COLLOQUIUM MATHEMATICUM

VOL. 103 2005 NO. 2

A WEIGHTED ERGODIC MAXIMAL EQUALITY FOR NONSINGULAR SEMIFLOWS

BY

LASHA EPHREMIDZE (Tbilisi) and RYOTARO SATO (Okayama)

Abstract. A weighted ergodic maximal equality is proved for a conservative and ergodic semiflow of nonsingular automorphisms.

1. Introduction. Let $(X, \mathbb{S}, \mu)$ be a σ-finite measure space and $\{T_t\}_{t\ge 0}$ a measurable semiflow of nonsingular automorphisms of $(X, \mathbb{S}, \mu)$. For $f \in L_1(\mu)$ and a weight g (we call a measurable function g a *weight* if it is positive almost everywhere and the function of variable $t \ge 0$,

$$t \mapsto g(T_t x)\,\frac{d\mu\circ T_t}{d\mu}(x),$$

is locally integrable for a.a. $x \in X$) the *weighted maximal ergodic function* f_g^* is defined by

$$f_g^*(x) = \sup_{b>0} \frac{\int_0^b f(T_t x)\,\frac{d\mu\circ T_t}{d\mu}(x)\,dt}{\int_0^b g(T_t x)\,\frac{d\mu\circ T_t}{d\mu}(x)\,dt}.$$

In this paper we assume that the semiflow $\{T_t\}$ is *conservative* and *ergodic*, and prove the weighted ergodic maximal equality:

THEOREM. *We have*

$$\alpha \int_{\{f_g^*>\alpha\}} g\,d\mu = \int_{\{f_g^*>\alpha\}} f\,d\mu \tag{1}$$

for all

$$\alpha > \frac{\int_X f\,d\mu}{\int_X g\,d\mu} \tag{2}$$

(*it is assumed that the right-hand side of* (2) *is* 0 *whenever* $\int_X g\,d\mu = \infty$).

2000 *Mathematics Subject Classification*: Primary 28D10.

Key words and phrases: one-parameter semigroup of nonsingular automorphisms, weighted ergodic maximal function.

For the case of measure-preserving transformations on a probability space and when $g \equiv 1$, the corresponding equality

$$\alpha\mu\{f^* > \alpha\} = \int\limits_{\{f^*>\alpha\}} f\,d\mu \tag{3}$$

was first proved in [5] (see also [6]). By using a different technique, a generalization of this result to infinite measure spaces was given in [7] where the equality (1) for the weighted ergodic maximal function was first established as well. A different proof of (3) was proposed in [1], applying the continuous version of the "filling scheme" method (see [6]).

Later the equality (3) was generalized to nonsingular flows in [8], i.e. the theorem was proved under the hypothesis $g \equiv 1$.

Recently a new simpler proof of (3) appeared in [2] and [3] (respectively, for probability and σ-finite measure spaces). In the present paper we show that the same method of proof can be applied to obtain a result in the most general setting. The above theorem covers all the previously known situations. At the same time it represents a new result as far as σ-finite measure spaces and arbitrary weights (even not integrable) are considered.

To characterize roughly the proof of the Theorem, the situation on the real line is transmitted to the general case by applying the continuous version of the Chacon–Ornstein theorem: *If U_t is a one-parameter measurable conservative and ergodic semigroup of contractions on $L_1(\mu)$ and $f, g \in L_1(\mu)$, $g > 0$, then*

$$\lim_{b\to\infty} \frac{\int_0^b U_t f\,dt}{\int_0^b U_t g\,dt} = \frac{\int_X f\,d\mu}{\int_X g\,d\mu}$$

(see [4]).

Further applications of equality (1) can be found in [7], [8].

2. Auxiliary lemmas. In this section we obtain two lemmas for the group of translations on the real line $\mathbb{R}$. These lemmas are known but we give their proofs for the sake of completeness.

Let $f, g \in L_{\mathrm{loc}}(\mathbb{R})$, $g > 0$, and define the weighted maximal function

$$M_g f(t) = \sup_{r>t} \frac{\int_t^r f(s)\,ds}{\int_t^r g(s)\,ds}.$$

Obviously $\{M_g f > \alpha\} = \{t \in \mathbb{R} : M_g f(t) > \alpha\}$ is an open set.

