COLLOQUIUM MATHEMATICUM

VOL. 103 2005 NO. 2

ORLICZ BOUNDS FOR OPERATORS OF RESTRICTED WEAK TYPE

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Abstract. It is shown that if T is a sublinear translation invariant operator of restricted weak type $(1,1)$ acting on $L^1(\mathbb{T})$, then T maps simple functions in $L\log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$.

Let $\mathbb{T}$ denote the unit circle. An operator T acting on $L^1(\mathbb{T})$ is said to be of *restricted weak type* $(1,1)$ if for some constant C the inequality

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq \frac{C}{\alpha}\, \|\chi_E\|_{L^1(\mathbb{T})}$$

holds for every measurable set $E \subset \mathbb{T}$ and $\alpha > 0$. Examples of restricted weak type $(1,1)$ operators include the Hardy–Littlewood maximal operator and the Hilbert transform. Now, it is well known [4] that the Hardy–Littlewood maximal operator as well as the Hilbert transform map $L\log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$. As both of these operators are also translation invariant, it is natural to consider whether or not every sublinear translation invariant restricted weak type $(1,1)$ operator maps $L\log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$.

The purpose of this paper is to show that all sublinear translation invariant restricted weak type $(1,1)$ operators acting on $L^1(\mathbb{T})$ do indeed map $L\log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$ and, moreover, that operators of this type are bounded on $L^p(\mathbb{T})$ for $1 < p < 2$. We remark that our methods of proof here have been strongly influenced by the work of E. M. Stein on limits of sequences of operators [5] as well as by the suggestive results of L. Colzani in his paper on translation invariant operators acting on Lorentz spaces [1].

THEOREM 1. *Let T be a translation invariant sublinear operator acting on $L^1(\mathbb{T})$. Also suppose that for any measurable set E in $\mathbb{T}$ and $\alpha > 0$ we have*

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq \frac{|E|}{\alpha}.$$

2000 *Mathematics Subject Classification*: Primary 42B35.

Key words and phrases: restricted weak type, sublinear operator, translation invariant operator.

If f is a simple function supported on $\mathbb{T}$, then

$$\|T(f)\|_{L^1(\mathbb{T})} \leq C\|f\|_{L\log L(\mathbb{T})},$$

where C is a universal constant.

Proof. We begin by gathering some lemmas that will be of use to us. The first is a Borel–Cantelli type lemma devised by E. M. Stein in his work on limits of sequences of operators.

LEMMA 1 ([5]). *Let $E_1, E_2, \ldots$ be a collection of sets in $\mathbb{T}$ such that $\sum |E_j| = \infty$. Then there exist sets $F_1, F_2, \ldots$ in $\mathbb{T}$ such that each F_j is a translate of E_j in $\mathbb{T}$ and almost every point of $\mathbb{T}$ belongs to infinitely many of the sets F_j.*

The second lemma involves a well known property of Rademacher functions.

DEFINITION 1. Let $r_n(t)$ denote the Rademacher functions on $\mathbb{R}$, defined by

$$r_n(t) = r_0(2^n t),$$

where $r_0(t) = 1$ if $0 \leq t \leq 1/2$, $r_0(t) = -1$ if $1/2 < t < 1$, and $r_0(t+1) = r_0(t)$.

LEMMA 2 ([6, 10]). *Let $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, and let $F(t) = \sum_{n=0}^{N} a_n r_n(t)$ be a Rademacher series. Let $0 < p < \infty$. Then there exist finite, positive constants $A(p)$, $B(p)$ such that*

$$A(p)\|F\|_{L^p([0,1])} \leq \Big(\sum_{n=0}^{N} |a_n|^2\Big)^{1/2} \leq B(p)\|F\|_{L^p([0,1])}.$$

The third lemma we shall use follows from the work of F. Soria on extrapolation theorems of Carleson–Sjölin type.

