ORLICZ BOUNDS FOR OPERATORS OF RESTRICTED WEAK TYPE

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Abstract. It is shown that if $T$ is a sublinear translation invariant operator of restricted weak type $(1,1)$ acting on $L^1(\mathbb{T})$, then $T$ maps simple functions in $L\log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$.

Let $\mathbb{T}$ denote the unit circle. An operator $T$ acting on $L^1(\mathbb{T})$ is said to be of restricted weak type $(1,1)$ if for some constant $C$ the inequality

$$\left| \{ x \in \mathbb{T} : |T\chi_E(x)| > \alpha \} \right| \leq \frac{C}{\alpha} \| \chi_E \|_{L^1(\mathbb{T})}$$

holds for every measurable set $E \subset \mathbb{T}$ and $\alpha > 0$. Examples of restricted weak type $(1,1)$ operators include the Hardy–Littlewood maximal operator and the Hilbert transform. Now, it is well known [4] that the Hardy–Littlewood maximal operator as well as the Hilbert transform map $L\log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$. As both of these operators are also translation invariant, it is natural to consider whether or not every sublinear translation invariant restricted weak type $(1,1)$ operator maps $L\log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$.

The purpose of this paper is to show that all sublinear translation invariant restricted weak type $(1,1)$ operators acting on $L^1(\mathbb{T})$ do indeed map $L\log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$ and, moreover, that operators of this type are bounded on $L^p(\mathbb{T})$ for $1 < p < 2$. We remark that our methods of proof here have been strongly influenced by the work of E. M. Stein on limits of sequences of operators [5] as well as by the suggestive results of L. Colzani in his paper on translation invariant operators acting on Lorentz spaces [1].

Theorem 1. Let $T$ be a translation invariant sublinear operator acting on $L^1(\mathbb{T})$. Also suppose that for any measurable set $E$ in $\mathbb{T}$ and $\alpha > 0$ we have

$$\left| \{ x \in \mathbb{T} : |T\chi_E(x)| > \alpha \} \right| \leq \frac{|E|}{\alpha}.$$

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If $f$ is a simple function supported on $\mathbb{T}$, then

$$\|T(f)\|_{L^1(\mathbb{T})} \leq C\|f\|_{L\log L(\mathbb{T})},$$

where $C$ is a universal constant.

Proof. We begin by gathering some lemmas that will be of use to us. The first is a Borel–Cantelli type lemma devised by E. M. Stein in his work on limits of sequences of operators.

**Lemma 1 ([5]).** Let $E_1, E_2, \ldots$ be a collection of sets in $\mathbb{T}$ such that $\sum |E_j| = \infty$. Then there exist sets $F_1, F_2, \ldots$ in $\mathbb{T}$ such that each $F_j$ is a translate of $E_j$ in $\mathbb{T}$ and almost every point of $\mathbb{T}$ belongs to infinitely many of the sets $F_j$.

The second lemma involves a well known property of Rademacher functions.

**Definition 1.** Let $r_n(t)$ denote the Rademacher functions on $\mathbb{R}$, defined by

$$r_n(t) = r_0(2^n t),$$

where $r_0(t) = 1$ if $0 \leq t \leq 1/2$, $r_0(t) = -1$ if $1/2 < t < 1$, and $r_0(t+1) = r_0(t)$.

**Lemma 2 ([6, 10]).** Let $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, and let $F(t) = \sum_{n=0}^{N} a_n r_n(t)$ be a Rademacher series. Let $0 < p < \infty$. Then there exist finite, positive constants $A(p), B(p)$ such that

$$A(p)\|F\|_{L^p([0,1])} \leq \left( \sum_{n=0}^{N} |a_n|^2 \right)^{1/2} \leq B(p)\|F\|_{L^p([0,1])}.$$

The third lemma we shall use follows from the work of F. Soria on extrapolation theorems of Carleson–Sjölin type.

