \right\},$$

where $\mathbb{Q}$ is the field of rational numbers and $\mathbb{Q}(\sqrt[3]{2})$ is a field extension of $\mathbb{Q}$ of degree 3. Then B is a 7-dimensional hereditary $\mathbb{Q}$-algebra whose valued

Gabriel quiver Q_B is the quiver

$$1 \xleftarrow{(1,3)} 2$$

of Dynkin type $\mathbb{G}_2$. Moreover, the Auslander–Reiten quiver Γ_B is of the form

$$\begin{tikzcd}[column sep=small]
 & P_2 \arrow[dr, "(3,1)"] & & \tau_B^{-1}P_2 \arrow[dr, "(3,1)"] & & \tau_B^{-2}P_2 = I_2 = S_2 \\
S_1 = P_1 \arrow[ur, "(1,3)"] & & \tau_B^{-1}P_1 \arrow[ur, "(1,3)"] & & \tau_B^{-2}P_1 = I_1 \arrow[ur, "(1,3)"] &
\end{tikzcd}$$

where P_i, I_i and S_i, for $i \in \{1,2\}$, denote the indecomposable projective, indecomposable injective and simple module in $\operatorname{mod} B$ at the vertex i. Observe that there is exactly one section in Γ_B without projective and injective modules, namely the full valued subquiver Δ of Γ_B with the vertices $\tau_B^{-1}P_1$ and $\tau_B^{-1}P_2$.

Consider the trivial extension algebra $A = \mathrm{T}(B) = B \ltimes D(B)$. Then the Auslander–Reiten quiver Γ_A is of the form

$$\begin{tikzcd}[column sep=tiny]
P(2) \arrow[dr] \arrow[dd, dash, dashed] & & & \tau_A^{-1}P_2 \arrow[d, equal] & & & & & & & P(2) \arrow[dd, dash, dashed] \\
 & P_2 \arrow[dr, "(3,1)"] & & \tau_B^{-1}P_2 \arrow[dr, "(3,1)"] & & S_2 \arrow[dr, "(3,1)"] & & \tau_A^{-1}S_2 \arrow[dr, "(3,1)"] & & \operatorname{rad} P(2) \arrow[ur] \arrow[dr, "(3,1)"] & \\
S_1 \arrow[ur, "(1,3)"] & & \tau_B^{-1}S_1 \arrow[ur, "(1,3)"] \arrow[d, equal] & & I_1 \arrow[ur, "(1,3)"] \arrow[dr] & & P(1)/S_1 \arrow[ur, "(1,3)"] & & \tau_A^{-1}(P(1)/S_1) \arrow[ur, "(1,3)"] & & S_1 \\
 & & \tau_A^{-1}S_1 & & & P(1) \arrow[ur] & & & & &
\end{tikzcd}$$

where $P(1)$ and $P(2)$ are the projective covers of S_1 and S_2 in $\operatorname{mod} A$, respectively (see [28], [45]). Then the full subquiver Δ of Γ_A of the form

$$\tau_B^{-1}S_1 \xrightarrow{(1,3)} \tau_B^{-1}P_2$$

is a double τ_A-rigid stable slice in Γ_A, which is moreover regular.

EXAMPLE 4.4. Let Q be the quiver

$$\begin{tikzcd}
 & 1 \arrow[ddl, "\alpha_1"'] & \\
 & 2 \arrow[dl, "\alpha_2"'] & \\
0 & & 5 \arrow[uul, "\beta_1"'] \arrow[ul, "\beta_2"'] \arrow[dl, "\beta_3"] \arrow[ddl, "\beta_4"] \\
 & 3 \arrow[ul, "\alpha_3"] & \\
 & 4 \arrow[uul, "\alpha_4"] &
\end{tikzcd}$$

