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KOSZUL DUALITY FOR N-KOSZUL ALGEBRAS

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Abstract. The correspondence between the category of modules over a graded algebra and the category of graded modules over its Yoneda algebra was studied in [8] by means of A_∞ algebras; this relation is very well understood for Koszul algebras (see for example [5], [6]). It is of interest to look for cases such that there exists a duality generalizing the Koszul situation. In this paper we will study N-Koszul algebras [1], [7], [9] for which such a duality exists.

Dualities for N-Koszul algebras. In [10], we studied a generalization of Yoshino's results [12] concerning the relation between the exterior algebra and the polynomial algebra, very close in line with the famous paper by Bernstein–Gelfand–Gelfand [2]–[4].

It was proved there that for Koszul algebras there exists a duality between graded modules and linear complexes of projective modules over the Yoneda algebra which restricts to a duality between Koszul modules and complexes $(P^\bullet, d^\bullet)$ of finitely generated projective modules over the Yoneda algebra such that $P^j = 0$ for $j < 0$ and $H^j(P^\bullet) = 0$ for $j \neq 0$. The aim of this paper is generalize this theorem to a particular class of N-Koszul algebras.

We will start by recalling some definitions and results from [1], [7], [9].

Definition 1. Let $\Lambda = KQ/I$ be a graded factor of a path algebra. Let N be a positive integer and $\delta : \mathbb{Z} \to \mathbb{Z}$ the function $\delta(2k) = kN$ and $\delta(2k+1) = kN + 1$. We say that a finitely generated graded module M is *N-Koszul* if it has a graded projective resolution $\to P^j \to P^{j-1} \to \cdots \to P^1 \to P^0 \to M \to 0$ such that each $P^{(j)}$ is finitely generated with generators in degree $\delta(j)$. If all graded simple modules with support in degree zero are N-Koszul, then we say that Λ is *N-Koszul.*

Definition 2. A graded factor $\Lambda = KQ/I$ of a path algebra is *N-homogeneous* if the ideal I is generated by homogeneous elements of degree N.

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N-Koszul algebras are a natural generalization of Koszul algebras. Indeed, 2-Koszul algebras are just the Koszul algebras. As in the Koszul situation, if $\Lambda = KQ/I$ is N-Koszul, then it is N-homogeneous and we may consider its *homogeneous dual algebra* $\Lambda^! = KQ^{\mathrm{op}}/\langle I_N^\perp \rangle$, where $\langle I_N^\perp \rangle$ is the orthogonal ideal constructed in a way similar to the quadratic case. There will be some differences from the classical situation:

(a) The algebra $\Lambda^! = KQ^{\mathrm{op}}/\langle I_N^\perp \rangle$ will not in general be N-Koszul.

(b) $\Lambda^!$ is not isomorphic to the Yoneda algebra for $N > 2$.

For (a) it is very easy to give examples such that $\Lambda^!$ is N-Koszul and examples where it is not [9].

For (b) we have the following result:

THEOREM 1 ([7]). *Let $\Lambda = KQ/I$ be an N-Koszul algebra with $N \geq 2$, and $\Lambda^! = KQ^{\mathrm{op}}/\langle I_N^\perp \rangle$ its homogeneous dual algebra. Then the Yoneda algebra $\Gamma = \bigoplus_{k\geq 0} \mathrm{Ext}_\Lambda^k(\Lambda_0, \Lambda_0)$ is isomorphic as a graded algebra to the algebra $B = \bigoplus_{j\geq 0} B_j$ defined in the following way: $B_n = \Lambda^!_{\delta(n)}$ as vector spaces, and multiplication in B is defined as follows: if $x \in B_n$ and $y \in B_m$, then $x \cdot y = 0$ if both m and n are odd, and $x \cdot y$ is the product in $\Lambda^!$ if either n or m is even.* ∎

In this paper we will consider N-Koszul algebras $\Lambda = KQ/I$ such that $\Lambda^!$ is N-Koszul. We will see that under mild restrictions on such algebras, there exists a natural generalization of Koszul duality.

We will need the following:

LEMMA 1. *Let Λ be any ring. Consider the following commutative diagram of Λ-modules of finite length*:

$$
\begin{tikzcd}
 & & & & & & 0 \arrow[d] & \\
 & & & & & 0 \arrow[r]\arrow[d] & A_{n,n} \arrow[d] & \\
 & & & & 0 \arrow[r] & A_{n-1,n-1} \arrow[r]\arrow[d] & A_{n-1,n} \arrow[r]\arrow[d] & 0 \\
 & & & 0 \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & & 0 \arrow[r]\arrow[d] & A_{3,3} \arrow[r]\arrow[d] & \cdots \arrow[r] & A_{3,n-1} \arrow[r]\arrow[d] & A_{3,n} \arrow[r]\arrow[d] & 0 \\
 & 0 \arrow[r]\arrow[d] & A_{2,2} \arrow[r]\arrow[d] & A_{2,3} \arrow[r]\arrow[d] & \cdots \arrow[r] & A_{2,n-1} \arrow[r]\arrow[d] & A_{2,n} \arrow[r]\arrow[d] & 0 \\
0 \arrow[r] & A_{1,1} \arrow[r] & A_{1,2} \arrow[r]\arrow[d] & A_{1,3} \arrow[r]\arrow[d] & \cdots \arrow[r] & A_{1,n-1} \arrow[r]\arrow[d] & A_{1,n} \arrow[r]\arrow[d] & 0 \\
 & & 0 & 0 & & 0 & 0 &
\end{tikzcd}
\tag{1}
$$

such that:

(i) *All columns are exact.*
(ii) *The row* $0 \to A_{1,1} \to A_{1,2} \to \cdots \to A_{1,n-1} \to A_{1,n} \to 0$ *is a complex.*
(iii) *All other rows are exact.*
(iv) $0 \to A_{1,1} \to A_{1,2}$ *is exact.*
(v) $A_{1,1} \cong A_{n,n}$.

