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ON WEAKLY MIXING AND DOUBLY ERGODIC NONSINGULAR ACTIONS

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Abstract. We study weak mixing and double ergodicity for nonsingular actions of locally compact Polish abelian groups. We show that if T is a nonsingular action of G, then T is weakly mixing if and only if for all cocompact subgroups A of G the action of T restricted to A is weakly mixing. We show that a doubly ergodic nonsingular action is weakly mixing and construct an infinite measure-preserving flow that is weakly mixing but not doubly ergodic. We also construct an infinite measure-preserving flow whose cartesian square is ergodic.

1. Preliminaries. In [17], Kakutani and Parry constructed an infinite measure-preserving invertible transformation T such that $T \times T$ is ergodic (and conservative) but $T \times T \times T$ is not conservative, hence not ergodic. They also constructed other examples including one where all finite cartesian products are ergodic. Since that time there has been interest in understanding dynamical properties for infinite measure-preserving and nonsingular transformations that are analogous to the weak mixing property for finite measure-preserving transformations.

In [1], Aaronson, Lin and Weiss studied the notion of weak mixing for nonsingular and infinite measure-preserving transformations. A nonsingular transformation T is said to be *weakly mixing* if whenever $f \circ T = \lambda f$ for $f \in L^\infty$ and $\lambda \in \mathbb{C}$, then f is constant a.e. They showed that T is weakly mixing if an only if for every ergodic finite measure-preserving transformation S, $T \times S$ is ergodic, and constructed an example of a weakly mixing transformation such that $T \times T$ is not conservative, hence not ergodic. In [2], Adams, Friedman and Silva showed that it can happen that T is weakly mixing with $T \times T$ conservative but still $T \times T$ not ergodic. Other unusual behavior has been shown to exist: there is an infinite measure-preserving

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transformation T such that all its finite cartesian products are ergodic but $T \times T^2$ is not conservative [3].

These examples have been extended to the case of infinite measure-preserving and nonsingular actions of countable discrete abelian groups by Danilenko [7]. More recently, these notions have been studied in the context of multiple recurrence by Danilenko and Silva [8]. We refer to [7] and [8] for a more detailed history of these problems. However, both [7] and [8] and earlier work consider only actions of countable discrete abelian groups. In this work we are interested in studying notions such as weak mixing and its generalizations for infinite measure-preserving and nonsingular actions of continuous groups such as $\mathbb{R}$.

We start with a section of preliminary definitions where we review equivalent characterizations of ergodicity. In our definitions and general theorems we treat actions of a locally compact Polish abelian group G, and in our examples we specialize to the case when $G = \mathbb{R}$. We then define double ergodicity for nonsingular actions of G and show it implies weak mixing. Weak mixing for finite measure-preserving actions of amenable groups was studied by Dye [10], and these characterizations were extended to the case of finite measure-preserving actions of σ-compact locally compact groups by Bergelson and Rosenblatt [4]. (For the weak mixing property of finite measure-preserving actions of groups the reader may refer to [10], [20], [4], and [5].)

In Section 4 we characterize weak mixing for nonsingular G-actions, extending an old result of Hopf [15], where he showed that a finite measure-preserving $\mathbb{R}$-action is weakly mixing if and only if each nonzero time transformation is ergodic, and a result of Bergelson and Rosenblatt [4, 1.15], where the theorem is shown in the finite measure-preserving case. In Section 5 we construct an $\mathbb{R}$-action that is weakly mixing but not doubly ergodic. In Section 6 we construct an infinite measure-preserving $\mathbb{R}$-action whose cartesian square is ergodic, and briefly discuss how to construct doubly ergodic $\mathbb{R}^d$-actions.

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2. Definitions. Let $(X, \mathcal{B}, \mu)$ denote a σ-finite nonatomic Lebesgue measure space. In our applications, (X, μ) will have infinite measure. A *non-*

singular automorphism ϕ on $(X, \mathcal{B}, \mu)$ is a measurable invertible map on X such that $\mu(A) = 0$ if and only if for all $A \in \mathcal{B}$, $\mu(\phi^{-1}(A)) = 0$; ϕ is *measure-preserving* if for all $A \in \mathcal{B}$, $\mu(\phi^{-1}(A)) = \mu(A)$. Let G be a locally compact Polish (separable completely metrizable) abelian topological group. An *action* T of G on X consists of a family of automorphisms $T = \{T^g : g \in G\}$ such that the map $G \times X \to X$, $(g, x) \mapsto T^g x$, is measurable and for all $x \in X_0$, $X_0 \subset X$, $\mu(X \backslash X_0) = 0$, we have $T^g T^h x = T^{gh} x$ and $T^e x = x$, where e is the identity element in G. We may and do assume (see e.g. [21]) that our actions are *continuous*, i.e., for all $A \in \mathcal{B}$, $\mu(T^g(A) \triangle A) \to 0$ as $g \to e$.

We say that a measurable set A is *almost invariant* if, for all $g \in G$, $\mu(T^g(A) \triangle A) = 0$. Similarly, we say that a measurable function f is *almost invariant* if for a.e. x, $f(T^g x) = f(x)$ for all $g \in G$. A measurable set $A \subset X$ has *partial measure* if $\mu(A) > 0$ and $\mu(A^c) > 0$.

An action T is *ergodic* if there are no sets of partial measure that are almost invariant under T.

The following proposition shows the equivalence of two definitions of ergodicity; its proof is standard and left to the reader.

PROPOSITION 2.1. *Let $T = \{T^g : g \in G\}$ be a nonsingular action of G on (X, μ). Then T is ergodic if and only if for every pair of measurable sets A and B in X there exists a $g \in G$ such that $\mu(T^g A \cap B) > 0$.*

REMARK 2.2. (a) It can be shown as in [21, Proposition 2.2.16] that T is ergodic if and only if T does not admit any strictly invariant set A (i.e., $T^g(A) = A$ for all $g \in G$) of partial measure.