Lemma 1. *If (a, b) is a bounded connected component of $\{M_g f > \alpha\}$, then*

$$\int\limits_a^b f(t)\,dt = \alpha \int\limits_a^b g(t)\,dt. \tag{4}$$

Proof. Since $a \notin \{M_g f > \alpha\}$, we have

$$\int_a^b f(t)\,dt \le \alpha \int_a^b g(t)\,dt.$$

We shall show the reverse inequality

$$\int_a^b f(t)\,dt \ge \alpha \int_a^b g(t)\,dt, \tag{5}$$

which completes the proof of (4).

In order to prove (5), let us show that for each $r \in (a, b)$ we have

$$\int_r^b f(t)\,dt \ge \alpha \int_r^b g(t)\,dt \tag{6}$$

(in fact, strict inequality holds in (6) but we do not need it here); then one can let r tend to a.

For each $s \in (a, b)$, there exists $s' \in (s, b]$ such that

$$\int_s^{s'} f(t)\,dt > \alpha \int_s^{s'} g(t)\,dt. \tag{7}$$

Indeed, the existence of $s' > s$ satisfying (7) is equivalent to the fact that $M_g f(s) > \alpha$. If $s' > b$, then

$$\int_b^{s'} f(t)\,dt \le \alpha \int_b^{s'} g(t)\,dt \tag{8}$$

(since $b \notin \{M_g f > \alpha\}$) and it follows from (7) and (8) that

$$\int_s^b f(t)\,dt = \int_s^{s'} f(t)\,dt - \int_b^{s'} f(t)\,dt > \alpha\Big(\int_s^{s'} g(t)\,dt - \int_b^{s'} g(t)\,dt\Big) = \alpha \int_s^b g(t)\,dt.$$

So we can take $s' = b$ in (7).

Take $r \in (a, b)$ and let

$$s = \sup\Big\{r' \in (r, b] : \int_r^{r'} f(t)\,dt \ge \alpha \int_r^{r'} g(t)\,dt\Big\}. \tag{9}$$

Obviously,

$$\int_r^s f(t)\,dt \ge \alpha \int_r^s g(t)\,dt$$

and we need to observe that $s = b$ (see (6)). Indeed, if we assume that $s < b$, then there exists $s' \in (s, b]$ satisfying (7) and we have

$$\int_r^{s'} f(t)\,dt = \int_r^s f(t)\,dt + \int_s^{s'} f(t)\,dt > \alpha\Big(\int_r^s g(t)\,dt + \int_s^{s'} g(t)\,dt\Big) = \alpha\int_r^{s'} g(t)\,dt,$$

which contradicts the maximality of s (see (9)). ■

COROLLARY. *If real numbers a and b do not belong to $\{M_g f > \alpha\}$, then*

$$\int_{(a,b)\cap\{M_g f>\alpha\}} f(t)\,dt = \alpha \int_{(a,b)\cap\{M_g f>\alpha\}} g(t)\,dt,$$

i.e.,

$$\int_a^b \mathbf{1}_{\{M_g f>\alpha\}} f(t)\,dt = \alpha\int_a^b \mathbf{1}_{\{M_g f>\alpha\}} g(t)\,dt. \tag{10}$$

Proof. $(a,b)\cap\{M_g f > \alpha\}$ is exactly a union of at most countably many bounded connected components of $\{M_g f > \alpha\}$. ■

LEMMA 2. *Let $f, g \in L_{\mathrm{loc}}(\mathbb{R})$, $g > 0$, and*

$$\int_0^\infty g(t)\,dt = \infty. \tag{11}$$

If $(a,\infty) \subset \{M_g f > \alpha\}$ for some a and α, then

$$\limsup_{b\to\infty} \frac{\int_0^b f(t)\,dt}{\int_0^b g(t)\,dt} \ge \alpha. \tag{12}$$

Proof. For each $s \in (a,\infty)$ there exists $s' \in (s,\infty)$ such that (7) holds and, as in the proof of Lemma 1,

$$\sup\Big\{s > a : \int_a^s f(t)\,dt > \alpha\int_a^s g(t)\,dt\Big\} = \infty.$$

Consequently,

$$\limsup_{b\to\infty} \frac{\int_a^b f(t)\,dt}{\int_a^b g(t)\,dt} \ge \alpha$$

and, by (11), inequality (12) holds as well. ■

3. Proof of Theorem. All sets and functions introduced below are assumed to be measurable; all relations are assumed to hold modulo sets of measure zero.