Then

$$0 \to A_{1,1} \to A_{1,2} \to A_{1,3} \to \cdots \to A_{1,n-1} \to A_{1,n} \to 0$$

is also exact.

Proof. By induction on n. For $n = 1$ or $n = 2$ there is nothing to prove. Consider the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & A_{3,3} \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & A_{2,2} \arrow[r] \arrow[d] & A_{2,3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{1,1} \arrow[r] & A_{1,2} \arrow[r] \arrow[d] & A_{1,3} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact columns, $0 \to A_{1,1} \to A_{1,2}$ exact and $A_{1,1} \cong A_{3,3}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[d] & 0 \arrow[d] & \\
 & A_{1,1} \arrow[r, dashed] \arrow[dr] & C \arrow[r] \arrow[d] & A_{3,3} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & A_{2,2} \arrow[r] \arrow[d] & A_{2,3} \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & A_{1,3} \arrow[r, "1"] \arrow[d] & A_{1,3} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where C is the kernel of $A_{2,2} \to A_{1,3}$. Hence $C \cong A_{3,3}$ and the induced map $A_{1,1} \to C$ is a monomorphism. It follows, by a length argument, that the map is an isomorphism. Therefore $0 \to A_{1,1} \to A_{1,2} \to A_{1,3} \to 0$ is exact.

Now assume the result is true for all diagrams of size $n - 1 \times n - 1$. We have the following commutative diagram:

$$
(2)\qquad
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & A_{1,1} & \to & C_{2,3} & \to & C_{2,4} & \to & \cdots & \to & C_{2,n-1} & \to & C_{2,n} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & A_{2,2} & \to & A_{2,3} & \to & A_{2,3} & \to & \cdots & \to & A_{2,n-1} & \to & A_{2,n} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & C & \to & A_{1,3} & \to & A_{1,4} & \to & \cdots & \to & A_{1,n-1} & \to & A_{1,n} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & & & 0 & & 0
\end{array}
$$

where $C_{2,j}$ is the kernel of $A_{2,j} \to A_{1,j}$ for $j \geq 3$ and C is the cokernel of $0 \to A_{1,1} \to A_{2,2}$.

We have an induced commutative diagram

$$
(3)\qquad
\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & \downarrow \\
 & & & & & & & & & & 0 & \to & A_{n,n} \\
 & & & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & 0 & \to & A_{n-1,n-1} & \to & A_{n-1,n} \to 0 \\
 & & & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & & & \vdots & & \vdots \\
 & & & & & & 0 & & & & & & \\
 & & & & & & \downarrow & & & & \downarrow & & \downarrow \\
 & & & & 0 & \to & A_{4,4} & \to & \cdots & \to & A_{4,n-1} & \to & A_{4,n} \to 0 \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & 0 & \to & A_{3,3} & \to & A_{3,4} & \to & \cdots & \to & A_{3,n-1} & \to & A_{3,n} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & A_{1,1} & \to & C_{2,3} & \to & C_{2,4} & \to & \cdots & \to & C_{2,n-1} & \to & C_{2,n} \to 0 \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0 & & & & 0 & & 0
\end{array}
$$

which is a diagram of size $n-1 \times n-1$ satisfying the conditions of the lemma, hence, by the induction hypothesis, the row $0 \to A_{1,1} \to C_{2,3} \to C_{2,4} \to \cdots \to C_{2,n-1} \to C_{2,n} \to 0$ is exact.

It follows that the diagram (2) is an exact sequence of complexes such that two of them are exact. Then, by the long homology sequence, the third one, $0 \to C \to A_{1,3} \to \cdots \to A_{1,n-1} \to A_{1,n} \to 0$, is also exact, as claimed. ■

Corollary 1. *Assume we have a commutative diagram* (1) *as in Lemma* 1 *such that*:

(i) *All columns are exact.*
(ii) $0 \to A_{2,2} \to A_{2,3} \to \cdots \to A_{2,n-1} \to A_{2,n} \to 0$ *is a complex and the remaining rows are exact.*

(iii) $A_{1,1} \cong A_{n,n}$.
(iv) $0 \to A_{2,2} \to A_{2,3}$ *is exact.*

Then $0 \to A_{2,2} \to A_{2,3} \to \cdots \to A_{2,n-1} \to A_{2,n} \to 0$ *is exact.*

Proof. As in the proof of the lemma, we have a commutative diagram (2) and a diagram of type (3). Hence, by the lemma the sequence $0 \to A_{1,1} \to C_{2,3} \to C_{2,4} \to \cdots \to C_{2,n-1} \to C_{2,n} \to 0$ is exact and in the diagram (2) we have an exact sequence of complexes and two of them are acyclic. Then, by the long homology sequence, the middle complex $0 \to A_{2,2} \to A_{2,3} \to A_{2,4} \to \cdots \to A_{2,n-1} \to A_{2,n} \to 0$ is exact. ∎

Given a graded quiver algebra $\Lambda = KQ/I = \bigoplus_{j\geq 0}\Lambda_j$, a finitely generated graded projective Λ-module P is isomorphic to $\bigoplus_{s=1}^m(\bigoplus_{j\geq 0}\Lambda_j)e_{k_s}$ where the e_{k_s} denote, not necessarily distinct, primitive idempotents of Λ. Hence

$$P \cong \bigoplus_{s=1}^{m}\Big(\bigoplus_{j\geq 0}\Lambda_j e_{k_s}\Big) \cong \bigoplus_{j\geq 0}\Big(\bigoplus_{s=1}^{m}\Lambda_j \underset{\Lambda_0}{\otimes} \Lambda_0 e_{k_s}\Big) \cong \bigoplus_{j\geq 0}\Lambda_j \underset{\Lambda_0}{\otimes} \Big(\bigoplus_{s=1}^{m}\Lambda_0 e_{k_s}\Big).$$

Consider the right Λ_0-module $V = \bigoplus_{s=1}^m e_{k_s}\Lambda_0$. Then $P \cong \Lambda \otimes_{\Lambda_0} V^*$, with $V^* = \mathrm{Hom}_{\Lambda_0}(V, \Lambda_0)$.