Let J be the ideal in the path algebra KQ generated by the elements

$$\beta_1\alpha_1 - \beta_2\alpha_2,\ \beta_2\alpha_2 - \beta_3\alpha_3,\ \beta_3\alpha_3 - \beta_4\alpha_4,$$

and $B = KQ/J$ be the associated bound quiver algebra. We denote by P_i and S_i, $i \in \{0,1,2,3,4,5\}$, the indecomposable projective and the simple module at the vertex i. Then the Auslander–Reiten quiver Γ_B has a connected generalized standard (in the sense of [32]) acyclic component $\mathcal{C}$ of the form

$$\begin{tikzcd}
 & & \tau_B S_1 \arrow[ddr] & & S_1 \arrow[ddr] & & \tau_B^{-1} S_1 \arrow[ddr] & & \\
 & & \tau_B S_2 \arrow[dr] & & S_2 \arrow[dr] & & \tau_B^{-1} S_2 \arrow[dr] & & \\
\cdots & \tau_B \operatorname{rad} P_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \operatorname{rad} P_5 \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & P_5 \arrow[r] & P_5/S_0 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau_B^{-1}(P_5/S_0) & \cdots \\
 & & \tau_B S_3 \arrow[ur] & & S_3 \arrow[ur] & & \tau_B^{-1} S_3 \arrow[ur] & & \\
 & & \tau_B S_4 \arrow[uur] & & S_4 \arrow[uur] & & \tau_B^{-1} S_4 \arrow[uur] & &
\end{tikzcd}$$

obtained by gluing of the preinjective component of the Auslander–Reiten quiver $\Gamma_{B'}$ of the hereditary algebra $B' = KQ'$ given by the quiver Q' with the vertices $0,1,2,3,4$, and the postprojective component of the Auslander–Reiten quiver $\Gamma_{B''}$ of the hereditary algebra $B'' = KQ''$ given by the quiver Q'' with the vertices $1,2,3,4,5$.

Observe that $\mathcal{C}$ admits a finite number of sections. Moreover, every section Δ of $\mathcal{C}$ contains the projective-injective module P_5 and satisfies the condition $\operatorname{Hom}_B(X, \tau_B Y) = 0$ for all modules X and Y lying on Δ. Then it follows from a criterion of Liu and Skowroński (see [3], [23], [31]) that B is a tilted algebra of the form $B = \operatorname{End}_H(T)$, where H is a hereditary algebra of wild type Δ^{op} and T is a tilting module in $\operatorname{mod} H$ such that $\mathcal{C}$ is the connecting component $\mathcal{C}_T$ and Δ is the canonical section Δ_T of $\mathcal{C}_T$ determined by T. We note that T has both an indecomposable projective and an indecomposable injective direct summands, because $\Delta = \Delta_T$ contains a projective-injective module.

Consider now the trivial extension algebra $A = \mathrm{T}(B) = B \ltimes D(B)$. We note that A is the bound quiver algebra $K\Omega/L$, where Ω is the quiver

$$\begin{tikzcd}
 & 1 \arrow[ddl, "\alpha_1"'] & \\
 & 2 \arrow[dl, "\alpha_2"'] & \\
0 \arrow[rr, "\gamma"] & & 5 \arrow[uul, "\beta_1"'] \arrow[ul, "\beta_2"'] \arrow[dl, "\beta_3"] \arrow[ddl, "\beta_4"] \\
 & 3 \arrow[ul, "\alpha_3"] & \\
 & 4 \arrow[uul, "\alpha_4"] &
\end{tikzcd}$$

and L is the ideal in $K\Omega$ generated by the elements

$$\beta_1\alpha_1 - \beta_2\alpha_2,\ \beta_2\alpha_2 - \beta_3\alpha_3,\ \beta_3\alpha_3 - \beta_4\alpha_4,\ \gamma\beta_1\alpha_1\gamma,\ \alpha_1\gamma\beta_2,\ \alpha_1\gamma\beta_3,\ \alpha_1\gamma\beta_4,$$
$$\alpha_2\gamma\beta_1,\ \alpha_2\gamma\beta_3,\ \alpha_2\gamma\beta_4,\ \alpha_3\gamma\beta_1,\ \alpha_3\gamma\beta_2,\ \alpha_3\gamma\beta_4,\ \alpha_4\gamma\beta_1,\ \alpha_4\gamma\beta_2,\ \alpha_4\gamma\beta_3.$$