LEMMA 3 ([3]). *Let T be a sublinear operator acting on $L^1(\mathbb{T})$. Suppose that, for any measurable subset E of $\mathbb{T}$ and $\alpha > 0$,*

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq \frac{|E|}{\alpha}.$$

If f is a simple function supported on $\mathbb{T}$ and $\alpha > 0$, then

$$|\{x \in \mathbb{T} : |Tf(x)| > \alpha\}| \leq C\,\frac{\|f\|_{L\log L(\mathbb{T})}}{\alpha},$$

where C is a universal constant.

We will now use these three lemmas to prove the following.

LEMMA 4. *Let T be a translation invariant sublinear operator acting on $L^1(\mathbb{T})$. Suppose also that, for any measurable set E in $\mathbb{T}$ and $\alpha > 0$,*

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \le \frac{|E|}{\alpha}.$$

If f is a simple function supported on $\mathbb{T}$ and $\alpha > 0$, then

$$|\{x \in \mathbb{T} : |Tf(x)| > \alpha\}| \le C\biggl(\frac{\|f\|_{L^2(\mathbb{T})}}{\alpha}\biggr)^2, \tag{1}$$

where C is a universal constant.

Proof. By contradiction. Suppose (1) were false. Then there would exist a sequence $\{f_n\}$ of simple functions and a sequence $\{E_n\}$ of sets such that

$$|Tf_n(x)| > 1 \quad \text{for } x \in E_n$$

and

$$|E_n| > n\|f_n\|^2_{L^2(\mathbb{T})}.$$

By taking subcollections of the original collections of $\{f_n\}$, $\{E_n\}$, with possible repetitions, we may obtain another set of collections, again denoted by $\{f_n\}$, $\{E_n\}$, such that $|Tf_n(x)| > 1$ if $x \in E_n$, $\sum |E_n| = \infty$, and $\sum \|f_n\|_2^2 < \infty$.

As $\sum \|f_n\|_2^2$ converges, we may find a sequence $\{R_n\}$ of positive numbers such that $R_n \to \infty$, but $\sum \|R_n f_n\|_2^2 = D < \infty$.

Now, for each $g \in \mathbb{T}$, we let τ_g denote the translation operator defined by

$$\tau_g f(x) = f(-g + x).$$

As $\sum |E_n| = \infty$, by Lemma 1 we see that there exists a sequence $\{F_n\}$ of sets in $\mathbb{T}$ such that each F_j is a translate of E_j in $\mathbb{T}$ and almost every point of $\mathbb{T}$ belongs to an infinite number of the sets F_n. We associate to each F_j an element $g_j \in \mathbb{T}$ such that

$$\chi_{F_j} = \tau_{g_j}\chi_{E_j}.$$

Let M be a positive integer. There exists a positive integer N and a subset $S \subset \mathbb{T}$ of measure greater than $1/2$ such that for all x in S, there exists an integer j_x such that $1 \le j_x \le N$ and

$$M < |R_{j_x} T(\tau_{g_{j_x}} f_{j_x})(x)|.$$

Now, define the function $h(x,t)$ on $\mathbb{T} \times [0,1]$ by

$$h(x,t) = \sum_{j=1}^{N} R_j \tau_{g_j} f_j(x) r_j(t).$$

If $g(x,t)$ is a measurable function on $\mathbb{T} \times [0,1]$, we define $Tg(x,t)$ by

$$Tg(x,t) = Tg_t(x),$$

where $g_t(x) = g(x,t)$.

Now, let $x_0 \in S$. For some j with $1 \le j \le N$ we have $|R_j T(\tau_{g_j} f_j)(x_0)| > M$. We assume without loss of generality that $j = 1$.