(b) For the remainder of this paper, equality means equality except on a set of measure zero where not specified.

A group action T is *weakly mixing* if whenever $f \in L^\infty(X, \mu)$ satisfies $T^g f = \lambda_g f$ a.e., for all $g \in G$, with $\lambda_g \in \mathbb{C}$, then f is constant a.e. This clearly implies that T does not admit almost invariant sets of partial measure, and so T must be ergodic.

We define a G-action T on (X, μ) to be *doubly ergodic* if for any measurable sets $A, B \subset X$ of positive measure, there exists an element $g \in G$ such that $\mu(T^g A \cap A) > 0$ and $\mu(T^g A \cap B) > 0$. It is easy to see that, in the definition of double ergodicity, one may assume $g \neq e$. In the finite measure-preserving transformation case, it was shown by Furstenberg [13] that double ergodicity is equivalent to weak mixing. However, as shown in [6], the situation in the infinite measure-preserving case is quite different. It is easy to see that the ergodic cartesian square property, both for transformations and group actions, implies double ergodicity, but it was shown in [6] that there exist infinite measure-preserving transformations that are doubly ergodic but have nonergodic cartesian square. It was also shown in [6] that for nonsingular transformations, double ergodicity implies weak mixing, and it was observed that the infinite measure-preserving transformation that was shown in [2] to be weakly mixing but with nonconservative cartesian square is not doubly ergodic. For the case of transformations the reader may refer to [2], [6], [14]; we discuss these implications for the case of group actions in the following sections.

3. Double ergodicity implies weak mixing. In this section we show that double ergodicity implies weak mixing for nonsingular G-actions. The idea of the proof is as in Furstenberg's proof [13, Theorem 4.31] for the finite measure-preserving transformation case. The fact that the converse does not hold is shown in Section 5. We note that in [6] it is shown directly that if T is a nonsingular transformation that is doubly ergodic then for all ergodic finite measure-preserving transformations S, $T \times S$ is ergodic.

PROPOSITION 3.1. *Let T be a nonsingular G-action. If T is doubly ergodic, then it is weakly mixing.*

Proof. Suppose T is not weakly mixing. Then there exists a nonconstant $f \in L^\infty$ such that $T^g f = \lambda_g f$ for all $g \in G$. Since T^g is an L^∞ isometry, $|\lambda_g| = 1$ for all $g \in G$. Note that $|f|$ is constant a.e., since $T^g|f| = |\lambda_g f| = |f|$ and T is ergodic, as it is doubly ergodic. Without loss of generality, take $|f| = 1$ a.e. Letting $*\lambda_g$ denote multiplication by λ_g on S^1, we have $f \circ T^g = *\lambda_g \circ f$. We get the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{T^g} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S^1 & \xrightarrow{*\lambda_g} & S^1 \end{array} \tag{1}$$

For this proof, consider S^1 under the canonical identification with $[0, 1)$. Let $B_{n,k} := [k/2^n, (k + 1)/2^n)$ for $0 \le k < 2^n$. Since f is nonconstant we may take n so large that $\mu(f^{-1}(B_{n,k_i})) \ne 0$ for at least two B_{n,k_i} which are not next to each other (mod 1). Let A and B be two such sets.

Since multiplication by λ_g in S^1 corresponds to translation in $[0, 1)$, we see that either $\lambda_g A \cap B = \emptyset$ or $\lambda_g A \cap A = \emptyset$, for all $g \in G$, since they are not next to each other. Thus, by the commutativity of the diagram, we see that for all $g \in G$, $\mu(T^g f^{-1}(A) \cap f^{-1}(A)) = 0$ or $\mu(T^g f^{-1}(A) \cap f^{-1}(B)) = 0$. Hence, T is not doubly ergodic, completing the proof. ∎

4. Weak mixing. In this section we study subactions of weakly mixing nonsingular group actions. In particular, if A is a cocompact subgroup of G (i.e., A is a closed subgroup of G such that G/A is compact), what can we learn by investigating the restriction of the group action to A? Our main result is Theorem 4.3, which generalizes a result of Bergelson and Rosenblatt [4, 1.15] from the finite measure-preserving case to the nonsingular case, though in our case we assume our groups are abelian, which [4] does not. In Corollary 4.4 we obtain an extension of an old result of Hopf [15], who showed that a finite measure-preserving flow is weakly mixing if and only if every nontrivial time of the flow is an ergodic transformation, to the case of nonsingular $\mathbb{R}^d$-actions. Our methods of proof are different from [4] and [15].

In this section we keep the same assumptions on T and G as in the rest of the paper. We require two lemmas.

LEMMA 4.1. *Let A be a cocompact subgroup of G, let ν be Haar measure on G/A, and let T be a nonsingular G-action on (X,μ). Suppose there exists $f \in L^\infty(X,\mu)$ such that $T^a f = f$ for all $a \in A$. For $\phi \in \widehat{G/A}$, define*

$$k_\phi(x) = \int_{G/A} \overline{\phi([g])} f(T^g x)\, d\nu([g]).$$

Then $k_\phi \in L^\infty(X,\mu)$ is well defined, and

$$T^h k_\phi(x) = \overline{\phi([h^{-1}])} k_\phi(x).$$

Proof. Fix $g \in G$ and $a \in A$. Then we see, for a.e. $x \in X$,

$$f(T^{ga}x) = f(T^g x). \tag{2}$$

It follows that $k_\phi \in L^\infty(X,\mu)$ is well defined. To complete the proof, it remains to change variables and use the invariance of Haar measure. ∎

LEMMA 4.2. *Let A be a cocompact subgroup of G. Let T be a nonsingular ergodic G-action on (X,μ). Suppose $T|_A$ is not ergodic, i.e., there exists a measurable set $B \subset X$ of partial measure such that for all $a \in A$, $T^a B = B$. Then T is not weakly mixing.*

Proof. Since $T^a \chi_B = \chi_B$ for all $a \in A$, letting ν be Haar measure on G/A, Lemma 4.1 gives $k_\phi(x) = \int \overline{\phi([g])}\chi_B(T^g x) d\nu([g])$ for each $\phi \in \widehat{G/A}$ and tells us $T^h k_\phi(x) = \overline{\phi([h^{-1}])} k_\phi(x)$. Suppose, for a contradiction, that T is weakly mixing. Hence, for all $\phi \not\equiv 1, k_\phi(x) = 0$. Thus $k_\phi(x) = 0$ a.e. for all $\phi \in \widehat{G/A}$. Since G/A is compact and metrizable, $\widehat{G/A}$ is countable, and so for a.e. x, $k_\phi(x) = 0$ for all $\phi \in \widehat{G/A}$.

