COLLOQUIUM MATHEMATICUM

VOL. 141 2015 NO. 1

AN ALPERN TOWER INDEPENDENT OF A GIVEN PARTITION

BY

JAMES T. CAMPBELL (Memphis, TN), JARED T. COLLINS (Henderson, TN), STEVEN KALIKOW (Memphis, TN), RAENA KING (Memphis, TN), and RANDALL MCCUTCHEON (Memphis, TN)

Abstract. Given a measure-preserving transformation T of a probability space $(X,\mathcal{B},\mu)$ and a finite measurable partition $\mathbb{P}$ of X, we show how to construct an Alpern tower of any height whose base is independent of the partition $\mathbb{P}$. That is, given $N\in\mathbb{N}$, there exists a Rokhlin tower of height N, with base B and error set E, such that B is independent of $\mathbb{P}$, and $TE\subset B$.

1. Introduction and statement of results. It has long been known that, given an ergodic invertible probability measure-preserving system, a Rokhlin tower may be constructed with base independent of a given partition of the underlying space [R1], [R2]. In [A], meanwhile, S. Alpern proved a "multiple" Rokhlin tower theorem (see [EP] for an easy proof) whose full statement we will not give, but which has the following corollary of interest:

THEOREM 1.1. *Let $N\in\mathbb{N}$ and $\epsilon>0$ be given. For any ergodic invertible measure-preserving transformation T of a Lebesgue probability space $(X,\mathcal{B},\mu)$, there exists a Rokhlin tower of height N with base B and error set E with $\mu(E)<\epsilon$ such that $TE\subset B$.*

A *Rokhlin tower of height N with base B and error set E* is characterized by the collection of sets $\{B,TB,\ldots,T^{N-1}B,E\}$ forming a partition of X. If in addition $TE\subset B$, we shall say *Alpern tower*. It is our goal to show that for ergodic transformations on $(X,\mathcal{B},\mu)$, given a finite measurable partition $\mathbb{P}$ of X, an Alpern tower may be constructed with base B independent of $\mathbb{P}$. Precisely:

MAIN THEOREM 1.2. *Let $(X,\mathcal{B},\mu)$ be a Lebesgue probability space, and suppose $\mathbb{P}$ is a finite measurable partition of X. For any ergodic invertible measure-preserving transformation T of X and any $N\in\mathbb{N}$, there exists a Rokhlin tower of height N with base B and error set E such that $T(E)\subset B$ and B is independent of $\mathbb{P}$.*

2010 *Mathematics Subject Classification*: Primary 28D05; Secondary 37M25, 60A10.

Key words and phrases: Rokhlin tower, Alpern tower, independent sets, measure-preserving transformation, probability space.

DOI: 10.4064/cm141-1-10

We do not specify the size of the error set; but the process of constructing our tower makes it clear that the error set may be made arbitrarily small.

2. Proof of main result. For the remainder of the paper, $(X, \mathcal{B}, \mu)$ will be a fixed Lebesgue probability space, and $T : X \to X$ will be an invertible ergodic measure-preserving transformation on X. All sets mentioned will be measurable, and we will adopt a cavalier attitude toward null sets. In particular, "partition" will typically mean "measurable partition modulo null sets".

DEFINITION 2.1. By a *tower over* B we will mean a set $B \subset X$, called the *base*, and a countable partition $B = B_1 \cup B_2 \cup \cdots$, together with the images $T^i B_j$, $0 \le i < j$, such that the family $\{T^i B_j : 0 \le i < j\}$ consists of pairwise disjoint sets. If this family partitions X, we will say that the tower is *exhaustive*.

If a tower over B is exhaustive and $B = B_N \cup B_{N+1}$, we shall speak of an *exhaustive Alpern tower of height* $\{N, N+1\}$, as in such a case, $\{B, TB, \ldots, T^{N-1}B, E = T^N B_{N+1}\}$ partitions X with $TE \subset B$. So we may rephrase Theorem 1.2 as:

THEOREM 1.2. *Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space and suppose $\mathbb{P}$ is a finite measurable partition of X. For any ergodic invertible measure-preserving transformation T of X and any $N \in \mathbb{N}$, one may find an exhaustive Alpern tower of height $\{N, N+1\}$ having base independent of $\mathbb{P}$.*

We require a lemma (and a corollary).

