COLLOQUIUM MATHEMATICUM

VOL. 103 2005 NO. 2

ESTIMATES FOR THE HARDY–LITTLEWOOD MAXIMAL FUNCTION ON THE HEISENBERG GROUP

BY

JACEK ZIENKIEWICZ (Wrocław)

Abstract. We prove the dimension free estimates of the $L^p \to L^p$, $1 < p \le \infty$, norms of the Hardy–Littlewood maximal operator related to the optimal control balls on the Heisenberg group $\mathbb{H}^n$.

Introduction. Let $\mathbb{H}^n$ be the Heisenberg Lie algebra, i.e. $\mathbb{R}^{2n+1}$ with the linear basis $h = \{e_{x_1}, \ldots, e_{x_n}, e_{y_1}, \ldots, e_{y_n}, e_z\}$ and commutator structure such that $[e_{x_i}, e_{y_i}] = e_z$ and other commutators are zero. We define the Lie group multiplication in $\mathbb{H}^n$ by the Campbell–Hausdorff formula. Then in the coordinates corresponding to h, for

$$g_1 = (\mathbf{x}^1, \mathbf{y}^1, z^1), \quad g_2 = (\mathbf{x}^2, \mathbf{y}^2, z^2), \quad \mathbf{x}^1, \mathbf{y}^1, \mathbf{x}^2, \mathbf{y}^2 \in \mathbb{R}^n,\ z^1, z^2 \in \mathbb{R},$$

the group product of g_1 and g_2 is given by the formula

$$g_1 g_2 = \left(\mathbf{x}^1 + \mathbf{x}^2, \mathbf{y}^1 + \mathbf{y}^2, z^1 + z^2 - \frac{1}{2}\,\mathbf{S}((\mathbf{x}^1, \mathbf{y}^1), (\mathbf{x}^2, \mathbf{y}^2))\right)$$

where the symplectic form $\mathbf{S}$ is defined by

$$\mathbf{S}((\mathbf{x}^1, \mathbf{y}^1), (\mathbf{x}^2, \mathbf{y}^2)) = \langle \mathbf{x}^1, \mathbf{y}^2 \rangle - \langle \mathbf{x}^2, \mathbf{y}^1 \rangle, \quad \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i.$$

Denote by $X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z$ the left invariant vector fields on $\mathbb{H}^n$ such that $X_1(0) = e_{x_1}, \ldots, Z(0) = e_z$. Then in coordinates we have

$$X_i = \partial_{x_i} + \frac{1}{2}\, y_i \partial_z, \quad Y_i = \partial_{y_i} - \frac{1}{2}\, x_i \partial_z, \quad Z = \partial_z. \tag{0.1}$$

For $t > 0$ let us define the isotropic dilations on $\mathbb{H}^n$ by $\delta_t(e_{x_i}) = te_{x_i}$, $\delta_t(e_{y_i}) = te_{y_i}$, $\delta_t(e_z) = t^2 e_z$ and extend these formulas by linearity to $\mathbb{H}^n$. Then the dilations δ_t form a one-parameter group of automorphisms of $\mathbb{H}^n$.

2000 *Mathematics Subject Classification*: 42B25, 43A15.

Key words and phrases: Hardy–Littlewood maximal function, Heisenberg group.

The author is supported by the KBN grant no. 1 PO3A 018 26.

An absolutely continuous curve $\gamma : [0,1] \to \mathbb{H}^n$ is called *admissible* if

$$\frac{d}{dt}\gamma(t) = \sum (a_i(t)X_i(\gamma(t)) + b_i(t)Y_i(\gamma(t))) \quad \text{and} \quad \gamma(0) = 0. \tag{0.2}$$

We define the length of γ (cf. [G]) by

$$|\gamma| = \int\limits_0^1 \Big(\sum_i (a_i^2(t) + b_i^2(t))\Big)^{1/2} dt$$

and then the *optimal control norm* by

$$d(g) = \inf\{|\gamma| : \gamma(1) = g \text{ and } \gamma \text{ is admissible}\}.$$

It follows directly from the definition that d is subadditive, symmetric on $\mathbb{H}^n$ and homogeneous of degree 1 with respect to δ_t, that is, $d(\delta_t(g)) = td(g)$. It is well known that d is everywhere finite and bounded on compact subsets of $\mathbb{H}^n$. By homogeneity d is mutually Hölder continuous with respect to the euclidean norm. We define a metric on $\mathbb{H}^n$ by $d(g_1, g_2) = d(g_1 g_2^{-1})$.

