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A NOTE ON A CLASS OF HOMEOMORPHISMS BETWEEN BANACH SPACES

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Abstract. This paper deals with homeomorphisms $F : X \to Y$, between Banach spaces X and Y, which are of the form

$$F(x) := \widetilde{F}x^{(2n+1)}$$

where $\widetilde{F} : X^{2n+1} \to Y$ is a continuous $(2n+1)$-linear operator.

1. Introduction. Let us assume that X, Y are real Banach spaces, and n a non-negative integer. Let $\widetilde{F} : X^{2n+1} \to Y$ be a continuous $(2n+1)$-linear operator, and let $F : X \to Y$ be defined by

$$F(x) := \widetilde{F}x^{(2n+1)}. \tag{1}$$

Suppose F is surjective and satisfies, for some constant $C > 0$, the condition

$$C\|x\|^{2n}\|h\| \le \|F(x+h) - F(x)\| \tag{2}$$

for $x, h \in X$. Then it is easy to see that F is a homeomorphism from X onto Y. Note that F which does not satisfy (2) may be a homeomorphism. The mapping $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ where $F(x) := x^{2n+1}$ is a simple example.

Nevertheless, the above described class is important because of the following two theorems from [3]:

THEOREM 1. *Assume $T : X \supset U \to Y$ is a C^{2n+1}-mapping from an open neighbourhood U of a point $x_0 \in X$ into Y. Suppose that the derivatives of T satisfy the condition*

$$T^{(k)}(x_0) = 0$$

for all $1 \le k \le 2n$. For $x \in X$, let

$$F(x) := T^{(2n+1)}(x_0).x^{(2n+1)}/(2n+1)!$$

and suppose that F maps X onto Y and satisfies condition (2) for some constant $C > 0$. Then T is a local homeomorphism in a neighbourhood of x_0,

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that is, there exist neighbourhoods $V \subset U$ *of* x_0 *and* W *of* $T(x_0)$ *such that* $T|V$ *is a homeomorphism from* V *onto* W.

THEOREM 2. *Assume that the norm of* X *is of class* C^{2n+1} *away from zero. Suppose that* $F : X \to Y$ *is of the form* (1), *with a continuous* $(2n+1)$-*linear operator* $\widetilde{F} : X^{2n+1} \to Y$, *and does not satisfy condition* (2). *Then there exists a* C^{2n+1}-*mapping* $r : X \to Y$ *with* $r(x) = o(\|x\|^{2n+1})$ *for which the mapping* $T : X \to Y$, *where* $T(x) := f(x) + r(x)$, *is not a local homeomorphism in a neighbourhood of zero.*

2. Main results.

In [3] the following theorem is proved:

THEOREM 3. *Suppose that* $F : X \to Y$ *is of the form* (1) *with a continuous* $(2n+1)$-*linear operator* $\widetilde{F} : X^{2n+1} \to Y$. *Assuming the invertibility of* F, *condition* (2) *holds if and only if there exists a constant* $C_1 > 0$ *such that*

$$C_1 \|x\|^{2n} \|h\| \le \|F'(x).h\| \tag{3}$$

for $x \in X$ *and*

$$F'(x).X = Y \quad \text{for } x \neq 0. \tag{4}$$

It is trivial that (2) implies the injectivity of F, so we can assume only the surjectivity of F in the "⇒" part of Theorem 3. We shall show that, in a particular case, one can weaken the assumption in the "⇐" part of the theorem.

THEOREM 4. *Suppose that* $F : X \to Y$ *is of the form* (1) *with a continuous* $(2n+1)$-*linear operator* $\widetilde{F} : X^{2n+1} \to Y$. *Assume that* $\dim X \ge 3$ *and* F *satisfies conditions* (3), (4) *and is proper, that is, the pre-image under* F *of any compact set in* Y *is compact. Then* F *is invertible.*

Proof. Observe that $F(x) = 0$ if and only if $x = 0$. Indeed, this follows from (3) and the equality

$$F(x) = F'(x).x/(2n+1).$$

Hence, it suffices to prove the invertibility of the mapping

$$F|X \setminus \{0\} : X \setminus \{0\} \to Y \setminus \{0\}.$$

We shall use the following theorem (see [1, Global Inversion Theorem 1.8]):