Since each T_t is nonsingular, the Radon–Nikodym theorem defines the positive (almost everywhere for each $t \ge 0$) function $\omega_t(x) = \frac{d\mu\circ T_t}{d\mu}(x)$ on X.

It can be assumed that the map $(x,t) \mapsto \omega_t(x)$ is measurable on $X \times \mathbb{R}_+$. The "chain rule" gives

$$\omega_{t+s}(x) = \omega_s(x)\omega_t(T_s x).$$

If we set $U_t f(x) = f(T_t x)\omega_t(x)$, then it can be easily seen that $\{U_t\}_{t\ge 0}$ becomes a one-parameter semigroup of positive linear isometries on $L_1(\mu)$. Hence, for any $\phi \in L_1(\mu)$, the function of variable $t \ge 0$, $t \mapsto \phi(T_t x)\omega_t(x)$, is locally integrable for almost all $x \in X$. It follows that every strictly positive function $g \in L_1(\mu)$ becomes a weight.

Since $\{T_t\}$ is conservative, for any positive $h \in L_1(\mu)$ we have

$$\int_0^\infty h(T_t x)\omega_t(x)\,dt = \infty \tag{13}$$

for a.a. $x \in X$.

Let $\phi_x(t) = \phi(T_t x)$ and $\omega_x(t) = \omega_t(x)$, $t \ge 0$. Note that for a.a. $x \in X$ the equation

$$f_g^*(T_s x) = M_{g_x\omega_x}(f_x\omega_x)(s) \tag{14}$$

holds for a.a. $s \in \mathbb{R}_+$ since

$$f_g^*(T_s x) = \sup_{b>0} \frac{\int_0^b f(T_t(T_s x))\omega_t(T_s x)\,dt}{\int_0^b g(T_t(T_s x))\omega_t(T_s x)\,dt} = \sup_{b>0} \frac{\int_0^b f_x(t+s)\,\frac{\omega_{t+s}(x)}{\omega_s(x)}\,dt}{\int_0^b g_x(t+s)\,\frac{\omega_{t+s}(x)}{\omega_s(x)}\,dt}$$

$$= \sup_{b>0} \frac{\int_0^b f_x(t+s)\omega_x(t+s)\,dt}{\int_0^b g_x(t+s)\omega_x(t+s)\,dt} = \sup_{b>0} \frac{\int_s^{s+b} f_x(t)\omega_x(t)\,dt}{\int_s^{s+b} g_x(t)\omega_x(t)\,dt} = M_{g_x\omega_x}(f_x\omega_x)(s).$$

Set $E = \{f_g^* > \alpha\}$ and take an arbitrary nonnegative function h satisfying

$$\int_X h\,d\mu = 1.$$

Fix any $x \in X$ for which (13) holds, (14) is valid for a.a. $s \in \mathbb{R}_+$ and the following relations hold:

$$\lim_{b\to\infty} \frac{\int_0^b f_x(t)\omega_t(x)\,dt}{\int_0^b g_x(t)\omega_t(x)\,dt} = \frac{\int_X f\,d\mu}{\int_X g\,d\mu}, \tag{15}$$

$$\lim_{b\to\infty} \frac{\int_0^b \mathbf{1}_E g(T_t x)\omega_t(x)\,dt}{\int_0^b h(T_t x)\omega_t(x)\,dt} = \int_X \mathbf{1}_E g\,d\mu = \int_E g\,d\mu, \tag{16}$$

$$\lim_{b\to\infty} \frac{\int_0^b \mathbf{1}_E f(T_t x)\omega_t(x)\,dt}{\int_0^b h(T_t x)\omega_t(x)\,dt} = \int_X \mathbf{1}_E f\,d\mu = \int_E f\,d\mu \tag{17}$$

(these relations hold for a.a. $x \in X$ according to Chacon–Ornstein's above mentioned theorem).