Pick a representative from each coset, say $b_g \in [g] \subset G$. Let $j([g],x) \equiv \chi_B(T^{b_g}x)$. By the well definedness of k_ϕ, we see, for a.e. x, $0 = k_\phi(x) = \langle j(\cdot,x), \phi\rangle$, for all $\phi \neq 1$. So, by the Peter–Weyl theorem [as $\widehat{G/A}$ forms an orthonormal basis for $L^2(G/A)$], for a.e. $[g]$ and a.e. x, we have

$$j([g],x) = \sum_{\phi \in \widehat{G/A}} k_\phi(x)\phi([g]) = k_1(x).$$

Here we use the fact that $\chi_B(T^{b_g}) \in L^2(G/A, d\nu)$. This is not hard to see, since ν is finite and $\chi_B(T^{b_g}) \le 1$ for all g and x, so it suffices to see that $\chi_B(T^{b_g})$ is ν-measurable. Since T acts continuously on the measurable functions by assumption (in the topology of convergence of measure) and since $\chi_B(T^g)$ is well defined on the quotient space G/A, $\chi_B(T^{b_g}) = \chi_B(T^g)$ is continuous and therefore measurable.

Since T is ergodic, we see $k_1(x)$ is constant almost everywhere. Thus, for ν-a.e. $[g]$ and μ-a.e. x, $j([g], x) = c$. But equation (2) from Lemma 4.1 tells us that $c = j([g], x) = \chi_B(T^g x)$ a.e. in $[g]$, a.e. in x. This contradicts the fact that T is nonsingular and ergodic. ∎

THEOREM 4.3. *Let T be a nonsingular G-action on (X, μ). Then T is weakly mixing if and only if for all cocompact subgroups A of G, T restricted to A is weakly mixing, i.e., if $f \in L^\infty(X, \mu)$ is such that $T^a f = \gamma(a) f$ for all $a \in A$, then f is constant a.e.*

Proof. Suppose that T is weakly mixing and take a subgroup A of G such that G/A is compact. Then Lemma 4.2 implies that $T|_A$ is ergodic. As pointed out to us by the referee, the result now follows from the well known fact that if T is a weakly mixing G_1 action and S is an ergodic G_2 action that commutes with T, then S is weakly mixing.

For the converse we note that if f is a nonconstant eigenfunction of T, it is also an eigenfunction of $T|_A$ for each closed subgroup A of G. ∎

COROLLARY 4.4. *Let T be a nonsingular $\mathbb{R}^d$-action on a possibly infinite measure space (X, μ). Then the following are equivalent*:

(i) *there exists a basis $\{a_1, \dots, a_d\}$ of $\mathbb{R}^d$ and a measurable set $A \subset X$ of partial measure such that $T^{a_1} A = \dots = T^{a_d} A = A$,*
(ii) *there exists a basis $\{a_1, \dots, a_d\}$ of $\mathbb{R}^d$ and a nonconstant $f \in L^\infty(X, \mu)$ such that $T^{a_i} f = \lambda_i f$, $i = 1, \dots, d$,*
(iii) *T is not weakly mixing.*

Proof. Clearly (i)⇒(ii). Taking $G = \mathbb{R}^d$, $A = \langle a_1, \dots, a_d \rangle$ from the theorem, we get (ii)⇒(iii). Suppose that T is not weakly mixing. Say $T^g f = \lambda_g f$, f nonconstant, $f \in L^\infty(X, \mu)$. In particular, $T^{te_i} f = e^{2\pi i \lambda_i t} f$, where e_i is the standard basis. Taking $t_i = 1/\lambda_i$, we get $T^{t_i e_i} f = f$. Since f is nonconstant, either $\mathrm{Re}(f)$ or $\mathrm{Im}(f)$ is nonconstant. Let g be this nonconstant function. Note that $T^{t_i e_i} g = g$. Take α such that $A \equiv \{x : g(x) > \alpha\} \ne X, \emptyset$. Note $T^{t_i e_i} A = A$. Thus (iii)⇒(i). ∎

The following corollary in the case when the flow is finite measure-preserving was shown by Hopf [15].

COROLLARY 4.5. *Let T be a nonsingular $\mathbb{R}$-action on (X, μ). Then the following are equivalent*:

(i) *T is weakly mixing,*
(ii) *for all $a \in \mathbb{R} \setminus \{0\}$, T^a is an ergodic $\mathbb{Z}$-action,*
(iii) *for all $a \in \mathbb{R} \setminus \{0\}$, T^a is a weakly mixing $\mathbb{Z}$-action.*

Proof. Let $d = 1$ in Corollary 4.4. ∎

COROLLARY 4.6. *Let T be a nonsingular $\mathbb{Z}^d$-action on (X, μ). Suppose there exists a basis $\{a_1, \dots, a_d\} \subset \mathbb{Z}^d$ of $\mathbb{R}^d$ and a nonconstant $f \in L^\infty(X, \mu)$ such that $T^{a_i} f = \lambda_i f$, $i = 1, \dots, d$. Then T is not weakly mixing.*

Proof. Let $G = \mathbb{Z}^d, A = \langle a_1, \dots, a_d \rangle$ in the theorem. ∎

5. Weakly mixing but not doubly ergodic. In this section we construct an infinite measure-preserving flow that is weakly mixing but not doubly ergodic. The construction is by the process of cutting and stacking rectangles in the plane. Cutting and stacking techniques for constructing rank-one transformations are well known (see e.g. [12]). A standard way to construct flows is using the notion of a flow built under a function, such as in [16], where the Chacon finite measure-preserving flow is constructed and shown to be weakly mixing, and to have the stronger property of minimal self-joinings that we do not study here.