THEOREM 5. *Let* f *be a continuous mapping from a metric space* M *into a metric space* N. *Assume that* f *is proper and locally invertible, that is, invertible in a neighbourhood of any point of* M. *Suppose that* M *is* *arcwise connected and N is simply connected, that is, N is arcwise connected and every closed path in N is homotopic to a constant. Then f is a homeomorphism from M onto N.*

It is clear that $F|X \setminus \{0\} : X \setminus \{0\} \to Y \setminus \{0\}$ is proper. The local invertibility of $F|X \setminus \{0\}$ follows from (3) and (4) by the Local Inversion Theorem. The metric spaces $M = X\setminus\{0\}$ and $Y\setminus\{0\}$ satisfy the assumptions of Theorem 4. Thus $F|X\setminus\{0\} : X\setminus\{0\} \to Y\setminus\{0\}$ is invertible. Consequently, $F : X \to Y$ is invertible. ∎

REMARK 1. Let us inspect the case of $\dim X \le 2$. The case of $\dim X = \dim Y = 0$ is trivial since the unique F is the zero mapping. In the case $\dim X = \dim Y = 1$, F is of the form

$$F(x) = \alpha x^{2n+1},$$

hence it satisfies conditions (3), (4), and is invertible if $\alpha \neq 0$. The case of $\dim X = \dim Y = 2$ is more interesting. Consider the mapping $F : \mathbb{C} \to \mathbb{C}$,

$$F(x) = x^{2n+1},$$

where the space $\mathbb{C}$ of complex numbers is treated as a real (two-dimensional) Banach space. We see that F satisfies conditions (3), (4). Nevertheless, for $n \ge 1$, it is not invertible.

REMARK 2. In the case of $\dim X = \dim Y < \infty$, if F satisfies condition (4), then $F'(x)$ is invertible for $x \neq 0$, and condition (3) holds. Indeed, we have

$$\inf_{x,h\neq 0} \|F'(x).h\|\,\|x\|^{-2n}\|h\|^{-1} = \inf_{x,h\neq 0} \|F'(x/\|x\|).h/\|h\|\,\| =: C_1 > 0,$$

since the function $(x,h) \mapsto \|F'(x).h\|$ is positive on the Cartesian square of the unit sphere, which is compact.

In this case, by (3),

$$C_1\|x\|^{2n+1} \le \|F'(x).x\| = (2n+1)\|F(x)\|.$$

Hence the pre-image under F of a compact set is bounded, and being closed, it is compact. Thus F is proper. From Theorem 4 and Remark 1, for the case of $\dim X \neq 3$, F is a homeomorphism from X onto Y.

Now, we shall show that conditions (3), (4) imply the surjectivity of F also in the case of infinite-dimensional spaces:

THEOREM 6. *Suppose that $F : X \to Y$ is of the form* (1) *with a continuous* $(2n+1)$*-linear operator $\widetilde{F} : X^{2n+1} \to Y$. Assume that F satisfies conditions* (3), (4)*. Then F is surjective.*

Proof. Without loss of generality, we may assume that $\dim X > 1$. Using the above argument, it suffices to prove that $F|X \setminus \{0\} : X \setminus \{0\} \to Y \setminus \{0\}$

is surjective. The local invertibility of $F|X \setminus \{0\}$ implies that $F(X \setminus \{0\})$ is open in $Y \setminus \{0\}$. Let (y_j) be a sequence of points from $F(X \setminus \{0\})$ tending to a point $y_0 \in Y \setminus \{0\}$. By Propositions 2 and 3 from [4], there exists $R > 0$ such that every ball with centre at y_j and radius R is included in $F(X \setminus \{0\})$. Thus, it is easy to see that $y_0 \in F(X \setminus \{0\})$.

Then the set $F(X \setminus \{0\})$ is open and closed in the connected space $Y \setminus \{0\}$, hence $F(X \setminus \{0\}) = Y \setminus \{0\}$, which ends the proof. ■

The above considerations lead to the following non-trivial problem:

PROBLEM 1. Find (if any) a continuous $(2n+1)$-linear mapping

$$\widetilde{F} : X^{2n+1} \to Y,$$

where X, Y are infinite-dimensional Banach spaces and $n \geq 1$, such that, for $F(x) := \widetilde{F}x^{(2n+1)}$, conditions (3) and (4) hold and $F : X \to Y$ is not injective.

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