We will identify $\mathbb{H}^n$ with $\mathbb{C}^n \times \mathbb{R}$ putting $z_j = x_j + iy_j$. Let $\mathbb{U}^n$ denote the group of unitary matrices on $\mathbb{C}^n$. In what follows we will consider $\mathbb{U}^n$ as a subgroup of $\mathbb{O}^{2n}(\mathbb{R})$. Then the natural action of $\mathbb{U}^n$ on $\mathbb{R}^{2n}$ preserves the symplectic form:

$$\mathbf{S}((\mathbf{x}^1, \mathbf{y}^1), (\mathbf{x}^2, \mathbf{y}^2)) = \mathbf{S}(\mathbf{U}(\mathbf{x}^1, \mathbf{y}^1), \mathbf{U}(\mathbf{x}^2, \mathbf{y}^2)) \quad \text{for } \mathbf{U} \in \mathbb{U}^n. \tag{0.3}$$

Consequently, the formula $\mathbf{Ug} = (\mathbf{U}(\mathbf{x}, \mathbf{y}), z)$ for $\mathbf{U} \in \mathbb{U}^n$ and $g = (\mathbf{x}, \mathbf{y}, z)$ defines an automorphic action of $\mathbb{U}^n$ on $\mathbb{H}^n$. See [F] for the proofs.

The Lebesgue measure $dg = d\mathbf{x}d\mathbf{y}dz$ is a bi-invariant measure on $\mathbb{H}^n$.

Acknowledgments. The author wishes to thank Fulvio Ricci and Waldemar Hebisch for some discussions on the subject of the paper, Jacek Dziubański for careful reading of the manuscript, and the referee for improving the presentation of the paper.

Main theorem. Let $B(r) = \{g : d(g) \le r\}$. The aim of this note is to prove the following result:

THEOREM 1. *Let $p > 1$ and*

$$B^* f(g) = \sup_{r>0} \frac{1}{|B(r)|} \int\limits_{B(r)} |f(gy^{-1})|\, dy$$

where $|\ |$ denotes the Lebesgue measure on $\mathbb{H}^n$. Then the operator B^* is bounded on $L^p(\mathbb{H}^n)$ with the norm which is controlled independently of n.

For the classical Hardy–Littlewood maximal function on $\mathbb{R}^n$ the above theorem is due to E. M. Stein (see [B], [S2] and [S3] for this and similar results). The question considered in Theorem 1 is due to M. Cowling.

We note that since the measure metric space $(\mathbb{H}^n, d, dg)$ is a space of homogeneous type, the maximal function B^* is bounded on L^p, $p > 1$, and in fact is of weak type $(1,1)$ (cf. [S1]).

The idea of our proof of Theorem 1 follows [S3]. We begin with the following standard fact:

LEMMA 1. *The metric d is radial: it satisfies*

$$d(\mathbf{x}^1, \mathbf{y}^1, z) = d(\mathbf{x}^2, \mathbf{y}^2, z) \quad \textit{for } \|\mathbf{x}^1\|^2 + \|\mathbf{y}^1\|^2 = \|\mathbf{x}^2\|^2 + \|\mathbf{y}^2\|^2$$

where $\| \ \|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Proof. In order to prove the lemma it suffices to observe that the unitary matrices act transitively on the unit sphere in $\mathbb{C}^n$, map admissible curves onto admissible curves and preserve $|\gamma|$. The statements above follow easily from the formula

$$\frac{d}{dt}\Gamma(t) = \sum A_i(t)X_i(\Gamma(t)) + B_i(t)Y_i(\Gamma(t))$$

where for a unitary matrix $\mathbf{U}$ and vectors $\mathbf{a}(t) = (a_1(t), \ldots, a_n(t))$, $\mathbf{b}(t) = (b_1(t), \ldots, b_n(t))$ we have

$$\Gamma(t) = \mathbf{U}(\gamma(t)), \qquad (A(t), B(t)) = \mathbf{U}(\mathbf{a}(t), \mathbf{b}(t)).$$