LEMMA 2.2. *Let $M \in \mathbb{N}$ and let $\mathbb{P} = \{P_1, \ldots, P_t\}$ be a partition of X with $\mu(P_i) > 0$ for each i. There exists a set S of positive measure such that if $x \in S$ with first return $n(x) = n$, say, then $|\{x, Tx, \ldots, T^{n-1}x\} \cap P_i| \ge M$ for $1 \le i \le t$.*

Proof. For almost every x we may find $K(x)$ such that for each i between 1 and t we have $|\{x, Tx, \ldots, T^{K(x)-1}x\} \cap P_i| \ge M$. Since almost all of X is the countable union (over $k \in \mathbb{N}$) of $\{x : K(x) = k\}$, there exists some fixed K such that the set $A = \{x : K(x) \le K\}$ has positive measure. If $C \subset A$ has very small measure ($\mu(C) < 1/K$), then the average first-return time of $x \in C$ to C is $1/\mu(C) > K$, so we can find $S \subset C$ with $\mu(S) > 0$ such that $S, TS, \ldots, T^{K-1}S$ are pairwise disjoint. ∎

COROLLARY 2.3. *Let $M \in \mathbb{N}$ and let $\mathbb{P} = \{P_1, \ldots, P_t\}$ be a partition of X with $\mu(P_i) > 0$ for each i. There is a tower having base*

$$S = S_{tM} \cup S_{tM+1} \cup \cdots$$

where for each $x \in S_r$,

$$|\{x, Tx, \dots, T^{r-1}x\} \cap P_i| \geq M \quad \textit{for all } 1 \leq i \leq t.$$

Proof. Let S, K be as in Lemma 2.2 and choose any $k \geq K$. ∎

We turn now to the proof of Theorem 1.2.

Fix a partition $\mathbb{P} = \{P_1, \dots, P_t\}$, an arbitrary natural number N, and $\epsilon > 0$. Set $m_i = \mu(P_i)$, and assume (without loss of generality) that $0 < m_1 \leq \cdots \leq m_t$. Select and fix $M > 3N^3t/m_1$. Let S be as in Corollary 2.3 for this M; hence $S = S_{tM} \cup S_{tM+1} \cup \cdots$. (Some S_i may be empty, of course.) For each non-empty S_R, partition S_R by $\mathbb{P}$-name of length R. (Recall that x, y in S_R *have the same* $\mathbb{P}$*-name of length* R if $T^i x$ and $T^i y$ lie in the same cell of $\mathbb{P}$ for $0 \leq i < R$.) Let C be the base of one of the resulting columns; hence, every $x \in C$ has the same $\mathbb{P}$-name of length R (for some $R \geq tM$), and the length R orbit of each $x \in C$ meets each P_i at least M times.

Partition C into pieces $C^{(1)}, \dots, C^{(t)}$ whose measures will be determined later. Then partition each $C^{(i)}$ into N equal measure pieces, $C^{(i)} = C_1^{(i)} \cup \cdots \cup C_N^{(i)}$.

Now we fix (R, C) and focus our attention on the height R *column* over a single $C^{(i)}$ and its height R *subcolumns* over $C_j^{(i)}$, $1 \leq j \leq N$. We refer to the sets $T^r C^{(i)}$, $0 \leq r < R$, as *levels* and to the sets $T^r C_j^{(i)}$ as *rungs*. We are going to build a portion of B by carefully selecting some rungs from the subcolumns under consideration. As we move through the various subcolumns, we need to have gaps of length N or $N+1$ between selections. Now to specifics. We want to have our $C^{(i)}$-selections form a “staircase” of height N starting at level $N^2 - N$. That is, at height $(N-1)N$, the rung over $C_1^{(i)}$ is the only one selected; at height $N(N-1)+1$, the rung over $C_2^{(i)}$ is the only one selected, etc., so that at height $N^2 - 1$, the rung over $C_N^{(i)}$ is the only one selected.

This is easy to accomplish. First, we select each base rung $C_j^{(i)}$, $j = 1, \dots, N$ (i.e. the rungs in the zeroth level). Over $C_1^{(i)}$, we then select $N-1$ additional rungs with gaps of length N; that is, we select the rungs at heights N, $2N, \dots, (N-1)N$. Over $C_2^{(i)}$ we select $N-2$ rungs with gap N, then a rung with gap $N+1$. We continue in this fashion, choosing one less gap of length N and one more of length $N+1$ in each subsequent subcolumn. In the last subcolumn (that over $C_N^{(i)}$) we are thus choosing rungs with gaps of length $N+1$ a total of $N-1$ times. See the left side of Figure 1 for the case $N = 4$.

