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THE GENERALIZED SCHOENFLIES THEOREM FOR ABSOLUTE SUSPENSIONS

BY

DAVID P. BELLAMY (Newark, DE) and JANUSZ M. LYSKO (Chester, PA)

Abstract. The aim of this paper is to prove the generalized Schoenflies theorem for the class of absolute suspensions. The question whether the finite-dimensional absolute suspensions are homeomorphic to spheres remains open. Partial solution to this question was obtained in [Sz] and [Mi]. Morton Brown gave in [Br] an ingenious proof of the generalized Schoenflies theorem. Careful analysis of his proof reveals that modulo some technical adjustments a similar argument gives an analogous result for the class of absolute suspensions.

1. A brief history of the problem. The original question whether finite-dimensional absolute suspensions are topological spheres was asked by de Groot in [Gr]. It was also observed by de Groot that in order to provide the positive answer to the above question one needs "only" prove that such spaces are topological manifolds. Szymański proved in [Sz] that the answer to de Groot's question was positive for spaces of dimension not exceeding three. Finite-dimensional absolute suspensions are homogeneous ANR's (see e.g. [Mi], [Sz]). We see therefore that the positive answer to de Groot's question would provide a partial answer to the long standing question of R. H. Bing and K. Borsuk who asked in [BB] whether finite-dimensional, homogeneous ANR's are topological manifolds. The answer to the Bing–Borsuk question is known only for homogeneous ANR's of dimension not exceeding two. In a very interesting paper [Ja] W. Jakobsche indicates the level of difficulty of the Bing–Borsuk question by showing that the positive answer to their question would provide the solution to the Poincaré conjecture.

2. Terminology, definitions and the statements of results used in the proof. All spaces considered in this paper will be finite-dimensional and metric. If Y is a compact space then the *suspension* over Y, denoted by $S(Y)$, is the quotient of $Y \times [-1, 1]$ with two nondegenerate equivalence classes: $Y \times \{1\}$ and $Y \times \{-1\}$. The two points obtained by the identification of these sets are called the *vertices* of the suspension. The quotient of

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$Y \times [0,1]$ with the only nondegenerate equivalence class $Y \times \{1\}$ will be called the *cone* over Y and will be denoted by $C(Y)$. We will always assume that Y is a subset of $S(Y)$ as well as of $C(Y)$ and we will identify Y with the set of points (y,t) for which $t = 0$. A compact space X is called an *absolute suspension* (abbr. $X \in \mathrm{AS}$) provided that for every pair of different points $p, q \in X$ there exists a space Y and a homeomorphism $h : X \to S(Y)$ such that $h(p)$ and $h(q)$ are the vertices of $S(Y)$. We say that the homeomorphism h determines the absolute suspension structure on X. It should be noted immediately that the space Y appearing in the definition of the absolute suspension is not uniquely determined. For example, R. Edwards [Ed] has shown that the double suspension of a certain 3-dimensional homology sphere is the topological sphere S^5. Therefore there exists a space that is not a sphere but whose suspension is a sphere.

Throughout the paper X will denote an n-dimensional absolute suspension. Every space $Y \subset X$ for which $X \overset{\text{top}}{=} S(Y)$ will be called an *equator* of X. The family of all equators of the space X will be denoted by E_X. Following M. Brown we will call a subset M of W an *inverse set* of a mapping $f : W \to Z$ if M contains more than one point and there is a point $z \in Z$ for which $M = f^{-1}(z)$. If X is an absolute suspension then every space homeomorphic to the cone $C(Y)$ over any $Y \in E_X$ will be called a *Y-cell.* If Q is a Y-cell and $h : Q \to C(Y)$ is a homeomorphism then the *interior* $\mathring{Q}$ of Q is defined as $h^{-1}(C(Y) \setminus Y)$ and the *boundary* $\dot{Q}$ of Q is defined as $h^{-1}(Y)$. A set M will be called *Y-cellular* in an n-dimensional compact space S if there is a sequence Q_n of Y-cells in S such that $Q_{n+1} \subset \mathring{Q}_n$ for all $n = 1, 2, \ldots$ and $M = \bigcap_{n=1}^{\infty} Q_n$. The following provides a list of the most important properties of the absolute suspensions which will be used in the proof of our main result.

