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COMPLETE GRADIENT RICCI SOLITONS

BY

UDO SIMON (Berlin)

Abstract. For complete gradient Ricci solitons we state necessary conditions for a non-trivial soliton structure in terms of intrinsic curvature invariants.

1. Introduction. In [5] Hamilton introduced the notion of Ricci solitons, generalizing the concept of Einstein spaces; for dimension $n \geq 3$, this generalization amounts to the following definitions:

Definition 1. (a) Let (M, g) be a connected, oriented Riemannian manifold with metric g of dimension $\dim M =: n \geq 2$. The quadruple (M, g, X, λ), where X is a vector field and $\lambda \in \mathbb{R}$, is called a *Ricci soliton* if the following equation for the $(0, 2)$ Ricci tensor Ric is satisfied:

$$\mathrm{Ric} + \tfrac{1}{2}\mathcal{L}_X g = \lambda g; \tag{1}$$

here $\mathcal{L}_X g$ denotes the Lie derivative of the metric g with respect to X.

(b) When X is the gradient vector field of a *potential function* $f \in C^\infty$ on M (we write $X = \operatorname{grad} f$), then (M, g, f, λ) is called a *gradient Ricci soliton*; in this case the previous equation reads

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g, \tag{2}$$

where $\mathrm{Hess}\, f := \mathrm{Hess}_g\, f$ stands for the *covariant Hessian* of f.

(c) A soliton (M, g, X, λ) will be called *expanding*, *steady* or *shrinking* if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively; analoguous terminology is used for gradient Ricci solitons.

(d) We call a gradient Ricci soliton *trivial* if f is constant.

In this paper we proceed with our investigations from [8]; in the introduction of [8] we summarized some known results for compact and also for complete Ricci solitons. Recall that Perelman [9] proved that compact Ricci solitons are always gradient.

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Here we present results for complete (in our paper *complete* always means *complete, but non-compact*) gradient Ricci solitons with particular emphasis on low dimensions. As in the foregoing paper [8], our interest is in the behaviour of intrinsic curvature invariants of complete Ricci solitons. As motivation, we recall a statement of A. Derdziński [4]:

Statement. *Non-trivial compact Ricci solitons are only possible if:*

- *the dimension n satisfies $n \geq 4$;*
- *the scalar curvature $\mathcal{R}$ is non-constant and positive;*
- *the soliton constant λ in the definition of Ricci solitons is in a certain real interval, namely*

$$0 < n\lambda \in (\min \mathcal{R}, \max \mathcal{R}).$$

In this paper our aim is to prove similar statements for complete gradient Ricci solitons. As a typical example of such a result we state

Theorem 1. *Let (M, g, f, λ) be a complete, non-compact gradient Ricci soliton of dimension $n \geq 2$; assume that:*

(i) *the Ricci curvature is bounded below, say* $\mathrm{Ric} \geq \delta g$ *for some* $\delta \in \mathbb{R}$;
(ii) $\mathcal{R} \neq 0$;
(iii) $\lambda > 0$;
(iv) f *is bounded above.*

Then $n\lambda \in [\inf \mathcal{R}, \sup \mathcal{R}]$.

An essential tool for the proofs is the maximum principle of Omori–Yau, a powerful tool for complete Riemannian manifolds.

2. Preliminaries

2.1. Basic notation for Riemannian manifolds. Throughout the paper let (M, g) be a connected, oriented Riemannian manifold of dimension $n \geq 2$.

2.1.1. *Levi-Civita connection and derivatives.* Denote by ∇ the Levi-Civita connection of (M, g), by $\operatorname{grad} f$ the *gradient* and by Δ the *Laplacian* acting on functions,

$$\Delta f := \operatorname{trace}_g \operatorname{Hess} f$$

for $f \in C^\infty(M)$.

2.1.2. *Curvature.* Denote by $\mathcal{R}ic$ the g-self adjoint Ricci operator, and by $\mathcal{R} := \operatorname{trace} \mathcal{R}ic$ the *scalar curvature*. Recall that the second Bianchi identity for the Riemannian curvature tensor implies

$$2\nabla_i R^i_j = \nabla_j \mathcal{R}; \qquad (3)$$

moreover, denote by ρ_i the eigenvalues and by e_i the associated orthonormal eigenvectors of $\mathcal{R}ic$. We write κ_{ij} for the sectional curvature of the 2-plane $\operatorname{span}(e_i, e_j)$ where $i \neq j$. Recall that $\rho_i = \sum_{j\neq i} \kappa_{ij}$, thus

$$\mathcal{R} = \sum_i \rho_i = \sum_{j\neq i} \kappa_{ij}.$$

Note that, for $n \geq 3$, if the sectional curvature is non-negative, then

$$\mathcal{R} - 2 \max_k \rho_k \geq 0.$$

2.1.3. *Standard local notation.* We adopt the standard local notation, raise and lower indices as usual, and apply the Einstein convention. In local notation we write g_{ij} and R_{ij} for the components of the metric tensor g and the Ricci tensor Ric, respectively. Considering local coordinates u^i and a Gauß basis $\{\partial_i\}$, we write partial derivatives of a function $f \in C^\infty(M)$ in the form $f_i := \partial_i f := \frac{\partial f}{\partial u^i}$, while we write covariant derivatives of f as $\nabla_j f_i = \nabla_j \nabla_i f$.

2.2. The maximum principle of Omori–Yau [14]. Let (M, g) be a complete, non-compact Riemannian n-manifold with Ricci curvature bounded below, $\operatorname{Ric} \geq \delta g$ for some $\delta \in \mathbb{R}$. Let $f \in C^2(M)$ be bounded below. Then there is a sequence of points $\{p_k \in M\}_{k\in\mathbb{N}}$ such that the following **O-Y**-*relations* are satisfied:

(1) $\lim_k f(p_k) = \inf f$;
(2) $\lim_k \|\operatorname{grad} f\|(p_k) = 0$;
(3) $\lim_k (\Delta f)(p_k) \geq 0$.