We denote by $P(0)$ and $P(5)$ the indecomposable projective modules in $\operatorname{mod} A$ with the tops S_0 and S_5, respectively. Then it follows from the results of [10] that Γ_A admits an acyclic connected component $\mathcal{D}$ of the form

$$\begin{tikzcd}
 & \tau_A S_1 \arrow[rdd] & P(5) \arrow[rdd] & S_1 \arrow[rdd] & P(0) \arrow[rdd] & \tau_A^{-1} S_1 \arrow[rdd] & & \\
 & \tau_A S_2 \arrow[rd] & & S_2 \arrow[rd] & & \tau_A^{-1} S_2 \arrow[rd] & & \\
\cdots \arrow[r] \arrow[ruu] \arrow[ru] \arrow[rd] \arrow[rdd] & \operatorname{rad} P(5) \arrow[r] \arrow[ruu] & \tau_A(P_5/S_0) \arrow[r] \arrow[ruu] \arrow[ru] \arrow[rd] \arrow[rdd] & P_5 \arrow[r] \arrow[ruu] & P_5/S_0 \arrow[r] \arrow[ruu] \arrow[ru] \arrow[rd] \arrow[rdd] & P(0)/S_0 \arrow[r] & \tau_A^{-1}(P_5/S_0) \arrow[r, dashed] & \cdots \\
 & \tau_A S_3 \arrow[ru] & & S_3 \arrow[ru] & & \tau_A^{-1} S_3 \arrow[ru] & & \\
 & \tau_A S_4 \arrow[ruu] & & S_4 \arrow[ruu] & & \tau_A^{-1} S_4 \arrow[ruu] & &
\end{tikzcd}$$

with $\tau_A S_i = \tau_B S_i$, $\tau_A^{-1} S_i = \tau_B^{-1} S_i$, for $i \in \{1,2,3,4\}$, containing exactly two projective modules, namely $P(0)$ and $P(5)$. Then, for any section Δ of $\mathcal{C}$, the quiver Δ is a stable slice of Γ_A but is not double τ_A-rigid. Indeed,

$$\operatorname{Hom}_A(P_5, \tau_A P_5) = \operatorname{Hom}_A(P(5)/S_5, \operatorname{rad} P(5)) \neq 0,$$
$$\operatorname{Hom}_A(\tau_A^{-1} P_5, P_5) = \operatorname{Hom}_A(P(0)/S_0, \operatorname{rad} P(0)) \neq 0.$$

On the other hand, taking a shift $\tau_A^m \Delta$ of such a section Δ of $\mathcal{C}$ inside $\mathcal{D}$ with $m \geq 2$, we obtain a regular double τ_A-rigid stable slice of Γ_A. Similarly, for $m \geq 2$, $\tau_A^{-m}\Delta$ is also a regular double τ_A-rigid stable slice of Γ_A. Therefore, $\mathrm{T}(B)$ is isomorphic to the trivial extension algebra $\mathrm{T}(B^*)$ of a tilted algebra $B^* = \operatorname{End}_{H^*}(T^*)$ of a hereditary algebra H^* and a tilting module T^* in $\operatorname{mod} H^*$ without nonzero projective or injective direct summands (see [10] for more details).

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Andrzej Skowroński
Faculty of Mathematics and
Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: skowron@mat.uni.torun.pl

Kunio Yamagata
Department of Mathematics
Tokyo University of Agriculture and Technology
Nakacho 2-24-16, Koganei
Tokyo 184-8588, Japan
E-mail: yamagata@cc.tuat.ac.jp

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