Property 2.1 ([Bo, p. 191]). *Let X be an n-dimensional, homogeneous ANR and let B be a compact subset of X cyclic in dimension $n-1$ and contractible to a point in a proper subset X' of X which is contractible in X. Then the set $X \setminus B$ is not connected.* (A compact set B is called *cyclic in dimension k* if $H_k(X) \neq 0$, where $H_k(X)$ denotes the kth Vietoris homology group of X.)

Property 2.2 ([Sz] and [Mi]). *If X is an n-dimensional absolute suspension, then X is a homogeneous ANR homotopy equivalent to the n-dimensional sphere S^n. Every equator of X is also an ANR homotopy equivalent to the $(n-1)$-dimensional sphere S^{n-1}.*

Property 2.3 ([Ly]). *If U and V are two homeomorphic subsets of a homogeneous finite-dimensional ANR then U is an open subset of X if and only if V is an open subset of X.*

3. Main result. The main result of [Br] states that if h is a homeomorphic embedding of $S^{n-1} \times [0,1]$ into S^n then the closure of each complementary domain of $h(S^{n-1} \times \{1/2\})$ is an n-cell. We are going to prove the following theorem, which generalizes this result.

THEOREM 3.1. *Let X be an n-dimensional absolute suspension, $Y \in E_X$ and let $h : Y \times [0,1] \to X$ be a homeomorphic embedding. Then the closure of any complementary domain of $h(Y \times \{1/2\})$ is homeomorphic to $C(Y)$.*

Note that even in the case of spheres, this generalizes Brown's result, since as noted above, there exists a continuum Y such that Y is not a sphere but the suspension of Y is a sphere. The proof of this theorem will follow from a sequence of lemmas. The lemmas (except the first) and their proofs are closely patterned after Brown's original ones and are repeated primarily for the reader's convenience.

LEMMA 3.2. *Every equator $Y \in E_X$ separates X into two complementary domains. If Y is bi-collared in X (this means that there is a homeomorphic embedding $h : Y \times [0,1] \to X$ such that $h(Y \times \{1/2\}) = Y$) then Y is the common boundary of its complementary domains.*

Proof. Let Y be a copy of an equator of X in X. Clearly $X \setminus Y \neq \emptyset$. Every proper compact subset of X is contractible to a point in a proper subset of X that is contractible in X. Since Y has the homotopy type of S^{n-1}, Y is cyclic in dimension $n-1$. According to Property 2.1, Y disconnects X. More precisely, if z is an $(n-1)$-dimensional cycle in Y that is not homologous to zero in Y, then there exists a compactum $A \subsetneq X$ such that $Y \subset A$, z is homologous to zero in A, and A is irreducible with respect to this property. According to the proof of Theorem 16.1 in [Bo], Y separates X between every pair of points of $A \setminus Y$ and $X \setminus A$, respectively. If p is an arbitrary point of $A \setminus Y$, then p has an open neighborhood U in X such that $U \cap Y = \emptyset$ and Y is contractible to a point in $X \setminus U$. Thus there exists a compactum $B \subset X \setminus U$ such that $Y \subset B$, z is homologous to zero in B, and B is irreducible with respect to this property. By using Theorem 16.1 of [Bo, p. 192] again we can claim that Y separates X between every pair of points in $B \setminus Y$ and $X \setminus B$. We claim that $A \cup B = X$. If this were not true then $A \cup B$ would contain, by the Brouwer–Phragmén theorem (see, for example, Proposition 3.6 in [Bo, p. 40]), an n-dimensional cycle not homologous to zero in $A \cup B$ and homologous to zero in X, which contradicts the assumption that X is n-dimensional.

If Y is bi-collared then, by Property 2.3, $h(Y \times (0,1))$ is an open neighborhood of $Y = h(Y \times \{1/2\})$ in X. Therefore, Y is the common boundary of both complementary domains by the first part of Lemma 3.2. It is interesting to ask whether the assumption that Y is bi-collared is essential in the last claim.