3. Gradient Ricci solitons. We recall some relations for gradient Ricci solitons.

3.1. Basic relations for gradient Ricci solitons. From the definition we get

(4) $\Delta f + \mathcal{R} = n\lambda$,

(5) $\|\operatorname{Hess} f\|^2 = n\lambda^2 - 2\lambda\mathcal{R} + \|\operatorname{Ric}\|^2 = \frac{1}{n}(n\lambda - \mathcal{R})^2 + \left\|\operatorname{Ric} - \frac{1}{n}\mathcal{R}g\right\|^2$.

The foregoing equations imply

$$\lambda(2\mathcal{R} - n\lambda) \leq \|\operatorname{Ric}\|^2.$$

Differentiating (2), Hamilton [5] derived the following important equation:

$$\mathcal{R} + \|\operatorname{grad} f\|^2 - 2\lambda f = \text{const} =: c, \tag{6}$$

where $c \in \mathbb{R}$. Relations (4) and (6) together give

$$\Delta f - \|\operatorname{grad} f\|^2 + 2\lambda f = n\lambda - c =: \theta = \text{const}, \tag{7}$$

where $\theta \in \mathbb{R}$.

REMARK 1. We add some simple observations:

- If $\mathcal{R} \geq 0$ then $2\lambda f + c \geq 0$. In particular, if additionally $\lambda > 0$ then f is bounded below.
- As a consequence of (2) and Hamilton's equation (6) we get

$$2R_{ij} f^j = \partial_i \mathcal{R} = \mathcal{R}_i, \tag{8}$$

$$2R^{ij} f_i \mathcal{R}_j = \|\operatorname{grad} \mathcal{R}\|^2 \geq 0. \tag{9}$$

4. PDEs for gradient Ricci solitons

4.1. Known PDEs. We refer to [8, Section 3] and recall the following two propositions; of course, some of the following PDEs, e.g. the ones for $\Delta\mathcal{R}$ and $\Delta\|\operatorname{grad} f\|^2$, already appeared in the literature before.

PROPOSITION 1. *The scalar curvature of a gradient Ricci soliton with dimension $n \geq 2$ satisfies the following PDEs:*

$$\tfrac{1}{2}\Delta\mathcal{R} = \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f) + (\lambda\mathcal{R} - \|\operatorname{Ric}\|^2), \tag{10}$$

$$\tfrac{1}{4}\Delta\mathcal{R}^2 = \mathcal{R} \cdot \big(\operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f) + (\lambda\mathcal{R} - \|\operatorname{Ric}\|^2)\big) + \tfrac{1}{2}\|\operatorname{grad} \mathcal{R}\|^2, \tag{11}$$

$$\tfrac{1}{2}\Delta\|\operatorname{grad} f\|^2 = \|\operatorname{Hess} f\|^2 - \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f), \tag{12}$$

$$\Delta(\|\operatorname{grad} f\|^2 + \mathcal{R}) = 2\lambda\Delta f = 2\lambda(n\lambda - \mathcal{R}). \tag{13}$$

PROPOSITION 2. *The Ricci tensor of a gradient Ricci soliton with dimension $n \geq 2$ satisfies the following PDE:*

$$\begin{aligned}\tfrac{1}{2}\Delta\|\operatorname{Ric}\|^2 = 2\sum_{i<j} \kappa_{ij}(\rho_i - \rho_j)^2 &+ \tfrac{1}{2}((4-n)\lambda + \mathcal{R})\|\operatorname{Ric}\|^2 \\ &+ \big\|\nabla\big(\operatorname{Ric} - \tfrac{1}{n}\mathcal{R}g\big)\big\|^2 + \tfrac{1}{n}\|\operatorname{grad} \mathcal{R}\|^2 - 2\sum(\rho_i)^3 + \operatorname{div},\end{aligned}$$

where div *denotes a divergence term, namely* (*see* [8])

$$\operatorname{div} := \tfrac{1}{2}\nabla_j(\|\operatorname{Ric}\|^2 f^j).$$

REMARK 2. (a) We calculated the foregoing formula using results from [11]. Our version of Proposition 2 is a minor modification of the result in [8].

(b) Since Perelman [9] has shown that compact Ricci solitons are gradient, the above PDEs are satisfied by all compact Ricci solitons.

4.2. The Hamilton constant and the Hamilton function. We add some remarks on Hamilton's equation (6). Let us rewrite it in the form

$$H := \mathcal{R} - 2\lambda f = c - \|\operatorname{grad} f\|^2,$$

and call c the *Hamilton constant* and H the *Hamilton function.* First we state some simple

Observations on compact Ricci solitons

(i) If f is stationary at $p \in M$ then $H(p) = H_{\max} = c$ is maximal and $\|\operatorname{grad} f\|^2(p) = 0$. Moreover, at p we have $\operatorname{grad} H = 0$ and $\operatorname{grad} \mathcal{R} = 0$.

(ii) If, at $q \in M$, $H(q) = H_{\min}$ is minimal then

$$\|\operatorname{grad} f\|^2(q) = \max_M \|\operatorname{grad} f\|^2,$$

thus $\operatorname{grad} H = 0$ and $2\lambda \operatorname{grad} f(q) = \operatorname{grad} \mathcal{R}(q)$.