There is a natural isomorphism between the cutting and stacking rectangles constructions in the plane and the constructions using a flow built under a function, but we find the first more geometric, in particular when constructing $\mathbb{R}^d$-actions. Finite measure-preserving flows have been constructed earlier using the process of cutting and stacking rectangles in the plane in [18] and [19]. They were constructed as examples of finite measure-preserving weakly mixing flows that are not mixing. However, as we show in Remark 5.2, the flow constructed in [18] is not weakly mixing. The flow in [19] is weakly mixing, but the choice of the "spacers" is different from ours, in addition to the fact that our examples are infinite measure-preserving. More recently, Fayad [11] has constructed, in the finite measure-preserving case, smooth rank-one mixing flows.

We now describe our construction. Let α be a positive irrational number. We recursively define a sequence of columns C_n, $n \geq 0$; each column will be a well defined rectangle in the plane. Let $C_0 = [0, 1) \times [0, 1)$ of height $h_0 = 1$ and width $w_0 = 1$. A column partially defines a flow in the following way: for $(x, y) \in [0, 1) \times [0, 1)$ and $r \geq 0$ define $T^r(x, y) = (x, y + r)$ if $y + r < 1$, and otherwise $T^r(x, y)$ remains undefined. Now given C_n of height h_n and width w_n, define C_{n+1} by first (vertically) cutting C_n into two equal subcolumns, and placing a spacer of height $2h_n + \alpha$ and width $w_n/2$ over the right hand subcolumn. By a *spacer* we mean a rectangle in the plane of the specified width and height that is disjoint from the current column. Now move the left hand column underneath the right hand column to make a column of height $h_{n+1} = 4h_n + \alpha$ and width $w_{n+1} = w_n/2$. Call this new column C_{n+1} (see Figure 1). In C_n we define $L_{n,h,s}$ to be a rectangle of height h of full width in C_n, starting at a distance of s from the top of C_n.

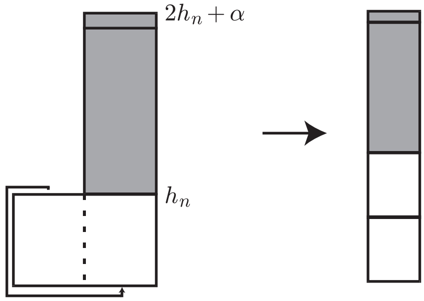

Fig. 1. C_{n+1} out of C_n

The partial flow $T^r_{C_n}$ for $r \in \mathbb{R}$ is defined by the translation that maps $(x, y) \in C_n$ to $(x, y + r)$ if $(x, y + r) \in C_n$, otherwise the flow remains undefined. Note that $T_{C_{n+1}}$ is defined wherever T_{C_n} is defined, and they agree wherever both are defined. Now let $X = \bigcup_{n \geq 0} C_n$ and define

$$T^r := \lim_{n \to \infty} T^r_{C_n}.$$

PROPOSITION 5.1. *T is an infinite measure-preserving weakly mixing flow.*

Proof. It is clear from the construction that T is measure-preserving and ergodic, and that the space X where it is defined has infinite measure. Suppose f is an eigenfunction of T with eigenfrequency other than 0. So $T^r f = e^{2\pi i \lambda r} f$, $\lambda \neq 0$, for all $r \in \mathbb{R}$. Clearly T is ergodic, so $|f| = 1$ a.e. Let d be the metric on S^1 given by identifying S^1 with $[0, 1)$ and using the usual Euclidean metric on $[0, 1)$. Note that d is rotationally invariant, i.e. rotation is an isometry with respect to d. Also note that if $d(z, 1) = \delta < 1/2m$, then $d(z^m, 1) = m\delta$. Fix $\varepsilon > 0$. Find a constant $c \in S^1$ such that $A := f^{-1}(\{z \in S^1 : d(z, c) < \varepsilon\})$ has positive measure. We may find some $R := L_{n,h,s}$ which is $\frac{3}{4}$ full of A (i.e., $\mu(R \cap A) \geq \frac{3}{4}\mu(R)$). Note that, for $m \geq n$, $\mu(T^{h_m} R \cap R) = \frac{1}{2}\mu(R)$. This is clear for $m = n$ as the left half of R is moved to the right half of R. For $m > n$ we may think of R as the union of level rectangles, L_{m,h,s_i}, and the result follows.

Let $m \geq n$. Now, since R was $\frac{3}{4}$ full of A, at least one half of R must be $\frac{3}{4}$ full of A, and more than $\frac{1}{2}$ of R hits itself under T^{h_m}, we see that $\mu(T^{h_m} A \cap A) > 0$. And so for a fixed $m \geq n$, on a set of positive measure, we see that $d(f(x), c) < \varepsilon$ and $d(f(T^{h_m} x), c) < \varepsilon$. Thus,

$$d(e^{2\pi i \lambda h_m}, 1) = d(e^{2\pi i \lambda h_m} f(x), f(x)) = d(f(T^{h_m} x), f(x)) < 2\varepsilon =: \varepsilon'. \tag{3}$$

We know that $h_{n+1} = 4h_n + \alpha$, and thus we get $d(e^{2\pi i\lambda(4h_n+\alpha)}, c) < \varepsilon'$. Using this in addition to $d(e^{2\pi i\lambda h_n}, c) < \varepsilon'$, and rotational invariance of d, we get $d(1, e^{2\pi i\lambda\alpha}) = d(e^{8\pi i\lambda h_n}, e^{2\pi i\lambda(4h_n+\alpha)}) < 5\varepsilon'$. Letting $\varepsilon' \to 0$, we see that $\lambda = k/\alpha$ for some integer k. Let $\kappa := e^{2\pi ik/a}$.