LEMMA 3.3. *Let $X \in \mathrm{AS}$, $Y \in E_X$ and let Q be a Y-cell. Suppose that $f : Q \to X$ is a continuous mapping such that f has only a finite number of inverse sets and that all these sets are in $\mathring{Q}$. Then $f(Q)$ is the union of $f(\dot{Q})$ and one of the complementary domains of $f(\dot{Q})$.*

Proof. First notice that $f(Q) \not\subseteq f(\dot{Q})$. Otherwise the map $(f|\dot{Q})^{-1} \circ f : Q \to \dot{Q}$ would give a retraction from Q to $\dot{Q}$, which is impossible since Q is contractible and $\dot{Q}$ being homeomorphic to Y is not, by Property 2.2. Since the inverse sets of f are in $\mathring{Q}$, the mapping $(f|\dot{Q})^{-1}$ is properly defined. Thus $f(Q)$ intersects at least one complementary domain, say D, of $f(\dot{Q})$. The set $\dot{Q}$ clearly does not separate Q, and $f(\dot{Q})$ separates X. (It is obvious that every proper, compact subset of X is contractible to a point in a subset which is contractible in X, thus Properties 2.1 and 2.2 together imply the second part of the above claim.) Thus $f(Q) \subset \overline{D}$, since otherwise there would be inverse sets of f intersecting $\dot{Q}$. If $f(Q)$ does not contain $\overline{D}$ then $f(Q)$ has infinitely many boundary points in D. But Property 2.3 guarantees that only a finite number of points of $f(Q) \cap D$ are boundary points of $f(Q)$. Therefore $f(Q) = \overline{D}$.

LEMMA 3.4. *Let $Q = C(Y)$, $h : Y \to Y$ be a homeomorphism, and A be a compact subset of Q not intersecting the base Y of $C(Y)$. If d denotes a metric on Q then for every positive ε there is a homeomorphism $h' : Q \to Q$ such that $h'|Y = h$ and $\operatorname{diam}(h'(A)) < \varepsilon$.*

Proof. Given arbitrary $\varepsilon, \delta \in (0,1)$ take a homeomorphism $g : [0,1] \to [0,1]$ such that $g(0) = 0$, $g(1) = 1$ and $g([\delta, 1]) \subset [\varepsilon, 1]$. Define h' by the formula $h'(y,t) = (h(y), g(t))$. It is obvious that for a suitable choice of ε and δ the homeomorphism h' satisfies the conditions of the lemma.

LEMMA 3.5. *Let Q be a Y-cell and let M be a Y-cellular subset of $\mathring{Q}$. Then there exists a map f of Q onto itself such that f is the identity on $\dot{Q}$ and M is the only inverse set of f.*

Proof. Let Q_i be a sequence of Y-cells in $\mathring{Q}$ such that $M = \bigcap_{i=1}^{\infty} Q_i$ and such that $Q_{i+1} \subset \mathring{Q}_i$ for all $i = 1, 2, \ldots$. By Lemma 3.4 there is a homeomorphism $h_1 : Q \to Q$ such that $h_1|\dot{Q} = \mathrm{id}_{\dot{Q}}$ and $\operatorname{diam}(h_1(Q_1)) < 1$. By induction, using repeatedly Lemma 3.4, we can construct a homeomorphism h_{i+1} of Q onto itself such that $h_{i+1} = h_i$ on $Q \setminus \mathring{Q}_i$ and such that $\operatorname{diam}(h_{i+1}(Q_{i+1})) < 1/(i+1)$. From the construction it follows immediately that the limit map $f = \lim_{i\to\infty} h_i$ is a properly defined transformation of Q onto itself. Since all h_i's are the identity on $\dot{Q}$ so is f. It is also obvious that $f(M)$ is a point and that M is the only inverse set of f.

LEMMA 3.6. *Let A be a topological copy of some $Y \in E_X$ in X and let D be one of the complementary domains of A. Suppose that f maps $\overline{D}$ onto a Y-cell Q such that the only inverse set of f is a Y-cellular subset M of D. Then $\overline{D}$ is a Y-cell.*

Proof. Let Q_1 be a Y-cell in D such that $M \subset \mathring{Q}_1$. Then M is Y-cellular in Q_1. Therefore by Lemma 3.5 there is a map g of Q_1 onto itself such that g is the identity on $\dot{Q}_1$ and the only inverse set of g is M. Let g' be the map of $\overline{D}$ onto itself which is the identity on $\overline{D} - Q_1$ and g on Q_1. Then the map $f \circ (g')^{-1}$ of $\overline{D}$ onto Q is a homeomorphism. Hence $\overline{D}$ is a Y-cell.