(iii) Equation (7) implies

PROPOSITION 3. *Let (M, g, f, λ) be a compact gradient Ricci soliton. Then the extrema of f satisfy*

$$2\lambda f_{\min} \leq n\lambda - c \leq 2\lambda f_{\max}.$$

REMARK 3. For a compact gradient Ricci soliton we have:

(a) If the soliton is trivial then $2\lambda f = \text{const} = n\lambda - c$.
(b) If $H = \text{const}$ then the soliton is trivial.
(c) If the soliton is non-trivial then there exists $q \in M$ such that

$$R^{ij} f_i f_j(q) > 0.$$

Proof. We prove (c): Assume that (M, g, f, λ) is non-trivial; then the integral of $R^{ij} f_i f_j$ is positive since, by (12), it equals the integral of $\|\operatorname{Hess} f\|^2$, which is positive (or else $\operatorname{Hess} f$ as well as Δf would vanish, implying that the soliton is trivial). Thus, the assertion follows. ∎

Observations on complete Ricci solitons

PROPOSITION 4. *Assume that (M, g, f, λ) satisfies:*

(i) *(M, g) is complete;*
(ii) *$\operatorname{Ric} \geq \delta g$ for some $\delta \in \mathbb{R}$;*
(iii) *f is bounded;*
(iv) *$\lambda \neq 0$.*

Then

$$2\lambda \inf f \leq n\lambda - c \leq 2\lambda \sup f.$$

Proof. From the assumptions we can apply the maximum principle of Omori–Yau: there exists a sequence $\{p_k\}_k$ of points such that

$$0 \leq \lim_k \Delta f = \lim_k \|\operatorname{grad} f\|^2 - 2\lambda \lim_k f + (n\lambda - c) = -2\lambda \inf f + (n\lambda - c).$$

Now consider analogously the function $-f$ with $\inf(-f) = \sup f$. ∎

Diagonalizable quadratic forms. Assume that, for orthonormal eigenvectors $e_1, \ldots, e_n$ of the Ricci tensor,

$$\operatorname{Ric}(e_i, e_j) = \rho_i \delta_{ij} \quad \text{and} \quad g(e_i, e_j) = \delta_{ij}.$$

Then equation (2) implies that the Hessian must have diagonal form, too: $\mathrm{Hess}(e_i, e_j) = \sigma_i \delta_{ij}$, where $\sigma_1, \dots, \sigma_n$ denote the eigenfunctions of the Hessian. Equation (2) gives $\rho_i + \sigma_i = \lambda$ for all $i = 1, \dots, n$.

5. Gradient Ricci solitons in dimension $n \geq 2$

5.1. The scalar curvature and λ. In the following subsections we collect some relations between $\mathcal{R}$ and λ. One can find the first lemma in several papers, e.g. [3], [1], [2]:

LEMMA 1. *Let (M, g, f, λ) be a complete gradient Ricci soliton with $\lambda > 0$. Then $\mathcal{R} \geq 0$.*

One of the main results in [10] is the following

THEOREM 2. *Let (M, g, f, λ) be a complete gradient Ricci soliton of dimension $n \geq 2$. Then:*

- *If $\lambda < 0$ then $n\lambda \leq \inf \mathcal{R} \leq 0$, and the equality $n\lambda = \inf \mathcal{R}$ implies that the gradient soliton is trivial.*
- *If $\lambda > 0$ then $n\lambda \geq \inf \mathcal{R} \geq 0$, and the equality $n\lambda = \inf \mathcal{R}$ implies that the gradient soliton is trivial. Moreover, the equality $\inf \mathcal{R} = 0$ implies that (M, g) is isometric to the standard flat $\mathbb{R}^n$.*

Now we recall Proposition 6.2 from [8]:

PROPOSITION 5. *Consider a complete gradient Ricci soliton (M, g, f, λ) of dimension $n \geq 2$ with $\lambda \neq 0$. Assume that:*

(i) *the Ricci curvature is bounded below, say $\mathrm{Ric} \geq \delta g$ for some $\delta \in \mathbb{R}$;*
(ii) $\lambda \mathcal{R} \leq \|\mathrm{Ric}\|^2$.

Then $\lambda \inf \mathcal{R} \geq 0$.

LEMMA 2. *Consider a complete gradient Ricci soliton (M, g, f, λ) of dimension $n \geq 2$ such that:*

(i) $\mathrm{Ric} \geq \delta g$ *for some $\delta \in \mathbb{R}$;*
(ii) *f is bounded below.*

Then:

(a) $\inf \mathcal{R} \leq n\lambda$.
(b) $\delta \leq \lambda$.

Proof. (a) In view of our assumptions, there exists a sequence of points $\{p_k \in M\}_{k \in \mathbb{N}}$ such that relations (1)–(3) of Subsection 2.2 are satisfied. Consequently, from equation (4),

$$0 \leq \lim_k (\Delta f)(p_k) = n\lambda - \lim_k \mathcal{R}(p_k).$$

This gives $\inf \mathcal{R} \leq \lim \mathcal{R} \leq n\lambda$.

(b) The proof of this part uses the last inequality with $n\delta \leq \lim \mathcal{R}$. ∎

Theorem 2 and Lemma 2 imply the following

COROLLARY 1. *Consider a complete gradient Ricci soliton* (M, g, f, λ) *of dimension* $n \geq 2$ *for which:*

(i) $\mathrm{Ric} \geq \delta g$ *for some* $\delta \in \mathbb{R}$*;*
(ii) f *is bounded below;*
(iii) $\lambda < 0$.

Then the gradient soliton is trivial.

Proof. The preceding results together imply that $\inf \mathcal{R} = n\lambda$. Thus, the soliton is trivial. ∎

5.2. Compact gradient Ricci solitons in dimension $n \geq 3$

REMARK 4. Let (M, g, f, λ) be compact with $n \geq 3$. Assume that f satisfies the PDE

$$\mathrm{Hess}\, f - \tfrac{1}{n}\Delta f \cdot g = 0. \tag{14}$$

Then f is constant, and the soliton has constant sectional curvature.

Proof. Assume that f is non-constant. Then the PDE (14) implies that (M, g) is conformally equivalent to a standard sphere, moreover (M, g) is an Einstein space as

$$0 = \mathrm{Hess}\, f - \tfrac{1}{n}\Delta f \cdot g = -\big(\mathrm{Ric} - \tfrac{1}{n}\mathcal{R} g\big).$$

Both properties together imply that (M, g) is a Riemannian sphere of curvature $\frac{1}{n-1}\lambda$ (see [13]). ∎

5.2.1. *Shrinking gradient solitons*

LEMMA 3. *Consider a complete gradient Ricci soliton* (M, g, f, λ) *of dimension* $n \geq 2$ *such that:*

(i) *the scalar curvature is bounded below, say* $\mathcal{R} \geq n\delta$ *for some* $\delta \in \mathbb{R}$*;*
(ii) $\lambda > 0$.