Solving the recurrence relation for h_m, we get $h_m = (1+\alpha/3)4^{m-1} - \alpha/3 = 4^{m-1} + p_m\alpha$, where p_m is some integer depending on m. For $m \geq n$, we get $d(\kappa^{4^m}, \kappa^{4^{m-1}}) = d(e^{2\pi i\lambda h_{m+1}}, e^{2\pi i\lambda h_m}) < 2\varepsilon'$. Using the rotational invariance of d (dividing by $\kappa^{4^{m-1}}$), we get $d(\kappa^{4^{m-1}3}, 1) < 2\varepsilon'$. And so, $d(\kappa^{4^{m-1}}, 1) < \frac{2}{3}\varepsilon'$. Since we started this argument with $d(\kappa^{4^{m-1}}, 1) = d(e^{2\pi i\lambda h_m}, 1) < \varepsilon'$ (see equation (3)), repeating, we get $d(\kappa^{4^{m-1}}, 1) < \left(\frac{2}{3}\right)^j \varepsilon'$ for all $j \geq 1$. Thus $\kappa^{4^{m-1}} = 1$, which contradicts the irrationality of α.

Hence, the only eigenfunctions of T have eigenvalue 1. By the ergodicity of T, these eigenfunctions are all constant; therefore T is weakly mixing. ∎

Remark 5.2. (a) In [18] a similar example is constructed. Instead of spacers of height $2h_n+\alpha$, [18] uses spacers of height 1. However, the example in [18] is not weakly mixing. This can be seen by finding a nonergodic nonzero time. In fact, let A be the union of all level sets, in any fixed column, of height $\frac{1}{4}$ whose lowest x coordinate is an integer. This set is clearly fixed under the time 1 map, and is clearly not the whole space. Thus the map cannot be weakly mixing. (We note that [18] uses an incorrect definition for weak mixing.)

(b) If in our example we take spacers of height α we obtain a finite measure-preserving flow and the same proof applies to show that it is weakly mixing.

Proposition 5.3. *T is not doubly ergodic.*

Proof. This proof is a modification of one found in [2]. Take A and B to be thin level rectangles in C_1, separated by d and of height less than d, as shown in Figure 2.

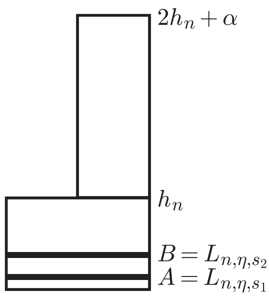

Fig. 2. The sets A and B

Define an $\mathbb{R}$-action R_n on C_n which agrees with T_{C_n} wherever T_{C_n} is defined and maps the very top row back to the bottom, i.e. a rotation on C_n.

Define

$$I_n(A,L) := \{r : 0 \le r < h_n,\, \mu(R_n^r A \cap L) > 0\}, \qquad I_n := I_n(A,A) \cap I_n(A,B).$$

We proceed by induction to show $I_n = \emptyset$ for all $n \in \mathbb{N}$. Clearly $I_1 = \emptyset$, and for the induction, assume $I_j = \emptyset$ for $j \in \mathbb{N}$. Because, for all $r \in [2h_j, 2h_j+\alpha]$, $R_{j+1}^r A$ is contained in the spacers placed on the right subcolumn of C_j, we have the following inclusion for $L = A$ or $L = B$:

$$\begin{aligned} I_{j+1}(A,L) \subset I_j(A,L) &\cup (I_j(A,L) + h_j) \\ &\cup (I_j(A,L) + 2h_j + \alpha) \cup (I_j(A,L) + 3h_j + \alpha). \end{aligned}$$

Hence,

$$\begin{aligned} I_{j+1} &= I_{j+1}(A,A) \cap I_{j+1}(A,B) \\ &\subset I_j(A,A) \cap I_j(A,B) \\ &\quad \cup (I_j(A,A) + h_j) \cap (I_j(A,B) + h_j) \\ &\quad \cup (I_j(A,A) + 2h_j + \alpha) \cap (I_j(A,B + 2h_j + \alpha) \\ &\quad \cup (I_j(A,A) + 3h_j + \alpha) \cap (I_j(A,B) + 3h_j + \alpha). \end{aligned}$$

By induction, each row in the above expression is the empty set; therefore $I_{j+1} = \emptyset$. Noting that for all $r \in \mathbb{R}$, there exists an $n > 0$ such that $T^r A = R_n^r A$, we have $\mu(T^r A \cap A)\mu(T^r A \cap B) = 0$. This completes the proof. ∎

6. Cartesian square ergodicity for an infinite measure-preserving $\mathbb{R}$-action. We now construct a rank-one, infinite measure-preserving flow T such that $T \times T$ is ergodic. The construction of the flow is similar to the construction of the infinite measure-preserving transformation in [9] that has all finite cartesian products of nonzero powers ergodic. However, the proof is significantly different, as it is not clear how to extend the approximation arguments in [9] to the case of flows.