Given a graded Λ-module M, the module $M[n]$ is defined by $M[n]_j = M_{n+j}$.

The following proposition is a consequence of [9]; we give the proof for completeness.

PROPOSITION 1. *Let* $\Lambda = KQ/I$ *be an* N*-Koszul algebra with* $N \geq 2$, *and* $\Lambda^!$ *its homogeneous dual algebra. Then we have exact sequences*

$$\begin{aligned}\to \Lambda\otimes(\Lambda^!_{3N})^*[-3N] \to \Lambda\otimes(\Lambda^!_{2N+1})^*[-(2N+1)] \to \Lambda\otimes(\Lambda^!_{2N})^*[-2N]\\ \to \Lambda\otimes(\Lambda^!_{N+1})^*[-(N+1)] \to \Lambda\otimes(\Lambda^!_{N})^*[-N]\\ \to \Lambda\otimes(\Lambda^!_{1})^*[-1] \to \Lambda\otimes(\Lambda^!_{0})^*[0] \to \Lambda_0 \to 0\end{aligned}$$

and

$$\begin{aligned}\to (\Lambda^!_{3N})^*\otimes\Lambda[-3N] \to (\Lambda^!_{2N+1})^*\otimes\Lambda[-(2N+1)] \to (\Lambda^!_{2N})^*\otimes\Lambda[-2N]\\ \to (\Lambda^!_{N+1})^*\otimes\Lambda[-(N+1)] \to (\Lambda^!_{N})^*\otimes\Lambda[-N] \to (\Lambda^!_{1})^*\otimes\Lambda[-1]\\ \to (\Lambda^!_{0})^*\otimes\Lambda[0] \to \Lambda_0 \to 0.\end{aligned}$$

Here for a right Λ_0*-module* V, $V^* = \mathrm{Hom}_{\Lambda_0}(V, \Lambda_0)$.

Proof. By hypothesis, there exists a minimal graded projective resolution of the Λ-module Λ_0:

$$\begin{aligned}\to \Lambda\otimes(V_4)^*[-2N] \to \Lambda\otimes(V_3)^*[-(N+1)] \to \Lambda\otimes(V_2)^*[-N]\\ \to \Lambda\otimes(V_1)^*[-1] \to \Lambda\otimes(V_0)^*[0] \to \Lambda_0 \to 0.\end{aligned}$$

Hence

$$\mathrm{Ext}^n_\Lambda(\Lambda_0,\Lambda_0)\cong \mathrm{Hom}_\Lambda(\Lambda\otimes (V_n)^*,\Lambda_0)\cong \mathrm{Hom}_\Lambda(\Lambda,(V_n)^{**})\cong (V_n)^{**}\cong V_n.$$

It was proved in [1] and [7] that $\Lambda^!_{\delta(n)}\cong \mathrm{Ext}^n_\Lambda(\Lambda_0,\Lambda_0)$, so the exactness of the first sequence follows. The exactness of the second sequence follows by using right modules and the fact proved in [1], [7] that the opposite algebra of an N-Koszul algebra is N-Koszul. ■

Omitting arrows and looking at proper degrees, we can display the left resolution of Λ_0 in a matrix as follows:

$$\begin{array}{ccccc}
 & & & & \Lambda_0\\
 & & & \Lambda_0\otimes(\Lambda^!_1)^* & \Lambda_1\\
 & & & \Lambda_1\otimes(\Lambda^!_1)^* & \Lambda_2\\
 & & & & \vdots\\
 & & & \Lambda_{N-2}\otimes(\Lambda^!_1)^* & \Lambda_{N-1}\\
 & & \Lambda_0\otimes(\Lambda^!_N)^* & \Lambda_{N-1}\otimes(\Lambda^!_1)^* & \Lambda_N\\
 & \Lambda_0\otimes(\Lambda^!_{N+1})^* & \Lambda_1\otimes(\Lambda^!_N)^* & \Lambda_N\otimes(\Lambda^!_1)^* & \Lambda_{N+1}\\
 & \Lambda_1\otimes(\Lambda^!_{N+1})^* & \Lambda_2\otimes(\Lambda^!_N)^* & \Lambda_{N+1}\otimes(\Lambda^!_1)^* & \Lambda_{N+2}\\
 & & & & \vdots\\
 & \Lambda_{N-2}\otimes(\Lambda^!_{N+1})^* & \Lambda_{N-1}\otimes(\Lambda^!_N)^* & \Lambda_{2N-2}\otimes(\Lambda^!_1)^* & \Lambda_{2N-1}\\
\Lambda_0\otimes(\Lambda^!_{2N})^* & \Lambda_{N-1}\otimes(\Lambda^!_{N+1})^* & \Lambda_N\otimes(\Lambda^!_N)^* & \Lambda_{2N-1}\otimes(\Lambda^!_1)^* & \Lambda_{2N}\\
\cdots & \cdots & \cdots & \cdots & \Lambda_{2N+1}
\end{array}$$

Hence we get exact sequences of Λ_0-modules

$$0\to(\Lambda^!_{kN})^*\to\Lambda_{N-1}\otimes(\Lambda^!_{(k-1)N+1})^*\to\cdots\to\Lambda_{(k-1)N-1}\otimes(\Lambda^!_{N+1})^*$$
$$\to\Lambda_{(k-1)N}\otimes(\Lambda^!_N)^*\to\Lambda_{kN-1}\otimes(\Lambda^!_1)^*\to\Lambda_{kN}\to 0$$

and

$$0\to(\Lambda^!_{kN+1})^*\to\Lambda_1\otimes(\Lambda^!_{kN})^*\to\Lambda_N\otimes(\Lambda^!_{(k-1)N+1})^*\to\cdots$$
$$\to\Lambda_{(k-1)N}\otimes(\Lambda^!_{N+1})^*\to\Lambda_{(k-1)N+1}\otimes(\Lambda^!_N)^*\to\Lambda_{kN}\otimes(\Lambda^!_1)^*\to\Lambda_{kN+1}\to 0.$$