Denote by $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}$, $\pi(\mathbf{x}, \mathbf{y}, z) = (\mathbf{x}, \mathbf{y})$, the projection onto the generating subspace of $\mathbb{H}^n$. To prove the formula, we apply $\mathbf{U}$ to (0.2). Then by (0.1) and (0.3),

$$\begin{aligned}\frac{d}{dt}\Gamma(t) &= \mathbf{U}\left(\frac{d}{dt}\gamma(t)\right) = \mathbf{U}\Big(\sum(a_i(t)X_i(\gamma(t)) + b_i(t)Y_i(\gamma(t)))\Big)\\ &= \mathbf{U}(\mathbf{a}, \mathbf{b}) + \mathbf{S}((\mathbf{a}, \mathbf{b}), \pi(\gamma))e_z\\ &= \mathbf{U}(\mathbf{a}, \mathbf{b}) + \mathbf{S}(\mathbf{U}(\mathbf{a}, \mathbf{b}), \mathbf{U}(\pi(\gamma)))e_z\\ &= \sum(A_i(t)X_i(\mathbf{U}(\gamma(t))) + B_i(t)Y_i(\mathbf{U}(\gamma(t)))).\end{aligned}$$

The lemma follows.

LEMMA 2. *We have*

$$d(\mathbf{x}, \mathbf{y}, z) \geq d(\mathbf{x}, \mathbf{y}, 0) = (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^{1/2}.$$

Proof. Fix $g = (\mathbf{x}, \mathbf{y}, z) \in \mathbb{H}^n$ and let γ be an admissible curve joining $(0,0,0)$ and g such that $|\gamma| \leq (1+\varepsilon)d(\mathbf{x}, \mathbf{y}, z)$. It follows directly from (0.1) that for every $g \in \mathbb{H}^n$ and $1 \leq i \leq n$ we have

$$\pi(X_i(g)) = \partial_{x_i}(\pi g), \qquad \pi(Y_i(g)) = \partial_{y_i}(\pi g).$$

Consequently, the projection $\widetilde{\gamma} = \pi\gamma$ of the curve γ on $\mathbb{R}^{2n}$ satisfies the equation

$$\frac{d}{dt}\widetilde{\gamma}(t) = \sum_j (a_j(t)\pi(X_j(\gamma)) + b_j(t)\pi(Y_j(\gamma))) = \sum_j (a_j(t)\partial_{x_j}(\widetilde{\gamma}) + b_j(t)\partial_{y_j}(\widetilde{\gamma}))$$

and $\widetilde{\gamma}(1) = (\mathbf{x}, \mathbf{y}, 0)$.

By the classical isoperimetric inequality we have

$$r = (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^{1/2} \le |\widetilde{\gamma}| \le (1+\varepsilon)d(\mathbf{x}, \mathbf{y}, z)$$

and consequently $r \le d(\mathbf{x}, \mathbf{y}, t)$.

Applying the above argument to $g = (\mathbf{x}, \mathbf{y}, 0)$ we get

$$r = (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^{1/2} \le d(\mathbf{x}, \mathbf{y}, 0).$$

In order to obtain the opposite inequality $d(\mathbf{x}, \mathbf{y}, 0) \le r$ it suffices to check that by (0.1) the line segment joining $(0, 0, 0)$ and $(\mathbf{x}, \mathbf{y}, 0)$ is an admissible curve and then to compute its length. We omit the calculations. The lemma follows.

The following simple fact will be crucial for our argument.

LEMMA 3. *Let $z \in \mathbb{R}$ and let*

$$\Gamma_z^r = \{(\mathbf{x}, \mathbf{y}, z) : (\mathbf{x}, \mathbf{y}, z) \in B(r)\} \quad \text{and} \quad m_r(z) = |\Gamma_z^r|$$

where $|\ |$ denotes the Lebesgue measure on $\mathbb{R}^{2n}$. Then $z \mapsto m_r(z)$ is decreasing for $z \ge 0$ and increasing for $z \le 0$ (and in fact symmetric, but we will not use it).

Proof. Let $\Gamma_z = \Gamma_z^1$, $m(z) = m_1(z)$. By homogeneity of d it suffices to prove the lemma for $m(z)$. Let $z \ge 0$. For a fixed $0 < t < 1$ consider the set

$$\delta_t(\Gamma_z) = \{(t\mathbf{x}, t\mathbf{y}, t^2 z) : (\mathbf{x}, \mathbf{y}, z) \in \Gamma_z\}.$$

By Lemma 2, for a fixed $(\mathbf{x}, \mathbf{y}, z) \in \Gamma_z$ with $(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^{1/2} = r$ we have $r = d(\mathbf{x}, \mathbf{y}, 0) \le d(\mathbf{x}, \mathbf{y}, z) \le 1$.