We define recursively a sequence of columns C_n, $n \ge 0$. Let $C_0 = [0,1) \times [0,1)$. Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{Q}$ and $h_0 = 1$. Assume a column C_n of height h_n has been defined. Let $\lceil h_n \rceil$ denote the least integer greater than or equal to h_n. Then to define C_{n+1}, cut C_n into four equal subcolumns, place a spacer of height $\lceil h_n \rceil \alpha$ on the second subcolumn and place a spacer of height α on the fourth subcolumn. We then stack the subcolumns from left to right with the first placed below the second, the second below the third (with the spacer on it) and the third below the fourth. Note that

$$h_{n+1} \ge 4h_n + (h_n + 1)\alpha.$$

Each column C_n defines a partial flow $T_{C_n}^g$ for $g \in \mathbb{R}$ by the translation that maps $(x,y) \in C_n$ to $(x, y+g)$ if $(x, y+g) \in C_n$. Otherwise $T_{C_n}^g$ remains undefined in C_n. Let $X = \bigcup_{n=0}^{\infty} C_n$. Then $X \subset \mathbb{R}^2$ has infinite measure as

$\mu(C_{n+1}) > (1+\alpha/4)\mu(C_n)$. Then define the flow as

$$T^g = \lim_{n\to\infty} T^g_{C_n}.$$

Let $0 < n$, $0 < \eta \le h_n$ and $0 < s \le h_n$. A *level* of C_n is defined to be a rectangle of height η of full width in C_n, starting at a distance of s from the top of C_n; we denote it by $L_{n,\eta,s}$. We now consider $T^{h_n}L_{n,\eta,s}$. The intersection of $T^{h_n}L_{n,\eta,s}$ with C_n is called the *crescent* of $L_{n,\eta,s}$. The crescent of $L_{n,\eta,s}$ contains the blackened region of Figure 3. (When $\alpha > 1$,

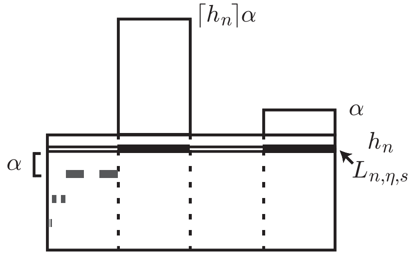

Fig. 3. $T^{h_n}L_{n,\eta,s}$

the crescent of $L_{n,\eta,s}$ is equal to the blackened region of Figure 3.) The crescent has levels of height η, separated by α and decreasing in measure by a factor of $\frac{1}{4}$ with each successive level. For $n \ge 0$, $l \ge 1$, we have

$$\mu(T^{h_n}L_{n,\eta,s} \cap L_{n,\eta,l\alpha+s}) \ge \mu(L_{n,\eta,s})/(2\cdot 4^l), \tag{4}$$

provided that $L_{n,\eta,l\alpha+s}$ is actually defined, i.e. provided that $l\alpha+s+\eta \le h_n$.

If we fix k and let $n > k$ then column C_k appears in C_n as 4^{n-k} disjoint levels of height h_k. Each of these is called a *copy* of C_k. Thus there are 4^{n-k} copies of C_k in C_n.

Definition 6.1. Let A be a measurable set in X and let $I = L_{k,\eta,s}$ be a level, for some k,η,s. Define

$$\mathrm{Full}(A,I) = \frac{\mu(A\cap I)}{\mu(I)}.$$

Analogously, for $A \subset X\times X$ define

$$\mathrm{Full}(A, I\times J) = \frac{\mu\times\mu(A\cap I\times J)}{\mu\times\mu(I\times J)}.$$

Theorem 6.2. *Let T be the flow defined above by cutting each column C_n into four subcolumns, adding spacers, and stacking. Then $T\times T$ is an ergodic flow.*

Proof. Let A and B be subsets of $X\times X$ of positive measure. Given $\frac{1}{15} > \delta > 0$, A and B can be approximated by the cartesian product of full levels I_1, I_2, J_1, J_2 of the form L_{k,η,x_i} for fixed $k \ge 0$ and $\eta > 0$ with

$x_i = x_{I_1}, x_{I_2}, x_{J_1}$ or x_{J_2}, where x_{I_i} stands for the distance from the top of C_k to the start of I_i, x_{J_i} stands for the distance from the top of C_k to the start of J_i, and

$$\mathrm{Full}(A, I_1 \times I_2) > 1 - \delta, \qquad \mathrm{Full}(B, J_1 \times J_2) > 1 - \delta.$$

For $n > k$ let $s_n = h_n + (x_{I_1} - x_{J_1})$. One can verify that $\mu(T^{s_n} I_1 \cap J_1) = \mu(I_1)/2$.

We are going to show that at one of the times s_n, $(T \times T)^{s_n} A$ intersects B in a positive measure set. To do this, we are going to consider $T^{s_n} I_1 \cap J_1$ and $T^{s_n} I_2 \cap J_2$. As previously noted, $\mu(T^{s_n} I_1 \cap J_1) = \mu(I_1)/2$, so our primary concern is $T^{s_n} I_2 \cap J_2$. Note that I_2 is a union of levels in C_n and that $|x_{I_1} - x_{J_1}|$ is smaller than h_k which in turn is smaller than h_n. Therefore, much of $T^{s_n} I_2$ is simply a translate in C_n of $T^{h_n} I_2 \cap C_n$ (expecially for large n) and $T^{h_n} I_2 \cap C_n$ is in turn a union of crescents.

For $T^{s_n} I_2$ to intersect J_2, we want the distance between a copy of I_2 in C_n translated by $x_{I_1} - x_{J_1}$ and a copy of J_2 in C_n to be close to a positive integer times α, because then we may apply (4). The distance between two copies of C_k in C_n is of the form $ah_k + b\alpha$ for positive integers a, b since all the spacers added are of heights which are integer multiples of α. Let $I_{2,n}$ denote a full level subset of I_2 in C_n and $J_{2,n}$ a full level subset of J_2 in C_n which lies under $I_{2,n}$. Let $\mathrm{dist}(I_{2,n}, J_{2,n})$ denote the distance from the top of $I_{2,n}$ to the top of $J_{2,n}$. Then for some positive integers a, b,

$$\mathrm{dist}(I_{2,n}, J_{2,n}) + x_{I_2} - x_{J_2} = ah_k + b\alpha.$$

And we desire to have $\mathrm{dist}(I_{2,n}, J_{2,n}) + (x_{I_1} - x_{J_1})$ close to a positive integer times α.