Then f *is bounded below.*

Proof. For $p \in M$ equation (6) implies $f(p) \geq \frac{1}{2\lambda}(n\delta - c)$. ∎

REMARK 5. (a) Hamilton's equation (6) has another immediate consequence: If $\lambda > 0$ and f is bounded above then $\mathcal{R}$ is bounded above.

(b) Note that, in Lemma 3 and the foregoing remark (a), the assumptions and Lemma 1 imply that $\mathcal{R} \geq 0$.

5.2.2. *Steady gradient Ricci solitons*

REMARK 6. We assume that:

(i) $\lambda = 0$;
(ii) the Ricci curvature is bounded below, say $\mathrm{Ric} \geq \delta g$ for some $\delta \in \mathbb{R}$;
(iii) f is bounded below.

Then we can apply the **O-Y**-relations to formulas (4) and (6), respectively, which gives:

$$-\lim \mathcal{R} = \lim \Delta f \geq 0 \quad \text{and} \quad \lim \mathcal{R} = c.$$

Hence $c \leq 0$. In particular, if we additionally assume $\mathcal{R} \geq 0$ then $c = 0$ and $\inf \mathcal{R} = 0$.

5.2.3. *Positive scalar curvature*

Proposition 6. *Let (M, g, f, λ) be a complete gradient Ricci soliton and assume that:*

(i) *the Ricci curvature is bounded below, say* $\mathrm{Ric} \geq \delta g$ *for some $\delta \in \mathbb{R}$;*
(ii) *$n\lambda \geq \mathcal{R}$;*
(iii) *$\lambda > 0$ and $\mathcal{R} \neq 0$ everywhere;*
(iv) *f is bounded above.*

Then $0 < n\lambda = \sup \mathcal{R}$.

Proof. First note that Lemma 1 and assumption (iii) imply that $\mathcal{R} > 0$. We continue the proof in the following steps.

(a) From the assumptions we have

$$\lambda \Delta f = \lambda(n\lambda - \mathcal{R}) \geq 0.$$

(b) Lemma 3 allows us to choose $\gamma \in (0, \infty)$ such that

$$F := (\lambda f + \gamma)^{-1/2}$$

is well-defined, positive, and $\inf F > 0$. Then

$$\operatorname{grad} F = -\tfrac{\lambda}{2} F^3 \operatorname{grad} f \quad \text{and} \quad F \Delta F = \tfrac{3}{4}\lambda^2 F^6 \|\operatorname{grad} f\|^2 - F^4 \lambda \Delta f.$$

(c) From the assumptions and Lemma 3 the function f is bounded above and below. By definition F is bounded below, and $\inf F > 0$. Again we apply the maximum principle of Omori–Yau: there exists a sequence $\{p_k\}_k \subset M$ such that relations (1)–(3) of Subsection 2.2 are satisfied by F. Note that $\lim_k \operatorname{grad} F = 0$ implies $\lim_k \operatorname{grad} f = 0$.

(d) We apply the PDE for F:

$$0 \leq \lim_k F \Delta F = -(\inf F)^4 \lambda \Big(n\lambda - \lim_k \mathcal{R} \Big) \leq 0;$$

this and the assumptions finally give $n\lambda = \lim_k \mathcal{R} = \sup \mathcal{R}$; thus Hamilton's equation implies that F takes its infimum where f and thus $\mathcal{R}$ takes its supremum. ∎

Proof of Theorem 1. Assume that (i)–(iv) in Theorem 1 are satisfied, and assume additionally that $n\lambda \notin (\inf \mathcal{R}, \sup \mathcal{R})$. Then

$$\text{either} \quad 0 < n\lambda \leq \inf \mathcal{R} \leq \mathcal{R}, \quad \text{or} \quad \mathcal{R} \leq \sup \mathcal{R} \leq n\lambda,$$

and one of the inequalities in the preceding results is satisfied. This leads to $n\lambda = \inf \mathcal{R}$ or $n\lambda = \sup \mathcal{R}$, and thus λ cannot be outside the closed interval appearing in the assertion. ∎

COROLLARY 2. *Let (M, g, f, λ) be complete with $\lambda \neq 0$. Assume that:*

(i) *the Ricci curvature is bounded below, say* $\mathrm{Ric} \geq \delta g$ *for some* $\delta \in \mathbb{R}$*;*
(ii) $\lambda \mathcal{R} \leq \|\mathrm{Ric}\|^2$*;*
(iii) $\inf \mathcal{R} > 0$*;*
(iv) f *is bounded above.*

Then $n\lambda \in [\inf \mathcal{R}, \sup \mathcal{R}]$.

Proof. The assumptions and Proposition 5 together imply $\lambda > 0$. Now apply Theorem 1. ∎

5.2.4. *Negative scalar curvature.* For non-positively bounded Ricci curvature we study various inequalities between $\mathcal{R}$ and λ.

PROPOSITION 7. *Let (M, g, f, λ) be complete and assume that:*

(i) *the Ricci curvature is non-positively bounded, say*

$$\delta_1 g \geq \mathrm{Ric} \geq \delta_2 g \quad \text{for some } 0 \geq \delta_1 \in \mathbb{R} \text{ and } 0 > \delta_2 \in \mathbb{R};$$

(ii) $\sup \mathcal{R} < 0$*;*
(iii) $n\lambda \geq \mathcal{R}$.