Using the fact that Λ^{op} is N-Koszul we obtain exact sequences

$$0\to(\Lambda^!_{kN})^*\to(\Lambda^!_{(k-1)N+1})^*\otimes\Lambda_{N-1}\to\cdots\to(\Lambda^!_{N+1})^*\otimes\Lambda_{(k-1)N-1}$$
$$\to(\Lambda^!_N)^*\otimes\Lambda_{(k-1)N}\to(\Lambda^!_1)^*\otimes\Lambda_{kN-1}\to\Lambda_{kN}\to 0,$$

$$0\to(\Lambda^!_{kN+1})^*\to(\Lambda^!_{kN})^*\otimes\Lambda_1\to(\Lambda^!_{(k-1)N+1})^*\otimes\Lambda_N\to\cdots$$
$$\to(\Lambda^!_{N+1})^*\otimes\Lambda_{(k-1)N}\to(\Lambda^!_N)^*\otimes\Lambda_{(k-1)N+1}\to(\Lambda^!_1)^*\otimes\Lambda_{kN}\to\Lambda_{kN+1}\to 0.$$

Dualizing the previous sequences, we obtain exact sequences

$$0 \to (\Lambda_{kN})^* \to \Lambda^!_1 \otimes (\Lambda_{kN-1})^* \to \cdots \to \Lambda^!_{(k-1)N} \otimes (\Lambda_N)^* \\ \to \Lambda^!_{(k-1)N+1} \otimes (\Lambda_{N-1})^* \to \Lambda^!_{kN} \to 0,$$

$$0 \to (\Lambda_{kN+1})^* \to \Lambda^!_1 \otimes (\Lambda_{kN})^* \to \cdots \to \Lambda^!_{(k-1)N+1} \otimes (\Lambda_N)^* \\ \to \Lambda^!_{kN} \otimes (\Lambda_1)^* \to \Lambda^!_{kN+1} \to 0,$$

and the corresponding exact sequences

$$0 \to (\Lambda_{kN})^* \to (\Lambda_{kN-1})^* \otimes \Lambda^!_1 \to \cdots \to (\Lambda_N)^* \otimes \Lambda^!_{(k-1)N} \\ \to (\Lambda_{N-1})^* \otimes \Lambda^!_{(k-1)N+1} \to \Lambda^!_{kN} \to 0,$$

$$0 \to (\Lambda_{kN+1})^* \to (\Lambda_{kN})^* \otimes \Lambda^!_1 \to \cdots \to (\Lambda_N)^* \otimes \Lambda^!_{(k-1)N+1} \\ \to (\Lambda_1)^* \otimes \Lambda^!_{kN} \to \Lambda^!_{kN+1} \to 0.$$

If we now assume that $\Lambda^!$ is also N-Koszul, then interchanging the roles of Λ and $\Lambda^!$ we get the corresponding exact sequences for Λ_{kN} and Λ_{kN+1}. We now prove the following by induction on k.

PROPOSITION 2. *Assume that $\Lambda = KQ/I$ and $\Lambda^!$ are N-Koszul with $N \geq 2$ and that the quiver Q is connected and has no sources. Then for any $k \geq 2$, there exist exact sequences*

$$0 \to (\Lambda_{kN-1})^* \to \Lambda^!_{N-1} \otimes (\Lambda_{(k-1)N})^* \to \cdots \to \Lambda^!_{(k-2)N} \otimes (\Lambda_{2N-1})^* \\ \to \Lambda^!_{(k-1)N-1} \otimes (\Lambda_N)^* \to \Lambda^!_{(k-1)N} \otimes (\Lambda_{N-1})^* \to \Lambda^!_{kN-1} \to 0.$$

Proof. If $k = 2$, then we have a commutative diagram of the form

$$\begin{array}{ccccccccccc}
 & & & & & & & & 0 & & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & 0 & \to & A_{5,5} & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \to & A_{4,4} & \to & A_{4,5} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & A_{3,3} & \to & A_{3,4} & \to & A_{3,5} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{2,2} & \to & A_{2,3} & \to & A_{2,4} & \to & A_{2,5} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to A_{1,1} & \to & A_{1,2} & \to & A_{1,3} & \to & A_{1,4} & \to & A_{1,5} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

with entries:

$A_{5,5} = A_{1,1} = (\Lambda_{2N})^*$, $\quad A_{4,4} = (\Lambda_{N+1})^* \otimes (\Lambda_{N-1})^*$,
$A_{4,5} = (\Lambda_{N+1})^* \otimes \Lambda^!_{N-1}$, $\quad A_{3,3} = (\Lambda_N)^* \otimes (\Lambda_N)^*$,
$A_{3,4} = (\Lambda_N)^* \otimes \Lambda^!_1 \otimes (\Lambda_{N-1})^*$, $\quad A_{3,5} = (\Lambda_N)^* \otimes \Lambda^!_N$,

$A_{2,2} = \Lambda^!_1 \otimes (\Lambda_{2N-1})^*$, $\quad A_{2,3} = \Lambda^!_1 \otimes \Lambda^!_{N-1} \otimes (\Lambda_N)^*$,
$A_{2,4} = \Lambda^!_1 \otimes \Lambda^!_N \otimes (\Lambda_{N-1})^*$, $\quad A_{2,5} = \Lambda^!_1 \otimes \Lambda^!_{2N-1}$,
$A_{1,2} = \Lambda^!_1 \otimes (\Lambda_{2N-1})^*$, $\quad A_{1,3} = \Lambda^!_N \otimes (\Lambda_N)^*$,
$A_{1,4} = \Lambda^!_{N+1} \otimes (\Lambda_{N-1})^*$, $\quad A_{1,5} = \Lambda^!_{2N}$.
Since both Λ and $\Lambda^!$ are N-Koszul the columns are exact. By the above observations, so are all rows except perhaps

$$(4) \qquad 0 \to \Lambda^!_1 \otimes (\Lambda_{2N-1})^* \to \Lambda^!_1 \otimes \Lambda^!_{N-1} \otimes (\Lambda_N)^* \\ \to \Lambda^!_1 \otimes \Lambda^!_N \otimes (\Lambda_{N-1})^* \to \Lambda^!_1 \otimes \Lambda^!_{2N-1} \to 0.$$

The product $\Lambda_N \otimes \Lambda_{N-1} \to \Lambda_{2N-1} \to 0$ induces a monomorphism $0 \to (\Lambda_{2N-1})^* \to \Lambda^!_{N-1} \otimes (\Lambda_N)^*$, hence a monomorphism $0 \to \Lambda^!_1 \otimes (\Lambda_{2N-1})^* \to \Lambda^!_1 \otimes \Lambda^!_{N-1} \otimes (\Lambda_N)^*$. Now Corollary 1 shows that the sequence (4) is exact.