Since α is irrational and h_k is of the form $M + N\alpha$ for some integers M and N, for all $\varepsilon > 0$, there exist integers m, r such that $m \geq 0$ and

$$|mh_k + r\alpha - (x_{I_2} - x_{J_2}) + (x_{I_1} - x_{J_1})| < \varepsilon\eta.$$

Fix ε such that $1 - 12\delta > \varepsilon > 0$ and let m, r be as above.

For any $n > k$, call an $I_{2,n}$ *good* if there exists a $J_{2,n}$ beneath it such that the distance between them is $mh_k + b\alpha - (x_{I_2} - x_{J_2})$ for some positive integer b. By construction, the distance between such an $I_{2,n}$ and $J_{2,n}$ is close to a positive integer multiple of α minus $x_{I_1} - x_{J_1}$, causing $T^{s_n} I_{2,n}$ to intersect $J_{2,n}$. A good $I_{2,n}$ with its corresponding $J_{2,n}$ is shown in Figure 4 for $m = 3$. For such an $I_{2,n}$ and $J_{2,n}$, we see by (4) that

$$\mu(T^{s_n} I_{2,n} \cap J_{2,n}) > (1 - \varepsilon)\, \frac{1}{2 \cdot 4^{M_n}}\, \mu(I_{2,n})$$

where M_n is greater than $(\mathrm{dist}(I_{2,n}, J_{2,n}) + (x_{I_1} - x_{I_2}))/\alpha$. For instance we can take $M_n = \lceil h_n/\alpha \rceil$.

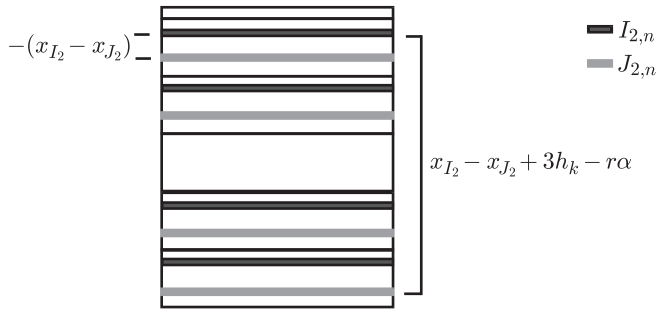

Fig. 4. The top $I_{2,n}$ has a partner in $J_{2,n}$

Note that for any $n > k$, a copy $I_{2,n}$ is good precisely when there are m copies of h_k below the copy of h_k in which $I_{2,n}$ is located. This is because all spacers inserted to make C_n from C_k are of height which is an integer multiple of α. Thus, in C_n all but m of the 4^{n-k} copies of I_2 are good.

Choose N such that $\delta/2 > m/4^{N-k}$. For all $l > N$, each $I_{2,l}$ is a copy of some $I_{2,N}$. Note that $I_{2,l}$ is good if it is a copy of a good $I_{2,N}$, but that there are some $I_{2,l}$ which are good but are not copies of a good $I_{2,N}$. We redefine good to ignore such copies, i.e. for $l > N$ a copy $I_{2,l}$ of I_2 is good precisely if $I_{2,l}$ is a copy of a good $I_{2,N}$. Thus M_N is greater than $(\mathrm{dist}(I_{2,l}, J_{2,l}) + (x_{I_1} - x_{J_1}))/\alpha$ for any good $I_{2,l}$ and its corresponding $J_{2,l}$. Thus, for all $l > N$ and all good $I_{2,l}$ and corresponding $J_{2,l}$,

$$\mu(T^{s_l} I_{2,l} \cap J_{2,l}) > (1-\varepsilon)\,\frac{1}{2\cdot 4^{M_N}}\,\mu(I_{2,l}).$$

Let

$$I_2' = \bigcup_{\text{good}\, I_{2,N}} I_{2,N}, \qquad J_2' = \bigcup_{\text{partner(good}\, I_{2,N})} J_{2,N}.$$

Then

$$\mu(I_2') > (1-\delta/2)\mu(I_2), \qquad \mu(J_2') > (1-\delta/2)\mu(J_2). \tag{5}$$

Take a good $I_{2,l}$ and its corresponding $J_{2,l}$. By definition of good, there is a positive integer d such that $d\alpha$ is close to

$$\mathrm{dist}(T^{x_{I_1}-x_{J_1}} I_{2,l}, J_{2,l}) = \mathrm{dist}(I_{2,l}, J_{2,l}) + (x_{I_1} - x_{J_1}).$$

More precisely, $|d\alpha - (\mathrm{dist}(I_{2,l}, J_{2,l}) + (x_{I_1} - x_{J_1}))| < \varepsilon\eta$. Note that $d \le M_n$. By examining the crescent, we see that the part of $I_{2,l}$ which is taken under T^{s_l} to $J_{2,l}$ is the part obtained by dividing $I_{2,l}$ into 4^{d+1} equal vertical pieces and taking the second and the fourth piece from the right. Call this part $I_{2,l}'$; it is illustrated by the two black squares in the upper right of Figure 5. Let $J_{2,l}'$ be the part of $J_{2,l}$ obtained by dividing $J_{2,l}$ into 4^{d+1} equal vertical pieces and taking the second and the fourth from the left. $J_{2,l}'$ is illustrated

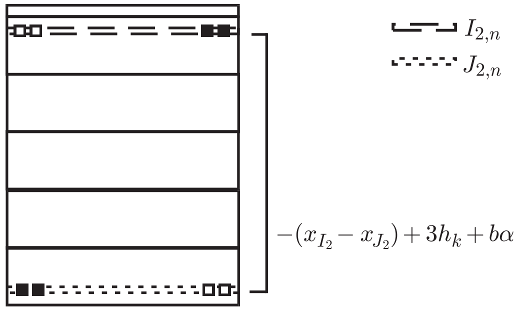

Fig. 5. Column C_n

by the two black squares in the lower left of Figure 5. Note that

$$\mu(T^{s_l}I'_{2,l}\cap J'_{2,l})=(1-\varepsilon)\mu(I'_{2,l})=(1-\varepsilon)\mu(J'_{2,l}). \tag{6}$$