**Lemma 3 ([3]).** Let $T$ be a sublinear operator acting on $L^1(\mathbb{T})$. Suppose that, for any measurable subset $E$ of $\mathbb{T}$ and $\alpha > 0$,

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq \frac{|E|}{\alpha}.$$

If $f$ is a simple function supported on $\mathbb{T}$ and $\alpha > 0$, then

$$|\{x \in \mathbb{T} : |Tf(x)| > \alpha\}| \leq C \frac{\|f\|_{L\log L(\mathbb{T})}}{\alpha},$$

where $C$ is a universal constant.

We will now use these three lemmas to prove the following.
Lemma 4. Let $T$ be a translation invariant sublinear operator acting on $L^1(\mathbb{T})$. Suppose also that, for any measurable set $E$ in $\mathbb{T}$ and $\alpha > 0$,
\[
|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq \frac{|E|}{\alpha}.
\]
If $f$ is a simple function supported on $\mathbb{T}$ and $\alpha > 0$, then
\[
|\{x \in \mathbb{T} : |Tf(x)| > \alpha\}| \leq C \left( \frac{\|f\|_{L^2(\mathbb{T})}}{\alpha} \right)^2,
\]
where $C$ is a universal constant.

Proof. By contradiction. Suppose (1) were false. Then there would exist a sequence $\{f_n\}$ of simple functions and a sequence $\{E_n\}$ of sets such that
\[
|Tf_n(x)| > 1 \quad \text{for } x \in E_n
\]
and
\[
|E_n| > n\|f_n\|_{L^2(\mathbb{T})}^2.
\]
By taking subcollections of the original collections of $\{f_n\}$, $\{E_n\}$, with possible repetitions, we may obtain another set of collections, again denoted by $\{f_n\}$, $\{E_n\}$, such that $|Tf_n(x)| > 1$ if $x \in E_n$, $\sum |E_n| = \infty$, and $\sum \|f_n\|_2^2 < \infty$.

As $\sum \|f_n\|_2^2$ converges, we may find a sequence $\{R_n\}$ of positive numbers such that $R_n \to \infty$, but $\sum \|R_nf_n\|_2^2 = D < \infty$.

Now, for each $g \in \mathbb{T}$, we let $\tau_g$ denote the translation operator defined by
\[
\tau_g f(x) = f(-g + x).
\]
As $\sum |E_n| = \infty$, by Lemma 1 we see that there exists a sequence $\{F_n\}$ of sets in $\mathbb{T}$ such that each $F_j$ is a translate of $E_j$ in $\mathbb{T}$ and almost every point of $\mathbb{T}$ belongs to an infinite number of the sets $F_n$. We associate to each $F_j$ an element $g_j \in \mathbb{T}$ such that
\[
\chi_{F_j} = \tau_{g_j} \chi_{E_j}.
\]
Let $M$ be a positive integer. There exists a positive integer $N$ and a subset $S \subset \mathbb{T}$ of measure greater than $1/2$ such that for all $x \in S$, there exists an integer $j_x$ such that $1 \leq j_x \leq N$ and
\[
M < |R_{j_x} T(\tau_{g_{j_x}} f_{j_x})(x)|.
\]
Now, define the function $h(x, t)$ on $\mathbb{T} \times [0, 1]$ by
\[
h(x, t) = \sum_{j=1}^N R_j \tau_{g_j} f_j(x)r_j(t).
\]
If $g(x, t)$ is a measurable function on $\mathbb{T} \times [0, 1]$, we define $Tg(x, t)$ by
\[
Tg(x, t) = Tg_t(x),
\]
where $g_t(x) = g(x, t)$. 
Now, let \( x_0 \in S \). For some \( j \) with \( 1 \leq j \leq N \) we have \( |R_j T(\tau_j f_j)(x_0)| > M \). We assume without loss of generality that \( j = 1 \).