Since we are assuming that Q has no sources, $\Lambda^!_1$ is a projective generator as a right Λ_0-module. It follows that the sequence

$$0 \to (\Lambda_{2N-1})^* \to \Lambda^!_{N-1} \otimes (\Lambda_N)^* \to \Lambda^!_N \otimes (\Lambda_{N-1})^* \to \Lambda^!_{2N-1} \to 0$$

is exact.

To illustrate the general situation, consider the case $k = 3$. As before we have a commutative diagram, which we write as a matrix without arrows:

$$\begin{array}{ccccccccc}
 & & & & & & & 0 & \\
 & & & & & & 0 & A_{7,7} & \\
 & & & & & 0 & A_{6.6} & A_{6,7} & 0 \\
 & & & & 0 & A_{5,5} & A_{5,6} & A_{5,7} & 0 \\
 & & & 0 & A_{4,4} & A_{4,5} & A_{4,6} & A_{4,7} & 0 \\
 & & 0 & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} & A_{3,7} & 0 \\
 & 0 & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} & A_{2,7} & 0 \\
0 & A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} & A_{1,7} & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}$$

with the following entries:
$A_{1,1} = A_{7,7} = (\Lambda_{3N})^*$, $\quad A_{6,6} = (\Lambda_{2N+1})^* \otimes (\Lambda_{N-1})^*$,
$A_{6,7} = (\Lambda_{2N+1})^* \otimes \Lambda^!_{N-1}$, $\quad A_{5,5} = (\Lambda_{2N})^* \otimes (\Lambda_N)^*$,
$A_{5,6} = (\Lambda_{2N})^* \otimes \Lambda^!_1 \otimes (\Lambda_{N-1})^*$, $\quad A_{5,7} = (\Lambda_{2N})^* \otimes \Lambda^!_N$,
$A_{4,4} = (\Lambda_{N+1})^* \otimes (\Lambda_{2N-1})^*$, $\quad A_{4,5} = (\Lambda_{N+1})^* \otimes \Lambda^!_{N-1} \otimes (\Lambda_N)^*$,
$A_{4,6} = (\Lambda_{N+1})^* \otimes \Lambda^!_N \otimes (\Lambda_{N-1})^*$, $\quad A_{4,7} = (\Lambda_{N+1})^* \otimes \Lambda^!_{2N-1}$,
$A_{3,3} = (\Lambda_N)^* \otimes (\Lambda_{2N})^*$, $\quad A_{3,4} = (\Lambda_N)^* \otimes \Lambda^!_1 \otimes (\Lambda_{2N-1})^*$,
$A_{3,5} = (\Lambda_N)^* \otimes \Lambda^!_N \otimes (\Lambda_N)^*$, $\quad A_{3,6} = (\Lambda_N)^* \otimes \Lambda^!_{N+1} \otimes (\Lambda_{N-1})^*$,
$A_{3,7} = (\Lambda_N)^* \otimes \Lambda^!_{2N}$, $\quad A_{2,2} = \Lambda^!_1 \otimes (\Lambda_{3N-1})^*$,
$A_{2,3} = \Lambda^!_1 \otimes \Lambda^!_{N-1} \otimes (\Lambda_{2N})^*$, $\quad A_{2,4} = \Lambda^!_1 \otimes \Lambda^!_N \otimes (\Lambda_{2N-1})^*$,
$A_{2,5} = \Lambda^!_1 \otimes \Lambda^!_{2N-1} \otimes (\Lambda_N)^*$, $\quad A_{2,6} = \Lambda^!_1 \otimes \Lambda^!_{2N} \otimes (\Lambda_{N-1})^*$,
$A_{2,7} = \Lambda^!_1 \otimes \Lambda^!_{3N-1}$, $\quad A_{1,2} = \Lambda^!_1 \otimes (\Lambda_{3N-1})^*$, $\quad A_{1,3} = \Lambda^!_N \otimes (\Lambda_{2N})^*$,

$A_{1,4} = \Lambda^!_{N+1} \otimes (\Lambda_{2N-1})^*, \quad A_{1,5} = \Lambda^!_{2N} \otimes (\Lambda_N)^*,$
$A_{1,6} = \Lambda^!_{2N+1} \otimes (\Lambda_{N-1})^*, \quad A_{1,7} = \Lambda^!_{3N}.$

Since we are assuming $\Lambda^!$ to be N-Koszul, all the columns are exact by the exactness of the sequences above and the case $k = 2$, and hence so are all rows except perhaps the row

$$0 \to \Lambda^!_1 \otimes (\Lambda_{3N-1})^* \to \Lambda^!_1 \otimes \Lambda^!_{N-1} \otimes (\Lambda_{2N})^* \to \Lambda^!_1 \otimes \Lambda^!_N \otimes (\Lambda_{2N-1})^* \\ \to \Lambda^!_1 \otimes \Lambda^!_{2N-1} \otimes (\Lambda_N)^* \to \Lambda^!_1 \otimes \Lambda^!_{2N} \otimes (\Lambda_{N-1})^* \to \Lambda^!_1 \otimes \Lambda^!_{3N-1} \to 0.$$

As in case $k = 2$, the map $0 \to \Lambda^!_1 \otimes (\Lambda_{3N-1})^* \to \Lambda^!_1 \otimes \Lambda^!_{N-1} \otimes (\Lambda_{2N})^*$ is mono, hence the above sequence is also exact.