Now, if $0 < t < 1$ and t is not of the form $k \cdot 2^j$ for some integers j, k, the sublinearity of T implies that

$$M < |T(R_1 \tau_{g_1} f_1)(x_0)| \le \frac{1}{2}\Big[\Big|T\Big(R_1\tau_{g_1} f_1(x) + \sum_{j=2}^{N} R_j \tau_{g_j} f_j(x) r_j(t)\Big)(x_0)\Big| + \Big|T\Big(R_1\tau_{g_1} f_1(x) + \sum_{j=2}^{N} R_j \tau_{g_j} f_j(x) r_j(1-t)\Big)(x_0)\Big|\Big].$$

So $|\{t \in [0,1] : |Th(x_0,t)| > M\}| \ge 1/4$. As $|S| > 1/2$, we then have

$$|\{(x,t) \in \mathbb{T} \times [0,1] : |Th(x,t)| > M\}| \ge 1/8. \tag{2}$$

Note that Lemma 2 implies

$$\begin{aligned}
\|h\|^2_{L^2(\mathbb{T}\times[0,1])} &= \int_{\mathbb{T}}\int_0^1 \Big(\sum_{j=1}^{N} R_j \tau_{g_j} f_j(x) r_j(t)\Big)^2 dt\,dx \\
&\le (A(2))^{-2} \int_{\mathbb{T}} \sum_{j=1}^{N} |R_j \tau_{g_j} f_j(x)|^2\,dx \\
&= (A(2))^{-2} \sum_{j=1}^{N} \|R_j \tau_{g_j} f_j\|^2_{L^2(\mathbb{T})} = (A(2))^{-2} \sum_{j=1}^{N} \|R_j f_j\|^2_{L^2(\mathbb{T})} \\
&\le (A(2))^{-1} \cdot D < \infty.
\end{aligned}$$

For our notational convenience, if L^Φ is a normed space on $[0,1]$ and L^Ψ is a normed space on $\mathbb{T}$, we define the mixed norm $\|\cdot\|_{L_t^\Phi(L^\Psi)_x}$ on functions on $[0,1] \times \mathbb{T}$ by

$$\|f(x,t)\|_{L_t^\Phi(L^\Psi)_x} = \big\|\|f(\cdot,t)\|_{L^\Psi(\mathbb{T})}\big\|_{L^\Phi([0,1])}.$$

Now note that

$$\begin{aligned}
\|h(x,t)\|_{L_t^1(L\log L)_x} &\le 10\|h(x,t)\|_{L_t^1(L^2)_x} \\
&\le 100\|h(x,t)\|_{L_t^2(L^2)_x} = 100\|h(x,t)\|_{L^2(\mathbb{T}\times[0,1])} \\
&\le 100 \cdot ((A(2))^{-1} \cdot D)^{1/2} = C' < \infty.
\end{aligned}$$

By Lemma 3 we then see that

$$|\{(x,t) : |Th(x,t)| > \alpha\}| \le \int_0^1 \frac{C\|h(\cdot,t)\|_{L\log L(\mathbb{T})}}{\alpha}\,dt \le \frac{C \cdot C'}{\alpha}.$$

This however is in contradiction to (2), which holds for arbitrarily large values of M. ∎

We now see that T is of restricted weak type $(1,1)$ and of restricted weak type $(2,2)$. By the extension of the Marcinkiewicz theorem to the case of restricted-weak endpoints (see [7] for details) we have, for $1 < p < 3/2$ and for all simple functions f,

$$\|Tf\|_{L^p(\mathbb{T})} \lesssim \frac{1}{p-1} \|f\|_{L^p(\mathbb{T})}.$$

Applying the Yano extrapolation theorem [9], we then deduce that

$$\|Tf\|_{L^1(\mathbb{T})} \lesssim \|f\|_{L \log L(\mathbb{T})}$$

for all simple functions f supported on $\mathbb{T}$, as desired. ∎

We emphasize that the following corollary arises from the proof above.

COROLLARY 1. *Suppose T is a sublinear translation invariant operator acting on $L^1(\mathbb{T})$ which is of restricted weak type $(1,1)$. If f is a simple function supported on $\mathbb{T}$, then*

$$\|Tf\|_{L^p(\mathbb{T})} \le C_p \|f\|_{L^p(\mathbb{T})}, \qquad 1 < p < 2,$$

where $C_p \sim 1/(p-1) + 1/(2-p)$.

A natural question for subsequent investigation is whether or not this corollary can be extended to encompass values of p greater than or equal to 2.

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Received 21 January 2003;
revised 14 October 2004 (4305)