Now, if \( 0 < t < 1 \) and \( t \) is not of the form \( k \cdot 2^j \) for some integers \( j, k \), the sublinearity of \( T \) implies that

\[
M < |T(R_1 \tau_1 f_1)(x_0)| \leq \frac{1}{2} \left[ T \left( R_1 \tau_1 f_1(x) + \sum_{j=2}^{N} R_j \tau_j f_j(x) r_j(t) \right)(x_0) \right] + \left[ T \left( R_1 \tau_1 f_1(x) + \sum_{j=2}^{N} R_j \tau_j f_j(x) r_j(1-t) \right)(x_0) \right].
\]

So \( \{|t \in [0, 1] : |Th(x_0, t)| > M \}| \geq 1/4 \). As \( |S| > 1/2 \), we then have

\[
(2) \quad \{|(x, t) \in T \times [0, 1] : |Th(x, t)| > M \}| \geq 1/8.
\]

Note that Lemma 2 implies

\[
\|h\|^2_{L^2(T \times [0, 1])} = \int_0^1 \left( \sum_{j=1}^{N} R_j \tau_j f_j(x) r_j(t) \right)^2 dt dx
\]

\[
\leq (A(2))^{-2} \left( \sum_{j=1}^{N} |R_j \tau_j f_j(x)|^2 \right) dx
\]

\[
= (A(2))^{-2} \sum_{j=1}^{N} \|R_j \tau_j f_j\|_{L^2(T)}^2 = (A(2))^{-2} \sum_{j=1}^{N} \|R_j f_j\|_{L^2(T)}^2
\]

\[
\leq (A(2))^{-1} \cdot D < \infty.
\]

For our notational convenience, if \( L^\Phi \) is a normed space on \([0, 1]\) and \( L^\Psi \) is a normed space on \( T \), we define the mixed norm \( \|\cdot\|_{L^\Psi(T^\Phi)} \) on functions on \([0, 1] \times T\) by

\[
\|f(x, t)\|_{L^\Psi(T^\Phi)} = \bigg\| \|f(\cdot, t)\|_{L^\Phi(T)} \bigg\|_{L^\Psi([0,1])}.
\]

Now note that

\[
\|h(x, t)\|_{L^1(T \log L)_x} \leq 10 \|h(x, t)\|_{L^1(L^2)_x}
\]

\[
\leq 100 \|h(x, t)\|_{L^2(L^2)_x} = 100 \|h(x, t)\|_{L^2(T \times [0, 1])}
\]

\[
\leq 100 \cdot ((A(2))^{-1} \cdot D)^{1/2} = C' < \infty.
\]

By Lemma 3 we then see that

\[
\{|(x, t) : |Th(x, t)| > \alpha\} \leq \frac{1}{C} \|h(\cdot, t)\|_{L^\Psi(T)} dx \leq \frac{C \cdot C'}{\alpha}.
\]

This however is in contradiction to (2), which holds for arbitrarily large values of \( M \). ■
We now see that $T$ is of restricted weak type $(1,1)$ and of restricted weak type $(2,2)$. By the extension of the Marcinkiewicz theorem to the case of restricted-weak endpoints (see [7] for details) we have, for $1 < p < 3/2$ and for all simple functions $f$,

$$\|Tf\|_{L^p(T)} \lesssim \frac{1}{p-1} \|f\|_{L^p(T)}.$$ 

Applying the Yano extrapolation theorem [9], we then deduce that

$$\|Tf\|_{L^1(T)} \lesssim \|f\|_{L\log L(T)}$$

for all simple functions $f$ supported on $T$, as desired. □

We emphasize that the following corollary arises from the proof above.

**Corollary 1.** Suppose $T$ is a sublinear translation invariant operator acting on $L^1(T)$ which is of restricted weak type $(1,1)$. If $f$ is a simple function supported on $T$, then

$$\|Tf\|_{L^p(T)} \leq C_p \|f\|_{L^p(T)}, \quad 1 < p < 2,$$

where $C_p \sim 1/(p-1) + 1/(2-p)$.

A natural question for subsequent investigation is whether or not this corollary can be extended to encompass values of $p$ greater than or equal to 2.

**REFERENCES**