From the fact that $\Lambda^!_1$ is a generator as a right Λ_0-module, it follows that the sequence

$$0 \to (\Lambda_{3N-1})^* \to \Lambda^!_{N-1} \otimes (\Lambda_{2N})^* \to \Lambda^!_N \otimes (\Lambda_{2N-1})^* \\ \to \Lambda^!_{2N-1} \otimes (\Lambda_N)^* \to \Lambda^!_{2N} \otimes (\Lambda_{N-1})^* \to \Lambda^!_{3N-1} \to 0$$

is exact. It is clear how to continue the induction. ∎

Now let M be an N-Koszul Λ-module and denote by $E^k(M)$ the Λ_0-module $\mathrm{Ext}^k_\Lambda(M, \Lambda_0)$. The minimal graded projective resolution of M can be displayed as a matrix witout arrows where we have put $\Lambda_n.E^k(M)^*$ instead of $\Lambda_n \otimes (\mathrm{Ext}^k_\Lambda(M, \Lambda_0))^*$ and $\Lambda_n.M_j$ instead of $\Lambda_n \otimes M_j$:

$$\tag{5} \begin{array}{cccccc} & & & 0 & \Lambda_0.M_0 & M_0 & 0 \\ & & 0 & \Lambda_0.E^1(M)^* & \Lambda_1.M_0 & M_1 & 0 \\ & & 0 & \Lambda_1.E^1(M)^* & \Lambda_2.M_0 & M_2 & 0 \\ & & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & 0 & \Lambda_{N-2}.E^1(M)^* & \Lambda_{N-1}.M_0 & M_{N-1} & 0 \\ & 0 & \Lambda_0.E^2(M)^* & \Lambda_{N-1}.E^1(M)^* & \Lambda_N.M_0 & M_N & 0 \\ & \Lambda_0.E^3(M)^* & \Lambda_1.E^2(M)^* & \Lambda_N.E^1(M)^* & \Lambda_{N+1}.M_0 & M_{N+1} & 0 \\ & \Lambda_1.E^3(M)^* & \Lambda_2.E^2(M)^* & \Lambda_{N+1}.E^1(M)^* & \Lambda_{N+2}.M_0 & M_{N+2} & 0 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & \Lambda_{N-2}.E^3(M)^* & \Lambda_{N-1}.E^2(M)^* & \Lambda_{2N-2}.E^1(M)^* & \Lambda_{2N-1}.M_0 & M_{2N-1} & 0 \\ \cdots & \Lambda_{N-1}.E^3(M)^* & \Lambda_N.E^2(M)^* & \Lambda_{2N-1}.E^1(M)^* & \Lambda_{2N}.M_0 & M_{2N} & 0 \\ \cdots & \Lambda_N.E^3(M)^* & \Lambda_{N+1}.E^2(M)^* & \Lambda_{2N}.E^1(M)^* & \Lambda_{2N+1}.M_0 & M_{2N+1} & 0 \end{array}$$

Dualizing the first two rows we obtain exact sequences $0 \to (M_0)^* \otimes \Lambda^!_0 \to \mathrm{Hom}_\Lambda(M, \Lambda_0) \to 0$ and $0 \to (M_1)^* \to (M_0)^* \otimes \Lambda^!_1 \to \mathrm{Ext}^1_\Lambda(M, \Lambda_0) \to 0$.

We can now prove our main theorem.

THEOREM 2. *Let $\Lambda = KQ/I$ be an N-Koszul algebra with $N \geq 2$ such that its homogeneous dual algebra $\Lambda^! = KQ^{\mathrm{op}}/\langle I_N^\perp\rangle$ is also N-Koszul and* *that the quiver Q is connected and has no sources. Let $M = \{M_j\}_{j\geq 0}$ be an N-Koszul module. Then for any $k \geq 0$, there exist exact sequences*

$$\begin{aligned}0 \to (M_{kN})^* \to (M_{(k-1)N+1})^* \otimes \Lambda^!_{N-1} \to \cdots \to (M_{2N})^* \otimes \Lambda^!_{(k-2)N} \\ \to (M_{N+1})^* \otimes \Lambda^!_{(k-1)N-1} \to (M_N)^* \otimes \Lambda^!_{(k-1)N} \\ \to (M_1)^* \otimes \Lambda^!_{kN-1} \to (M_0)^* \otimes \Lambda^!_{kN} \to \operatorname{Ext}^{2k}_\Lambda(M, \Lambda_0) \to 0\end{aligned}$$

and

$$\begin{aligned}0 \to (M_{kN+1})^* \to (M_{kN})^* \otimes \Lambda^!_1 \to \cdots \to (M_{2N})^* \otimes \Lambda^!_{(k-2)N+1} \\ \to (M_{N+1})^* \otimes \Lambda^!_{(k-1)N} \to (M_N)^* \otimes \Lambda^!_{(k-1)N+1} \to (M_1)^* \otimes \Lambda^!_{kN} \\ \to (M_0)^* \otimes \Lambda^!_{kN+1} \to \operatorname{Ext}^{2k+1}_\Lambda(M, \Lambda_0) \to 0.\end{aligned}$$

Proof. We illustrate the proof by looking at the cases $k = 0, 1, 2, 3, 4$, and leave the general argument to the reader.

The cases $k = 0, 1$ are clear.