Then $n\lambda = \sup \mathcal{R}$, *and the soliton is expanding.*

Proof. We have $\|\mathrm{Ric}\|^2 \geq \frac{1}{n}\mathcal{R}^2$ and we apply relation (11):

$$\begin{aligned}\tfrac{1}{4}\Delta\mathcal{R}^2 &= \mathcal{R} \cdot \mathrm{Ric}(\operatorname{grad} f, \operatorname{grad} f) + (-\mathcal{R})(\|\mathrm{Ric}\|^2 - \lambda\mathcal{R}) + \tfrac{1}{2}\|\operatorname{grad}\mathcal{R}\|^2 \\ &\geq \tfrac{1}{n}\mathcal{R}^2(n\lambda - \mathcal{R}).\end{aligned}$$

For $0 < \gamma \in \mathbb{R}$ we define $F := (\mathcal{R}^2 + \gamma)^{-1/2}$. From the assumptions we conclude that $\inf F > 0$; we calculate

$$F\Delta F = \tfrac{3}{4}F^6\|\operatorname{grad}\mathcal{R}^2\|^2 - \tfrac{1}{2}F^4\Delta\mathcal{R}^2.$$

Again we apply the maximum principle of Omori–Yau: there exists a sequence $\{p_k\}_k \subset M$ such that the three **O-Y**-relations are satisfied for F. Note that $\lim_k F(p_k) = \inf F > 0$.

We finally arrive at

$$\begin{aligned}0 &\leq \lim_k (F\Delta F)(p_k) \leq \tfrac{1}{2}(\inf F)^4(-\lim_k \Delta\mathcal{R}^2) \\ &\leq -\tfrac{1}{2n}(\inf F)^4 \sup \mathcal{R}^2 \cdot (n\lambda - \lim \mathcal{R}) \leq 0.\end{aligned}$$

The last series of inequalities together with the assumptions give $n\lambda = \lim_k \mathcal{R} = \sup \mathcal{R}$. ∎

PROPOSITION 8. *Let (M, g, f, λ) be complete, and assume that:*

(i) *the Ricci curvature is bounded, say $\delta_1 g \geq \mathrm{Ric} \geq \delta_2 g$ for some $\delta_1, \delta_2 \in \mathbb{R}$;*
(ii) $\lambda < 0$*;*
(iii) *f is bounded.*

Then $n\lambda = \inf \mathcal{R}$, and the gradient soliton is trivial.

Proof. Assumption (ii) and Theorem 2 imply that $n\lambda \leq \mathcal{R} \leq 0$. Then we have $\lambda(n\lambda - \mathcal{R}) \geq 0$, and we proceed as in Proposition 7 above. For an appropriate $0 < \gamma \in \mathbb{R}$ we define

$$F := (f + \gamma)^{-1/2},$$

and we choose γ such that $\inf F > 0$. We calculate

$$F\Delta F = \tfrac{3}{4} F^6 \|\mathrm{grad}\, f\|^2 - \tfrac{1}{2} F^4 \Delta f.$$

Again we apply the principle of Omori–Yau: there exists a sequence $\{p_k\}_k \subset M$ such that the three **O-Y**-relations are satisfied for F; we note that $\lim_k \mathrm{grad}\, F = 0$ implies $\lim_k \mathrm{grad}\, f = 0$. Finally, we get

$$0 \leq \inf F \cdot \lim_k (\Delta F)(p_k) = (\inf F)^4 \big(-\tfrac{1}{2}\big)(n\lambda - \lim_k \mathcal{R}) \leq 0$$

and $n\lambda = \lim_k \mathcal{R} = \inf \mathcal{R}$. From Theorem 2 the soliton must be trivial. ■

Following the lines of the proof of Theorem 1, Propositions 7 and 8 together give:

PROPOSITION 9. *Let (M, g, f, λ) be complete, and assume that:*

(i) *the Ricci curvature is bounded, say $\delta_1 g \geq \mathrm{Ric} \geq \delta_2 g$ for some $0 \geq \delta_1$, $0 > \delta_2 \in \mathbb{R}$;*
(ii) $\sup \mathcal{R} < 0$*;*
(iii) *f is bounded.*

Then $n\lambda \in [\inf \mathcal{R}, \sup \mathcal{R}]$.

COROLLARY 3. *Let (M, g, f, λ) be complete, and assume that:*

(i) *$\delta_1 g \geq \mathrm{Ric} \geq \delta_2 g$ for some $0 \geq \delta_1, 0 > \delta_2 \in \mathbb{R}$;*
(ii) $\sup \mathcal{R} < 0$*;*
(iii) *f is bounded.*

Then (M, g, f, λ) is trivial.

Proof. From the foregoing proposition we conclude that $\lambda < 0$. Now Corollary 1 implies the assertion. ■

5.3. Realization of Ricci-flat gradient Ricci solitons. In this subsection we state two remarks on Ricci-flat gradient Ricci solitons.

(1) Tashiro [12] proved: Let (M, g) be a complete Riemannian manifold and let $0 \neq \lambda \in \mathbb{R}$. Assume that there exists $f \in C^{\infty}(M)$ satisfying

$$\operatorname{Hess} f = \lambda g.$$

Then (M, g) is isometric to the standard flat $\mathbb{R}^n$.

(2) We sketch how we can realize a Ricci-flat gradient soliton as an affine graph immersion of M into $\mathbb{R}^{n+1}$. We refer to [6, Section 3.3.4] for such immersions in relative hypersurface theory.

The case of $\lambda = 0$ and $\operatorname{Ric} \equiv 0$ is trivial. For $\lambda \neq 0$, define $F := \frac{1}{\lambda} f$ and assume (M, g) to be Ricci-flat. Then $\operatorname{Ric} \equiv 0$ and (2) give $\operatorname{Hess} F = g$. A Riemannian metric g generated by a locally strongly convex function F is called a *Calabi* or *Hessian metric.*

Identify a chart $U \subset M$ with a subset $U \subset \mathbb{R}^n$; define locally an affine graph immersion

$$x : U \ni p \mapsto (p, F(p))$$

with locally strongly convex F and *relative* normalization (Y, y), where:

- We have a *constant transversal field* $y := (0, \ldots, 0, 1)$.
- We have a *conormal field* $Y := (-\partial_1 F, \ldots, -\partial_n F, 1)$.
- The *Gauß structure equations* read

  $$\bar{\nabla}_u dx(v) = dx(\nabla_u v) + \operatorname{Hess} F(u, v) y;$$

  here $\bar{\nabla}$ denotes the canonical flat connection of $\mathbb{R}^{n+1}$ and ∇ its tangential projection.
- We have the *affine invariants:*
  - the cubic form with components $C_{ijk} = \partial_k \partial_j \partial_i F$,
  - the relative shape operator $S \equiv 0$.