Applying the estimates

$$\text{(i)} \qquad d(\delta_t(\mathbf{x}, \mathbf{y}, z)) = t d(\mathbf{x}, \mathbf{y}, z) \le t$$

and

$$\text{(ii)} \qquad d(s\mathbf{x}, s\mathbf{y}, 0) = sr \le 1 - t, \qquad \text{valid for } s \in [0, r^{-1}(1-t)],$$

we get

$$d((t+s)\mathbf{x}, (t+s)\mathbf{y}, t^2 z) = d((t\mathbf{x}, t\mathbf{y}, t^2 z)(s\mathbf{x}, s\mathbf{y}, 0)) \le d(t\mathbf{x}, t\mathbf{y}, t^2 z) + d(s\mathbf{x}, s\mathbf{y}, 0) \le 1,$$

hence by Lemma 1,

$$A=\{(t(\mathbf{x},\mathbf{y})+s(\mathbf{x},\mathbf{y}),t^2z):s\in[0,r^{-1}(1-t)],\ \|x\|^2+\|y\|^2=r^2\}\subset\Gamma_{t^2z}.$$

In order to prove the lemma it suffices to observe that the orthogonal projection of Γ_z onto the plane containing Γ_{t^2z} is contained in A. Since A is a ring, this easily follows from the inequalities

$$rt\le r=rt+r(1-t)\le rt+1-t\qquad\text{for } r\le 1,\ t\le 1.$$

Since the proof for $z\le 0$ is similar, the lemma follows.

Denote by μ_r the uniform probability measure supported on the sphere $\{(\mathbf{x},\mathbf{y},0):\|\mathbf{x}\|^2+\|\mathbf{y}\|^2=1\}$ of radius r and by S_r the spherical average

$$S_rf(g)=\int f(g\delta_r(\omega))\,d\mu_1(\omega)=\mu_r*f(g).$$

Then by [NTh], the corresponding spherical maximal operator

$$S^*f(g)=\sup_{r>0}S_r|f|(g)$$

is bounded on L^p for $p>(2n-1)/(2n-2)$, $n\ge 2$.

Let M^* denote the one-dimensional Hardy–Littlewood maximal function along the central direction. The lemma below reduces our result to L^p estimates for S^*.

Lemma 4. *The following estimate holds*:

$$B^*f(g)\le M^*(S^*(f))(g),$$

Proof. Since B^* is left invariant, it suffices to prove the lemma for $g=0$. Observe that for the appropriate constant b_n, in the polar coordinates $(\mathbf{x},\mathbf{y})=(\mathbf{x}(\omega,\varrho),\mathbf{y}(\omega,\varrho))=\omega\varrho$, $(\omega,\varrho)\in\mathbb{S}^{2n-1}\times\mathbb{R}^+$, Lemma 1 implies

$$\begin{aligned}
&|B(r)|^{-1}\int\limits_{B(r)}|f|(g)\,dg\\
&=\iint b_n\varrho^{2n-1}\int\limits_{\substack{(\mathbf{x}(\omega,\varrho),\mathbf{y}(\omega,\varrho),z)\in\Gamma_z^r\\ \|\mathbf{x}(\omega,\varrho)\|^2+\|\mathbf{y}(\omega,\varrho)\|^2=\varrho^2}}|f|(\mathbf{x},\mathbf{y},z)\,d\mu(\omega)\,d\varrho\,dz\\
&\le\int\int\limits_{\{\exists_\omega(\mathbf{x}(\omega,\varrho),\mathbf{y}(\omega,\varrho),z)\in\Gamma_z^r,\|\mathbf{x}(\omega,\varrho)\|^2+\|\mathbf{y}(\omega,\varrho)\|^2=\varrho^2\}}b_n\varrho^{2n-1}S^*|f|(0,0,z)\,d\varrho\,dz\\
&=\int m_r(z)S^*f(0,0,z)\,dz\le M^*(S^*(f))(0).
\end{aligned}$$

The last inequality follows from the Stein theorem (see [S1, II, 2.1]), owing to Lemma 3 and the obvious fact $\int m_r dz=1$ (consider the constant function in the above calculations). The lemma follows.