Now we perform a similar procedure moving down from the top, so as to obtain a staircase starting at height $R - (N^2 - 1)$. Note that there are either N or $N-1$ unselected rungs at the top of each subcolumn. See the right side of Figure 1.

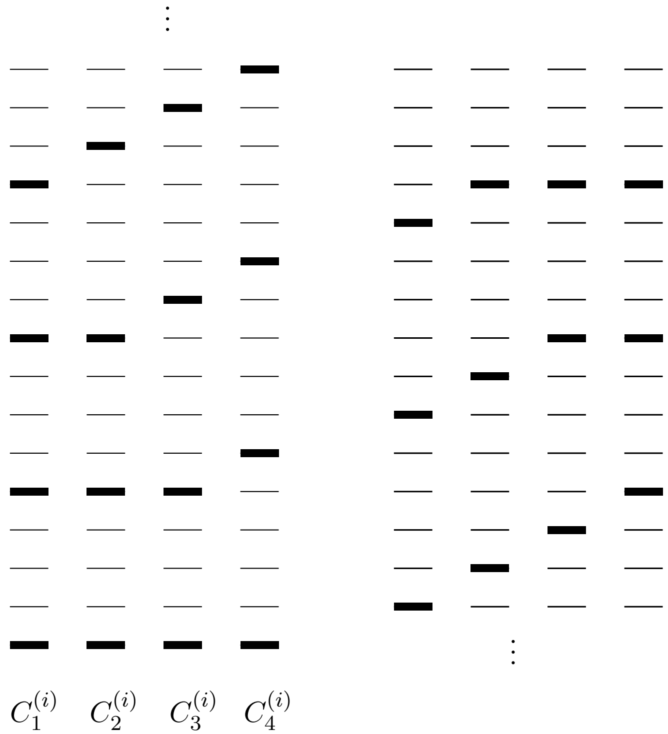

Fig. 1. Bottom and top of tower for $N = 4$

Next, we want to select rungs through the middle of the tower so as to iterate the staircase pattern all the way up, except that we will skip certain levels (i.e. not select any of their rungs), continuing the staircase pattern where we left off with the following rung. As we want to match stride with the staircase already selected at the top, the total number of levels skipped in the middle section will be constrained to a certain residue class modulo N, and as we want the selected rungs to form a portion of an Alpern tower of height $\{N, N+1\}$, we cannot skip any two levels with fewer than N levels between them.

Some terminology: an *appearance* of P_j in $C^{(i)}$ is just a level of $C^{(i)}$ that is contained in P_j. A *selection* of P_j is just a selected rung in a subcolumn of $C^{(i)}$ that is contained in P_j. The *net skip* of P_j in the tower over $C^{(i)}$ is defined as

$$S_j(C^{(i)}) = (\#\text{ of appearances of } P_j) - (\#\text{ of selections of } P_j).$$

For example, looking at Figure 1, one sees that four zeroth level rungs are selected. So if the zeroth level belongs to P_j, the zeroth level contribution to $S_j(C^{(i)})$ is -3 (one appearance and four selections).

Let $\delta = 2(N-1)(N-2)$ and choose γ with

$$\frac{\delta}{m_1} + N > \gamma \geq \frac{\delta}{m_1} \quad \text{and} \quad (t-1)\delta + \gamma \equiv R \pmod{N}.$$

Over $C^{(i)}$, we skip a quantity of "middle" levels belonging to each P_j (for $j \neq i$) sufficient to ensure that $S_j(C^{(i)}) = \delta$ for $j \neq i$ and $S_i(C^{(i)}) = \gamma$. (Note that P_j cannot have been skipped more than δ times in the outer rungs.) This is not delicate; one can just enact the selection greedily. That is to say, travel up the tower, beginning at level N^2, skipping rungs that belong to cells requiring additional skips whenever there has been no too-recent skip. Since each P_j appears at least $M > 3N^3t/m_1$ times, and we need only $\gamma + (t-1)\delta \leq 2N^2t/m_1$ net skips, we will find all the skips we need.

We have not specified the relative masses of the bases of the columns $C^{(i)}$. Set

$$b_j = \frac{\mu(P_j)(\gamma + (t-1)\delta) - \delta}{\gamma - \delta} \tag{2.1}$$

and set $\mu(C^{(i)}) = b_i\mu(C)$, $1 \leq i \leq t$. Our choice of γ ensures that $b_i \geq 0$ for each i, and one easily checks that $\sum b_i = 1$, so this is coherent.