Then $x(U)$ is part of an improper relative sphere with flat metric. An example is an elliptic paraboloid with Blaschke structure.

6. Gradient Ricci solitons in dimension $n = 2$. As before we consider a gradient Ricci soliton (M, g, f, λ). For $n = 2$ we denote the Gauß curvature of (M, g) by K. Then equation (8) reads

$$K \operatorname{grad} f = \operatorname{grad} K. \tag{15}$$

This and (9) imply

$$K^2 \|\operatorname{grad} f\|^2 = K g(\operatorname{grad} f, \operatorname{grad} K) = \|\operatorname{grad} K\|^2.$$

For $K \neq 0$ we get $f = \ln |K| + b$ for some $b \in \mathbb{R}$. Moreover, if $K \neq 0$, equation (15) is the basis for the following:

KEY LOCAL LEMMA FOR $n = 2$ (Derdziński–Nikčević–Simon; see [8, Lemma 2.1]). *Let* (M, g, f, λ) *be a gradient Ricci soliton. Then:*

(a) $|K| \exp(-f) = \text{const}$, *and thus, everywhere on* M, *either* $K \equiv 0$ *or* $K \neq 0$.
(b) *If* $K = \text{const} \neq 0$ *on* M *then* $\lambda = K$, $f = \text{const}$. *If* $K \equiv 0$ *on* M *then trivially* $\operatorname{Hess} f = \lambda g$.

The above result leads to the following

OBSERVATION. The Gauß curvature K has a constant sign on (M, g, f, λ); thus we have three possibilities: either $K > 0$, or $K \equiv 0$, or $K < 0$.

Additionally, we recall the following PDEs from [8, Section 3].

PROPOSITION 10. *Let* (M, g, f, λ) *be a gradient Ricci soliton,* $n = 2$, *with Gauß curvature* K. *Then:*

$$\Delta K = K \|\operatorname{grad} f\|^2 + 2K(\lambda - K), \tag{16}$$

$$K \Delta K = \|\operatorname{grad} K\|^2 + 2K^2(\lambda - K), \tag{17}$$

$$\Delta K^2 = 4\big[\|\operatorname{grad} K\|^2 + K^2(\lambda - K)\big]. \tag{18}$$

6.1. Complete gradient Ricci solitons for $n = 2$. Considering the above observation on the sign of the Gauß curvature, in each of the cases with $K \neq 0$ we discuss the relations between λ and the Gauß curvature K for complete gradient Ricci solitons.

6.1.1. *Dimension* $n = 2$ *and* $K > 0$. We study two cases of the relation between K and λ.

PROPOSITION 11. *Let* (M, g, f, λ) *be a complete gradient Ricci soliton.*

(i) *If* $\lambda \geq K > 0$ *then* $\lambda = \sup K$ *and* $\lambda = K = \text{const}$, $f = \text{const}$.
(ii) *If* $K > 0$ *and* $K > \lambda$ *then Proposition* 5 *implies* $\lambda \inf K \geq 0$, *thus* $\lambda \geq 0$. *There are two cases:*

- *If* $K \geq \lambda > 0$ *then Ric* $\geq \lambda g$, *thus* (M, g) *is compact from Myers' theorem, and* (M, g, f, λ) *is trivial.*
- *If* $K > 0$ *and* $\lambda = 0$ *then* $2Kg = \operatorname{Hess}_g(-f)$, *and on each chart the function* $-f$ *is locally strongly convex; more precisely,* $f = \ln K + \text{const}$, *thus*

$$\operatorname{Hess}_g \ln K + 2Kg = 0.$$

Moreover, in this case $\inf K = 0$.

Proof. For the proofs of (i)–(ii) see [8, Propositions 6.1, 6.2]. For (ii) with $K > 0$, $\lambda = 0$ it remains to prove that $\inf K = 0$. For this we note that K satisfies the assumptions of the Omori–Yau maximum principle. The foregoing PDE for $\mathrm{Hess}_g \ln K$ implies that

$$K\Delta K = \|\mathrm{grad}\, K\|^2 - 4K^3.$$

The **O-Y**-relations give

$$0 \leq \lim(K\Delta K) = \lim \|\mathrm{grad}\, K\|^2 - 4\lim K^3 = -4(\inf K)^3 \leq 0.$$

Thus $\inf K = 0$. ■

The last proposition and Lemma 1 give:

THEOREM 3. *Let (M, g, f, λ) be a complete gradient Ricci soliton with $n = 2$ and $K \neq 0$, $\lambda > 0$. If $\lambda \notin (\inf K, \sup K)$ then (M, g, f, λ) is trivial.*

A result of Naber [7, Theorem 1.2] states sufficient conditions for a Ricci soliton to be gradient; this gives the following corollary:

COROLLARY 4. *Let $(M, g, X, \lambda > 0)$ be a complete Ricci soliton with $n = 2$ and $0 < K < \delta$ for some $\delta \in \mathbb{R}$. If $\lambda \notin (\inf K, \sup K)$ then (M, g, X, λ) is trivial.*

6.1.2. *Dimension $n = 2$ and $K < 0$.* Again we discuss two different cases:

PROPOSITION 12. *Let (M, g, f, λ) be complete and $K < 0$.*

(i) *Assume that $\lambda \geq K \geq \delta$ for some $\delta \in \mathbb{R}$ and $\sup K \neq 0$. Then $\lambda = \sup K < 0$, thus (M, g, f, λ) is expanding.*
(ii) *Assume that $0 > K \geq \lambda$. Then $\lambda = \inf K$.*

Proof. We apply the Omori–Yau techniques developed above for the PDE

$$K^2\Delta K^2 = 4K^2(\|\mathrm{grad}\, K\|^2 + K^2(\lambda - K))$$

satisfied by the function K^2. ■

REMARK 7. Note that in the preceding proposition the Gauß curvature K is bounded, thus f is bounded by the Key Local Lemma. In both cases (i) and (ii) of Proposition 12 the soliton (M, g, f, λ) is expanding.