LEMMA 3.7. *Let Q be a Y-cell and suppose that f maps Q into X. Suppose that $M \subset \mathring{Q}$ is the only inverse set of f. Then M is a Y-cellular subset of Q.*

Proof. From Lemma 3.3 it follows that $f(Q) = f(\dot{Q}) \cup D$ where D is a complementary domain of $f(\dot{Q})$. Denote by U an open subset of $\mathring{Q}$ which contains M. Then $f(U)$ is open in D and contains the point $p = f(M)$. If we look at X as an absolute suspension and p as one of the vertices then we can easily construct a homeomorphism $h : X \to X$ for which $h(\overline{D}) \subset f(U)$ and which is the identity on some small open neighborhood V of the point p. Define the map $g : Q \to Q$ by

$$g(x) = \begin{cases} x & \text{if } x \in M, \\ f^{-1} \circ h \circ f(x) & \text{otherwise.} \end{cases} \tag{1}$$

Since $f^{-1} \circ h \circ f$ is the identity on $f^{-1}(V)$, the mapping g is a well defined homeomorphism. Hence $g(Q)$ is a Y-cell in U containing M in its interior.

LEMMA 3.8. *Let $f : X \to X$ be such that f has exactly two inverse sets A and B. Then both A and B are Y-cellular in X for some $Y \in E_X$.*

Proof. Consider $Y \subset X \setminus (A \cup B)$ such that $Y \in E_X$ and such that the closure of each complementary domain of Y in X is a Y-cell. It is easy to find such Y using a suspension structure on X with a vertex not belonging to $A \cup B$. If Y happens to disconnect X between A and B then the conclusion of the lemma follows immediately from the previous one. Therefore assume that Q is a Y-cell in X whose boundary is Y and which contains $A \cup B$ in its interior. Let $f(A) = \{a\}$ and $f(B) = \{b\}$. It follows from Lemma 3.3 that $f(Q) = f(\dot{Q}) \cup D$, where D is the complementary domain of $f(\dot{Q})$ which contains the points a and b. Let U be an open set in D which contains the point a but not b. There is a homeomorphism $h : X \to X$ which carries $f(Q)$ into U and which is the identity on a small neighborhood V of the point a. Consider the map $g : Q \to Q$ defined by

$$g(x) = \begin{cases} x & \text{if } x \in A, \\ f^{-1} \circ h \circ f(x) & \text{otherwise.} \end{cases} \tag{2}$$

Since $f^{-1} \circ h \circ f$ is the identity on $f^{-1}(V)$, the mapping g is well defined. One sees easily that the only inverse set of g is B. Hence by Lemma 3.7, B is Y-cellular. In a similar way one proves that A is Y-cellular.

Proof of the main theorem. Let $h : Y \times [0,1] \to X$ be a homeomorphic embedding. Denote by A the closure of the complementary domain of $h(Y \times \{1\})$ which does not contain $h(Y \times \{0\})$ and let B be the closure of the complementary domain of $h(Y \times \{0\})$ which does not contain $h(Y \times \{1\})$ (by Lemma 3.2, both $h(Y \times \{1\})$ and $h(Y \times \{0\})$ have two complementary domains in X). Let $f : X \to X$ be the map which carries A and B to the vertices of $S(Y)$ and $h(Y \times \{1/2\})$ on the equator Y and which has A and B as the only inverse sets. Denote by D_A and D_B the complementary domains of $h(Y \times \{1/2\})$ which contain A and B respectively. By Lemma 3.8, A and B are Y-cellular in D_A and D_B respectively. Hence by Lemma 3.6, $\overline{D}_A$ and $\overline{D}_B$ are Y-cells, which completes the proof.

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Department of Mathematical Sciences
University of Delaware
Newark, DE 19716, U.S.A.
E-mail: bellamy@math.udel.edu

Department of Mathematics
Widener University
Chester, PA 19013, U.S.A.
E-mail: lysko@maths.widener.edu

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