Dualizing the corresponding row of (5) we get an exact sequence

$$0 \to (M_N)^* \to (M_0)^* \otimes (\Lambda_N)^* \to \operatorname{Ext}^1_\Lambda(M, \Lambda_0) \otimes (\Lambda_{N-1})^* \to \operatorname{Ext}^2_\Lambda(M, \Lambda_0) \to 0.$$

With the same notation as above, we get a commutative diagram

$$\begin{array}{cccccccccc}
 & & & & & & & & 0 & \\
 & & & & & & & & \downarrow & \\
 & & & & & & 0 & \to & (M_N)^* & \\
 & & & & & & \downarrow & & \downarrow & \\
 & & & & 0 & \to & (M_1)^*.\Lambda^!_{N-1} & \to & (M_1)^*.\Lambda^!_{N-1} & \to 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & \to & (M_0)^*.(\Lambda_N)^* & \to & (M_0)^*.\Lambda^!_1.\Lambda^!_{N-1} & \to & (M_0)^*.\Lambda^!_N & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & (M_N)^* & \to & (M_0)^*.(\Lambda_N)^* & \to & E^1(M).\Lambda^!_{N-1} & \to & E^2(M) & \to 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 & & 0 &
\end{array}$$

such that:

(i) All rows are exact.
(ii) All columns but perhaps the last one are exact and this column is a complex.
(iii) $0 \to (M_N)^* \to (M_1)^* \otimes \Lambda^!_{N-1}$ is exact.

Then by symmetry, Lemma 1 applies and it follows that the last column is also exact.

Consider now the case $k = 3$. Dualizing the corresponding sequence in (5) we have an exact sequence

$$\begin{aligned} &0 \to (M_{N+1})^* \to (M_0)^* \otimes (\Lambda_{N+1})^* \\ &\ \to \operatorname{Ext}^1_\Lambda(M, \Lambda_0) \otimes (\Lambda_N)^* \to \operatorname{Ext}^2_\Lambda(M, \Lambda_0) \otimes \Lambda^!_1 \to \operatorname{Ext}^3_\Lambda(M, \Lambda_0) \to 0. \end{aligned}$$

We obtain as above a commutative diagram of the type

$$\begin{array}{ccccccccccc}
 & & & & & & & & & 0 & \\
 & & & & & & & & & \downarrow & \\
 & & & & & & & 0 & \to & A_{5,5} & \\
 & & & & & & & \downarrow & & \downarrow & \\
 & & & & & 0 & \to & A_{4,4} & \to & A_{4,5} & \to 0 \\
 & & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & 0 & \to & A_{3,3} & \to & A_{3,4} & \to & A_{3,5} & \to 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \to & A_{2,2} & \to & A_{2,3} & \to & A_{2,4} & \to & A_{2,5} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & A_{1,1} & \to & A_{1,2} & \to & A_{1,3} & \to & A_{1,4} & \to & A_{1,5} & \to 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

with the following entries:

$A_{1,1} = A_{5,5} = (M_{N+1})^*, \quad A_{4,4} = A_{4,5} = (M_N)^* \otimes \Lambda^!_1,$
$A_{3,3} = (M_1)^* \otimes (\Lambda_N)^*, \quad A_{3,4} = (M_1)^* \otimes \Lambda^!_{N-1} \otimes \Lambda^!_1,$
$A_{3,5} = (M_1)^* \otimes \Lambda^!_N, \quad A_{2,2} = (M_0)^* \otimes (\Lambda_{N+1})^*,$
$A_{2,3} = (M_0)^* \otimes \Lambda^!_1 \otimes (\Lambda_N)^*, \quad A_{2,4} = (M_0)^* \otimes \Lambda^!_N \otimes \Lambda^!_1,$
$A_{2,5} = (M_0)^* \otimes \Lambda^!_{N+1}, \quad A_{1,2} = (M_0)^* \otimes (\Lambda_{N+1})^*,$
$A_{1,3} = \operatorname{Ext}^1_\Lambda(M, \Lambda_0) \otimes (\Lambda_N)^*, \quad A_{1,4} = \operatorname{Ext}^2_\Lambda(M, \Lambda_0) \otimes \Lambda^!_1,$
$A_{1,5} = \operatorname{Ext}^3_\Lambda(M, \Lambda_0).$

The diagram satisfies the following conditions:

(i) All rows are exact.
(ii) The last column is a complex and the remaining columns are exact.
(iii) $0 \to (M_{N+1})^* \to (M_N)^* \otimes \Lambda^!_1$ is exact.

According to Lemma 1, the first column is also exact.

For $k = 4$, dualizing the corresponding columns of (5) we obtain an exact sequence

$$\begin{aligned} &0 \to (M_{2N})^* \to (M_0)^* \otimes (\Lambda_{2N})^* \to \operatorname{Ext}^1_\Lambda(M, \Lambda_0) \otimes (\Lambda_{2N-1})^* \\ &\quad \to \operatorname{Ext}^2_\Lambda(M, \Lambda_0) \otimes (\Lambda_N)^* \to \operatorname{Ext}^3_\Lambda(M, \Lambda_0) \otimes \Lambda^!_{N-1} \to \operatorname{Ext}^4_\Lambda(M, \Lambda_0) \to 0. \end{aligned}$$

We have as above a commutative diagram of the form

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & & & & & 0 & & \\
 & & & & & & & & & & & & \downarrow & & \\
 & & & & & & & & & & 0 & \to & A_{6,6} & & \\
 & & & & & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & & & 0 & \to & A_{5,5} & \to & A_{5,6} & \to & 0 \\
 & & & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & \to & A_{4,4} & \to & A_{4,5} & \to & A_{4,6} & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & \to & A_{3,3} & \to & A_{3,4} & \to & A_{3,5} & \to & A_{3,6} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & A_{2,2} & \to & A_{2,3} & \to & A_{2,4} & \to & A_{2,5} & \to & A_{2,6} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_{1,1} & \to & A_{1,2} & \to & A_{1,3} & \to & A_{1,4} & \to & A_{1,5} & \to & A_{1,6} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

with the following entries:

$$
\begin{aligned}
&A_{1,1} = A_{6,6} = (M_{2N})^*, \quad A_{5,5} = A_{5,6} = (M_{N+1})^* \otimes \Lambda^!_{N-1}, \\
&A_{4,4} = (M_N)^* \otimes (\Lambda_N)^*, \quad A_{4,5} = (M_N)^* \otimes \Lambda^!_1 \otimes \Lambda^!_{N-1}, \\
&A_{4,6} = (M_N)^* \otimes \Lambda^!_N, \quad A_{3,3} = (M_1)^* \otimes (\Lambda_{2N-1})^*, \\
&A_{3,4} = (M_1)^* \otimes \Lambda^!_{N-1} \otimes (\Lambda_N)^*, \quad A_{3,5} = (M_1)^* \otimes \Lambda^!_N \otimes \Lambda^!_{N-1}, \\
&A_{3,6} = (M_1)^* \otimes \Lambda^!_{2N-1}, \quad A_{2,2} = (M_0)^* \otimes (\Lambda_{2N})^*, \\
&A_{2,3} = (M_0)^* \otimes \Lambda^!_1 \otimes (\Lambda_{2N-1})^*, \quad A_{2,4} = (M_0)^* \otimes \Lambda^!_N \otimes (\Lambda_N)^*, \\
&A_{2,5} = (M_0)^* \otimes \Lambda^!_{N+1} \otimes \Lambda^!_{N-1}, \quad A_{2,6} = (M_0)^* \otimes \Lambda^!_{2N}, \\
&A_{1,2} = (M_0)^* \otimes (\Lambda_{2N})^*, \quad A_{1,3} = \operatorname{Ext}^1_\Lambda(M, \Lambda_0) \otimes (\Lambda_{2N-1})^*, \\
&A_{1,4} = \operatorname{Ext}^2_\Lambda(M, \Lambda_0) \otimes (\Lambda_N)^*, \quad A_{1,5} = \operatorname{Ext}^3_\Lambda(M, \Lambda_0) \otimes \Lambda^!_{N-1}, \\
&A_{1,6} = \operatorname{Ext}^4_\Lambda(M, \Lambda_0).
\end{aligned}
$$

As in the previous cases, the last column is a complex and all remaining columns are exact. All the rows are exact and the sequence $0 \to (M_{2N})^* \to (M_{N+1})^* \otimes \Lambda^!_{N-1}$ is exact. It follows as above that the last column is exact.

Now it is clear how to continue the induction. ∎

The following definition was given in [11]:

Definition 3. Let $\Lambda = KQ/I$ be a graded factor of a path algebra. A sequence of graded modules

$$\to M_n[-n] \xrightarrow{\delta} M_{n-1}[-(n-1)] \xrightarrow{\delta} \cdots \xrightarrow{\delta} M_k[-k] \xrightarrow{\delta}$$

is called an N-*complex* if N is a positive integer such that the composition of N maps δ is zero; this is written as $\delta^N = 0$.

If an N-complex is bounded above, for example if it is of the form

$$\to M_n[-n] \xrightarrow{\delta} M_{n-1}[-(n-1)] \xrightarrow{\delta} \cdots \xrightarrow{\delta} M_1[-1] \xrightarrow{\delta} M_0 \to 0,$$

then it induces by composition an ordinary complex:

$$\cdots \to M_{2N+1}[-(2N+1)] \xrightarrow{\delta} M_{2N}[-(2N)] \xrightarrow{\delta^{N-1}} M_{N+1}[-(N+1)] \xrightarrow{\delta} M_N[-(N)] \xrightarrow{\delta^{N-1}} M_1[-1] \xrightarrow{\delta} M_0 \to 0.$$

The following theorem was proved in [11]:

Theorem 3. *Let $\Lambda = KQ/I$ be an N-homogeneous graded factor of a path algebra with homogeneous dual $\Lambda^! = KQ^{\mathrm{op}}/\langle I_N^\perp\rangle$. Then there exists a duality between the category of locally finite graded Λ-modules*, lfgr_Λ, *and the category of N-complexes of finitely generated projective $\Lambda^!$-modules*, ${}_N\mathcal{L}c_{\Lambda^!}$. *The duality is given as follows: to a graded locally finite Λ-module $M = \{M_n\}_{n\in\mathbb{Z}}$ corresponds an N-complex*

$$\to D(M_n)\otimes\Lambda^![-n] \xrightarrow{\delta} D(M_{n-1})\otimes\Lambda^![-(n-1)] \xrightarrow{\delta} \cdots \xrightarrow{\delta} D(M_k)\otimes\Lambda^![-k] \to \cdots$$

where the maps δ are induced by the multiplication $\mu : \Lambda_1 \otimes M_n \to M_{n+1}$ and $D(M_n) = \mathrm{Hom}_{\Lambda_0}(M_n, \Lambda_0)$. ∎

The main theorem of the paper can be interpreted as follows:

Theorem 4. *Let $\Lambda = KQ/I$ be an N-Koszul algebra with $N \geq 2$ such that its homogeneous dual algebra $\Lambda^! = KQ^{\mathrm{op}}/\langle I_N^\perp\rangle$ is also N-Koszul and that the quiver Q is connected and has no sources. Let $M = \{M_j\}_{j\geq 0}$ be an N-Koszul module. Then the corresponding N-complex*

$$\to D(M_n)\otimes\Lambda^![-n] \xrightarrow{\delta} D(M_{n-1})\otimes\Lambda^![-(n-1)] \xrightarrow{\delta} \cdots \xrightarrow{\delta} D(M_1)\otimes\Lambda^![-1] \to D(M_0)\otimes\Lambda^! \to 0$$

induces an ordinary complex $(\mathcal{P}, d)$ of finitely generated graded projective $\Lambda^!$-modules:

$$\to D(M_{N+1})\otimes\Lambda^![-(N+1)] \xrightarrow{\delta} D(M_N)\otimes\Lambda^![-N] \xrightarrow{\delta^{N-1}} D(M_1)\otimes\Lambda^![-1] \xrightarrow{\delta} D(M_0)\otimes\Lambda^! \to 0$$

with homology $\mathcal{H}(\mathcal{P})^j_{\delta(k)} = 0$ for $j \neq 0$ and $\mathcal{H}(\mathcal{P})^0_{\delta(k)} = \mathrm{Ext}^k_\Lambda(M, \Lambda_0)$. ∎

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