From the statements in (i) and (ii) we get:

THEOREM 4. *If $K < 0$ with $\sup K \neq 0$ and if K is bounded below then:*

(i) *$\lambda < 0$, i.e. (M, g, f, λ) is expanding;*
(ii) *if $\lambda \notin [\inf K, \sup K]$ then (M, g, f, λ) is trivial.*

7. Gradient Ricci solitons in dimension $n = 3$. We recall Proposition 2 and calculate the right-hand side terms appearing in it for $n = 3$.

7.1. Sectional curvature and Ricci curvature. For $n = 3$, it is well known that the Ricci curvature determines the sectional curvature as follows:

$$2\kappa_{12} = \rho_1 + \rho_2 - \rho_3 = \mathcal{R} - 2\rho_3,$$
$$2\kappa_{13} = \rho_1 + \rho_3 - \rho_2 = \mathcal{R} - 2\rho_2,$$
$$2\kappa_{23} = \rho_2 + \rho_3 - \rho_1 = \mathcal{R} - 2\rho_1.$$

With elementary calculations one verifies the following relations.

LEMMA 4. *Let (M, g, f, λ) be a gradient Ricci soliton, and assume that $n = 3$. Then*

$$2\sum_{i<j}\kappa_{ij}(\rho_i - \rho_j)^2 = 4\sum_i \rho_i^3 + 6\rho_1\rho_2\rho_3 - 2\mathcal{R}\|\mathrm{Ric}\|^2,$$
$$\tfrac{1}{2}(\lambda + \mathcal{R})\|\mathrm{Ric}\|^2 + \tfrac{1}{2}(3\lambda - \mathcal{R})\|\mathrm{Ric}\|^2 = 2\lambda\|\mathrm{Ric}\|^2,$$
$$2\sum_{i<j}\kappa_{ij}(\rho_i - \rho_j)^2 - 2\sum_i(\rho_i)^3 + 2\lambda\|\mathrm{Ric}\|^2 = 2\sum_i(\rho_i)^3 + 6\rho_1\rho_2\rho_3 + 2(\lambda - \mathcal{R})\|\mathrm{Ric}\|^2.$$

7.2. The Laplacian $\Delta\,\|\mathrm{Ric}\|^2$. The calculations in the foregoing subsection give

PROPOSITION 13.

(a) *The Ricci tensor* Ric *of a gradient Ricci soliton in dimension $n = 3$ satisfies*

$$\begin{aligned}\tfrac{1}{2}\Delta\|\mathrm{Ric}\|^2 &= 2\sum_{i<j}\kappa_{ij}(\rho_i - \rho_j)^2 + \tfrac{1}{2}(\lambda + \mathcal{R})\|\mathrm{Ric}\|^2 + \|\nabla\,\mathrm{Ric}\,\|^2\\ &\quad - 2\sum_i(\rho_i)^3 + \tfrac{1}{2}\,\mathrm{grad}\,\|\mathrm{Ric}\|^2 \otimes \mathrm{grad}\,f + \tfrac{1}{2}\|\mathrm{Ric}\|^2\Delta\,f\\ &= 2\sum_i(\rho_i)^3 + 6\rho_1\rho_2\rho_3 + 2(\lambda - \mathcal{R})\|\mathrm{Ric}\|^2 + \|\nabla\,\mathrm{Ric}\,\|^2\\ &\quad + \tfrac{1}{2}\,\mathrm{grad}\,\|\mathrm{Ric}\|^2 \otimes \mathrm{grad}\,f.\end{aligned}$$

(b) *If* $\mathrm{Ric} \geq 0$ *then*

$$\tfrac{1}{2}\Delta\,\|\mathrm{Ric}\|^2 \geq 2(\lambda - (\mathcal{R} - \rho_{\inf}))\|\mathrm{Ric}\|^2 + 6\rho_1\rho_2\rho_3 + \tfrac{1}{2}\,\mathrm{grad}(\|\mathrm{Ric}\|^2) \otimes \mathrm{grad}\,f,$$

where $\rho_{\inf} := \inf_{p\in M}\{\rho_i(p) \mid i = 1, 2, 3\}$.

THEOREM 5. *Let (M, g) be a complete gradient Ricci soliton of dimension $n = 3$ with the following properties:*

- $\delta g \geq \mathrm{Ric} \geq 0$ *for some* $\delta \in \mathbb{R}$*;*
- $\lambda \geq \sup\,(\mathcal{R} - \rho_{\inf})$.

Then $\rho_{\inf} = 0$ *and* $0 \leq \lambda = \sup(\mathcal{R}) \leq \delta$.

Proof. For some positive $\gamma \in \mathbb{R}$ we define

$$F := (\|\mathrm{Ric}\|^2 + \gamma)^{-1/2}.$$

The function F has the following properties:

(i) $F > 0$ on M;
(ii) $\inf F > 0$ on M;
(iii) $\Delta F = \frac{3}{4} F^5 \big\|\mathrm{grad}\,\|\mathrm{Ric}\|^2\big\|^2 - F^3 \cdot \frac{1}{2}\Delta\|\mathrm{Ric}\|^2$;
(iv) $\mathrm{grad}\, F = 0 \Leftrightarrow \mathrm{grad}\,\|\mathrm{Ric}\|^2 = 0$.