Let B_C be the union of the rungs selected from the columns over C (this includes each of the rungs selected from each of the N subcolumns over $C^{(i)}$, $1 \leq i \leq t$) and set $B = \bigcup_C B_C$ (here C runs over the bases of the columns corresponding to every $\mathbb{P}$-name of length R for every $R \geq tM$). It is clear that B forms the base of an Alpern tower of height $\{N, N+1\}$. It remains to show that B is independent of $\mathbb{P}$, which we will do by constructing a set A, disjoint from B, such that both A and $A \cup B$ can be shown to be independent of $\mathbb{P}$.

Here is how A is constructed. Consider again the tower over $C^{(i)}$. This tower had R levels and RN rungs, some of which were selected for the base B. We now choose $\gamma + (t-1)\delta$ additional rungs for the set A. For each $j \neq i$, δ of these rungs should be contained in P_j, with the remaining γ contained in P_i. (We do not worry about gaps and whatnot; just choose any such collection of rungs disjoint from the family of B-selections.) Denote the union of these additional rungs (in all of the columns over $C^{(i)}$, $1 \leq i \leq t$) by A_C. Finally, set $A = \bigcup_C A_C$.

That $A \cup B$ is independent of $\mathbb{P}$ is a consequence of the fact that for each $C^{(i)}$, the number of appearances of P_j in the column over $C^{(i)}$ is precisely the number of B-selections from P_j plus the number of A-selections from P_j. Accordingly, the relative masses of the cells of $\mathbb{P}$ restricted to $A \cup B$ are equal to the relative frequencies of the appearances of the cells of $\mathbb{P}$ in the column over $C^{(i)}$. Therefore, since the proportion of the column that is selected for $A \cup B$ is independent of $C^{(i)}$ (in fact is always equal to $1/N$), and since the columns over the various $C^{(i)}$ exhaust X, we see that $A \cup B$ is independent of $\mathbb{P}$ (in fact $\mu(P_j \cap (A \cup B)) = \frac{1}{N}\mu(P_j)$, $1 \le j \le t$).

That A is independent of $\mathbb{P}$, meanwhile, is a consequence of (2.1). Fixing C and recalling that $b_i = \mu(C^{(i)})/\mu(C)$, that there were δ P_j-rungs in the column over $C^{(i)}$ selected for A, $i \neq j$, and that there were γ P_i-rungs in the column over $C^{(i)}$ selected for A, we see that the relative mass of P_i among the A-selections in the tower over C is

$$r_i = \frac{b_i\gamma + (1-b_i)\delta}{\gamma + (t-1)\delta}.$$

But, solving (2.1) for $\mu(P_i)$, one gets

$$\mu(P_i) = \frac{b_i\gamma + (1-b_i)\delta}{\gamma + (t-1)\delta}$$

as well. So the intersection of A with the column over C is independent of $\mathbb{P}$. That this is true for all C gives independence of A from $\mathbb{P}$ simpliciter.

REFERENCES

[A] S. Alpern, *Generic properties of measure preserving homeomorphisms*, in: Ergodic Theory, Lecture Notes in Math. 729, Springer, Berlin, 1979, 16–27.

[EP] S. J. Eigen and V. S. Prasad, *Multiple Rokhlin tower theorem: a simple proof*, New York J. Math., 3A (1997), 11–14.

[R1] V. A. Rokhlin, *On the fundamental ideas of measure theory*, Amer. Math. Soc. Transl. 71 (1952).

[R2] V. A. Rokhlin, *Generators in ergodic theory II*, Vestnik Leningrad. Univ. 20 (1965), no. 13, 68–72 (in Russian).

James T. Campbell, Steven Kalikow, Randall McCutcheon
Department of Mathematical Sciences
University of Memphis
Dunn Hall 373
Memphis, TN 38152, U.S.A.
E-mail: jcampbll@memphis.edu
skalikow@memphis.edu
rmcctchn@memphis.edu

Jared T. Collins
Department of Mathematics
and Computer Science
Freed-Hardeman University
Henderson, TN 38340, U.S.A.
E-mail: jtcollins@fhu.edu

Raena King
Department of Mathematics
and Computer Science
Christian Brothers University
Memphis, TN 38104, U.S.A.
E-mail: rking2@cbu.edu

Received 23 March 2015 (6581)