Lemma 5. *Let $p > 1$ be fixed. Then the maximal function S^* is bounded on $L^p(\mathbb{H}^n)$, $n \geq n(p) = 1+(2p-1)/(2p-2)$; its operator norm is controlled independently of n.*

Proof. Let $n \geq n(p)$ and let $A \in \mathbb{U}^n$. Denote by $dm(A)$ the right invariant probability measure on $\mathbb{U}^n$. Let $\widetilde{\mu}_r$ denote the uniform probability measure supported on the $2n(p)-1$-dimensional sphere of radius r contained in the plane $\Pi = \{(\mathbf{x},\mathbf{y},0) : x_{n(p)+1} = y_{n(p)+1} = \cdots = x_n = y_n = 0\} \subset \mathbb{R}^{2n}$ and centered at $(0,0,0)$. Observe that the fomula

$$\nu_r(E) = \int_{\mathbb{U}^n} \widetilde{\mu}_r(A(E))\, dm(A)$$

defines an $\mathbb{U}^n$-invariant probability measure on the sphere of radius r in $\mathbb{R}^{2n}$ so $\nu_r = \mu_r$. Hence

$$S_r * f(g) = \int_{\mathbb{U}^n} \widetilde{\mu}_r * A(f)(A^{-1}(g))\, dm(A)$$

and

$$\begin{aligned} S^* f(g) &\leq \int_{\mathbb{U}^n} \sup_{r\geq 0} \widetilde{\mu}_r * |A(f)|(A^{-1}(g))\, dm(A) \\ &= \int_{\mathbb{U}^n} \widetilde{\mu}^* |A(f)|(A^{-1}(g))\, dm(A) \end{aligned}$$

where $A(f)(g) = f(A(g))$ is an isometry in L^p, $p > 0$. Consequently, since for any Banach space X and strongly measurable X-valued function on $\mathbb{U}^n$ one has $\|\int f(A)\, dm(A)\|_X \leq \int \|f(A)\|_X\, dm(A)$ we get

$$\|S^*\|_{L^p(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} \leq \|\widetilde{\mu}^*\|_{L^p(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} \leq \|\widetilde{\mu}^*\|_{L^p(\mathbb{H}^{n(p)})\to L^p(\mathbb{H}^{n(p)})}.$$

To see the last estimate we will identify $\mathbb{H}^n = \mathbb{H}^{n(p)} \times \mathbb{R}^{2(n-n(p))}$ putting $g = (g_1, g_2)$ where

$$g_1 = (x_1, \ldots, x_{n(p)}, 0, \ldots, y_1, \ldots, y_{n(p)}, 0, \ldots, z) \in \Pi \times \mathbb{R} = \mathbb{H}^{n(p)}$$

and

$$g_2 = (0, \ldots, x_{n(p)+1}, \ldots, x_n, 0, \ldots, y_{n(p)+1}, \ldots, y_n, 0) \in \mathbb{R}^{2(n-n(p))}.$$

Then by the group multiplication formula

$$\widetilde{\mu}_r * f_{g_2}(g_1) = \widetilde{\mu}_r *_{\mathbb{H}^{n(p)}} f_{g_2}(g_1)$$

where $*_{\mathbb{H}^{n(p)}}$ denotes convolution on $\mathbb{H}^{n(p)}$ and $f_{g_2}(\cdot) = f(\cdot, g_2)$. Consequently, by Fubini's theorem,

$$\|\widetilde{\mu}^*\|_{L^p(\mathbb{H}^n)\to L^p(\mathbb{H}^n)} \leq \|\widetilde{\mu}^*\|_{L^p(\mathbb{H}^{n(p)})\to L^p(\mathbb{H}^{n(p)})}.$$

The lemma now follows from [NTh].

Proof of Theorem 1. The norm of M^* on $L^p(\mathbb{H}^n)$ is equal to the norm of the classical Hardy–Littlewood maximal operator on $L^p(\mathbb{R})$. Observe that then M^* acts only on f_2, which proves the statement. Applying Lemmas 4 and 5 we get a uniform bound of B^* on $L^p(\mathbb{H}^n)$, $n > n(p)$. Since B^* is bounded on $L^p(\mathbb{H}^n)$ for each n, the theorem follows.

REMARK. The proof of Theorem 1 works obviously for the family of Folland balls

$$B_r = \{(\mathbf{x}, \mathbf{y}, z) : (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^2 + |z|^2 \le r^4\}.$$

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Institute of Mathematics
Wrocław University
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: zenek@hera.math.uni.wroc.pl

Received 9 February 2004;
revised 3 November 2004 (4422)