Again we apply the maximum principle of Omori–Yau: There exists a sequence $\{p_k\}_k$ of points such that the three **O-Y**-relations for the function F are satisfied. We note:

$$\text{If} \quad \lim_k F(p_k) = \inf F \quad \text{then} \quad \lim_k \|\mathrm{Ric}\|^2 = \sup \|\mathrm{Ric}\|^2.$$

Moreover, under the assumptions of the theorem,

$$\lim_k \tfrac{1}{2}\Delta\|\mathrm{Ric}\|^2 \geq 2(\lambda - (\mathcal{R} - \rho_{\inf})) \lim_k \|\mathrm{Ric}\|^2 + 6 \lim_k (\rho_1\rho_2\rho_3) \geq 0.$$

As a consequence,

$$\begin{aligned} 0 \leq \lim_k \Delta F &= \lim_k\, (-F^3)\tfrac{1}{2}\Delta\|\mathrm{Ric}\|^2 \\ &\leq \lim_k\,(-F^3) \lim_k \big[2(\lambda - (\mathcal{R} - \rho_{\inf}))\|\mathrm{Ric}\|^2 + 6\rho_1\rho_2\rho_3\big] \leq 0. \end{aligned}$$

These inequalities and the assumptions together imply $\rho_{\inf} = 0$ and $\lambda = \sup \mathcal{R}$. ∎

Remark 8. From the assumptions and $\lambda = \sup \mathcal{R}$ we know that $\lambda \geq 0$. But $\lambda = 0$ leads to $\mathrm{Ric} \equiv 0$ and $\mathrm{Hess}\, f \equiv 0$, and therefore in the soliton equation (2) all terms vanish identically.

Thus only the case $\delta \geq \lambda > 0$ is left. First it follows from Hamilton's equation (6) and the assumptions that $2\lambda f \geq -c$, therefore f is bounded below. Then, as in Proposition 4, we can prove that

$$2\lambda \inf f \leq n\lambda - c.$$

8. Gradient Ricci solitons in dimension $n = 4$. In this section we apply the formula of Proposition 2 in dimension $n = 4$.

Proposition 14. *The Ricci tensor* $\mathrm{Ric} \geq 0$ *of a gradient Ricci soliton in dimension* $n = 4$ *satisfies*

$$\begin{aligned} \tfrac{1}{2}\Delta\|\mathrm{Ric}\|^2 = 2\sum_{i<j} \kappa_{ij}(\rho_i - \rho_j)^2 + \tfrac{1}{2}\mathcal{R}\|\mathrm{Ric}\|^2 + \|\nabla \mathrm{Ric}\|^2 \\ - 2\sum_i (\rho_i)^3 + \tfrac{1}{2}\,\mathrm{grad}(\|\mathrm{Ric}\|^2) \otimes \mathrm{grad}\, f + \tfrac{1}{2}\|\mathrm{Ric}\|^2 \Delta f \end{aligned}$$

$$= 2\sum_{i<j} \kappa_{ij}(\rho_i - \rho_j)^2 + 2\lambda\|\mathrm{Ric}\|^2 - 2\sum_i (\rho_i)^3 + \|\nabla \mathrm{Ric}\|^2 + \tfrac{1}{2}\,\mathrm{grad}\,\|\mathrm{Ric}\|^2 \otimes \mathrm{grad}\, f.$$

THEOREM 6. *Let (M, g, f, λ) be a complete gradient Ricci soliton of dimension $n = 4$ satisfying the following relations:*

- *the sectional curvature is non-negative;*
- $\mathrm{Ric} \le \delta g$ *for some* $0 < \delta \in \mathbb{R}$*;*
- $\lambda \ge \rho_{\sup} := \sup_{p\in M}\{\rho_i(p) \mid i = 1, 2, 3, 4\}$.

Then $\lambda = \rho_{\sup}$.

Proof. For some positive $\gamma \in \mathbb{R}$ we define

$$F := (\|\mathrm{Ric}\|^2 + \gamma)^{-1/2}.$$

As above, F has the following properties:

(i) $F > 0$ on M;
(ii) $\inf F > 0$ on M;
(iii) $\Delta F = \frac{3}{4} F^5 \big\|\mathrm{grad}\,\|\mathrm{Ric}\|^2\big\|^2 - F^3 \frac{1}{2}\Delta\|\mathrm{Ric}\|^2$;
(iv) $\mathrm{grad}\, F = 0 \Leftrightarrow \mathrm{grad}\,\|\mathrm{Ric}\,\|^2 = 0$.

From Proposition 14 we have

$$\tfrac{1}{2}\Delta\,\|\mathrm{Ric}\|^2 = 2\sum_{i<j} \kappa_{ij}(\rho_i - \rho_j)^2 + 2\lambda\|\mathrm{Ric}\|^2 - 2\sum_i (\rho_i)^3 + \|\nabla \mathrm{Ric}\|^2 + \tfrac{1}{2}\,\mathrm{grad}(\|\mathrm{Ric}\|^2) \otimes \mathrm{grad}\, f.$$

We calculate the term

$$2\lambda\|\mathrm{Ric}\|^2 - 2\sum_i (\rho_i)^3 \ge 2(\lambda - \rho_{\sup})\|\mathrm{Ric}\|^2.$$

Again we apply the maximum principle of Omori–Yau, which gives

$$0 \le \lim_k \Delta F = \lim_k\,(-F^3)\tfrac{1}{2}\Delta\|\mathrm{Ric}\|^2 \le \lim_k\,(-F^3)(\lambda - \rho_{\sup}) \le 0.$$

The assertion follows as before. ■

REMARK 9. Under the assumptions of the preceding Theorem 6, if additionally $\rho_{\sup} > 0$ then $\mathcal{R} \ge 0$; this follows from $\lambda \ge \rho_{\sup} > 0$ and Lemma 1.

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Udo Simon
Mathematisches Institut, MA 8-3
TU Berlin
Straße des 17. Juni 136
D-10623 Berlin, Germany
E-mail: simon@math.tu-berlin.de

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