Formally we can assume that $g_x\omega_x(t) = f_x\omega_x(t) = 1$ for $t < 0$ and set $e = \{t \in \mathbb{R} : M_{g_x\omega_x}(f_x\omega_x)(t) > \alpha\}$. By (14), $T_t x \in E$ is equivalent to $t \in e$ for a.a. $t \in \mathbb{R}_+$, i.e. $\mathbf{1}_E(T_t x) = \mathbf{1}_e(t)$ on $\mathbb{R}_+$, and we can substitute this equality in (16) and (17) to obtain

$$\lim_{b\to\infty} \frac{\int_0^b \mathbf{1}_e g_x(t)\omega_x(t)\,dt}{\int_0^b h(T_t x)\omega_t(x)\,dt} = \int_E g\,d\mu, \tag{18}$$

$$\lim_{b\to\infty} \frac{\int_0^b \mathbf{1}_e f_x(t)\omega_x(t)\,dt}{\int_0^b h(T_t x)\omega_t(x)\,dt} = \int_E f\,d\mu. \tag{19}$$

Since (15) and (2) hold we can apply Lemma 2 and conclude that there exists an increasing sequence of positive numbers $b_n \to \infty$ such that $b_n \notin \{M_{g_x\omega_x}(f_x\omega_x) > \alpha\}$ and, by (13), we can rewrite (18) and (19) as

$$\lim_{n\to\infty} \frac{\int_{b_1}^{b_n} \mathbf{1}_e g_x(t)\omega_x(t)\,dt}{\int_0^{b_n} h(T_t x)\omega_t(x)\,dt} = \int_E g\,d\mu,$$

$$\lim_{n\to\infty} \frac{\int_{b_1}^{b_n} \mathbf{1}_e f_x(t)\omega_x(t)\,dt}{\int_0^{b_n} h(T_t x)\omega_t(x)\,dt} = \int_E f\,d\mu.$$

Now, applying the Corollary of Lemma 1 to the functions $f_x(t)\omega_x(t)$ and $g_x(t)\omega_x(t)$, it remains to observe that

$$\alpha \int_{b_1}^{b_n} \mathbf{1}_e g_x(t)\omega_x(t)\,dt = \int_{b_1}^{b_n} \mathbf{1}_e f_x(t)\omega_x(t)\,dt$$

(see (10)) and the proof of equality (1) is complete. ∎

Acknowledgements. The first author wishes to express his gratitude to the Matsumae International Foundation for giving him the opportunity to visit Okayama University during the period of the above research.

REFERENCES

[1] L. Ephremidze, *On the majorant of ergodic means (continuous case)*, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruz. SSR 98 (1990), 112–124 (in Russian).

[2] —, *A new proof of the Ergodic Maximal Equality*, Real Anal. Exchange 29 (2003-04), 409–412.

[3] —, *On the Ergodic Maximal Equality*, Proc. A. Razmadze Math. Inst. 132 (2003), 89–92.

[4] M. Lin, *Semi-groups of Markov operators*, Boll. Un. Mat. Ital. (4) 6 (1972), 20–44.

[5] B. Marcus and K. Petersen, *Balancing ergodic averages*, in: Lecture Notes in Math. 729, Spinger, Berlin, 1979, 126–143.

[6] K. E. Petersen, *Ergodic Theory*, Cambridge Univ. Press, Cambridge, 1983.
[7] R. Sato, *Maximal functions for a semiflow in an infinite measure space*, Pacific J. Math. 100 (1982), 437–443.
[8] —, *An ergodic maximal equality for nonsingular flows and applications*, Acta Math. Hungar. 61 (1993), 51–58.

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
M. Aleksidze St.
Tbilisi 1093, Georgia
E-mail: lasha@rmi.acnet.ge

Department of Mathematics
Okayama University
Okayama 700-8530, Japan
E-mail: satoryot@math.okayama-u.ac.jp

Received 8 November 2004 (4526)