The rough idea for how to proceed is as follows: since the intersections $T^{s_l}I'_{2,l}\cap J'_{2,l}$ and $T^{s_l}I_1\cap J_1$ are large portions of the pieces involved, if $(T\times T)^{s_l}A$ and B do not intersect (in a nonnull set), then $I_{1,l}\times I'_{2,l}$ and $J_{1,l}\times J'_{2,l}$ cannot be very full of A and B respectively. Since $I_{1,l}\times I'_{2,l}$ and $J_{1,l}\times J'_{2,l}$ are small compared to $I_1\times I_2$ and $J_1\times J_2$, this will not directly produce a contradiction for any fixed l, and we must keep track of all the $I_{1,l}\times I'_{2,l}$ and $J_{1,l}\times J'_{2,l}$ for all l. To do this rigorously, we use a partnered partition, defined in the following manner:

A *partnered partition* of (I,J) is defined to be an ordered triple $(\mathcal{P}_I,\mathcal{P}_J,\sigma)$ such that $\mathcal{P}_I$ is a partition of I, $\mathcal{P}_J$ is a partition of J and σ is a measure-preserving bijection $\sigma:\mathcal{P}_I\to\mathcal{P}_J$, i.e. for all $\tilde I\in\mathcal{P}_I$, $\mu(\tilde I)=\mu(\sigma(\tilde I))$. We now define a partnered partition $(\mathcal{P}_{I'_2},\mathcal{P}_{J'_2},\sigma)$ by inductively defining partnered partitions $(\mathcal{P}_{I_2'^{,n}},\mathcal{P}_{J_2'^{,n}},\sigma_n)$ for $n=N,N+M_N,N+2M_N,N+3M_N,\dots$ where $I_2'^{,n}$ and $J_2'^{,n}$ are larger and larger portions of I'_2 and J'_2 respectively. For any good $I_{2,n}$, recall the definition of $I'_{2,n}$ and $J'_{2,n}$ given above and illustrated in black in Figure 5. Let $I^*_{2,n}$ denote the subset of $I_{2,n}$ directly above $J'_{2,n}$. Let $J^*_{2,n}$ denote the subset of $J_{2,n}$ directly below $I'_{2,n}$. The sets $I^*_{2,n}$ and $J^*_{2,n}$ are illustrated by white squares in the upper left and lower right, respectively, in Figure 5. Then σ_N is a partnered partition of

$$\Big(I_2'^{,N}=\bigcup_{\text{good }I_{2,N}}(I'_{2,N}\cup I^*_{2,N}),\ J_2'^{,N}=\bigcup_{\text{partner(good }I_{2,N})}(J'_{2,N}\cup J^*_{2,N})\Big),$$

$\mathcal{P}_{I_2'^{,N}}$ is the set of all $I^*_{2,N}\cup I'_{2,N}$ as $I_{2,N}$ runs over the good copies of I_2, $\mathcal{P}_{J_2'^{,N}}$ is the set of all the corresponding $J^*_{2,N}\cup J'_{2,N}$, and σ_N is defined by $\sigma_N(I'_{2,N}\cup I^*_{2,N})=J'_{2,N}\cup J^*_{2,N}$.

After we have defined the partnered partition $(\mathcal{P}_{I_2'^{,n}},\mathcal{P}_{J_2'^{,n}},\sigma_n)$ of $(I_2'^{,n},J_2'^{,n})$, we define σ_{n+M_N} as follows: for a given good copy $I_{2,n}$ of I_2

in C_n and its corresponding $I'_{2,n}$ and $J'_{2,n}$, recall the definition of d. For $l > n+d+1$, $I'_{2,n}$ and $I^*_{2,n}$ are unions of good copies of I_2 in C_l whose corresponding copies of J_2 are $J^*_{2,n}$ and $J'_{2,n}$ respectively. Since $M_n > d$, if $l > n + M_N$ then $I_2^{\prime,l} \setminus I_2^{\prime,n}$ is a union of good copies of I_2.

Let σ_{n+M_N} be a partnered partition of

$$\Big(I_2^{\prime,n+M_N} = \bigcup_{\text{good } I_{2,n+M_N} \subset I'_2 \setminus I_2^{\prime,n}} (I'_{2,n+M_N} \cup I^*_{2,n+M_N}),$$

$$J_2^{\prime,n+M_N} = \bigcup_{\text{partner(good } I_{2,n+M_N} \subset I'_2 \setminus I_2^{\prime,n})} (J'_{2,n+M_N} \cup J^*_{2,n+M_N})\Big).$$

Let $\mathcal{P}_{I_2^{\prime,n+M_N}}$ be the union of $\mathcal{P}_{I_2^{\prime,n}}$ and the set of all $I^*_{2,n+M_N} \cup I'_{2,n+M_N}$ as $I_{2,n+M_N}$ runs over the good copies of I_2. Let $\mathcal{P}_{J_2^{\prime,n+M_N}}$ be the union of $\mathcal{P}_{J_2^{\prime,n}}$ and the set of all the corresponding $J^*_{2,n+M_N} \cup J'_{2,n+M_N}$, and let σ_{n+M_N} extend σ_n by $\sigma_{n+M_N}(I'_{2,n+M_N} \cup I^*_{2,n+M_N}) = J'_{2,n+M_N} \cup J^*_{2,n+M_N}$. Since σ_{n+M_N} extends σ_n, we can define the partnered partition $(\mathcal{P}_{I'_2}, \mathcal{P}_{J'_2}, \sigma)$ of $(\bigcup_{j=0}^{\infty} I_2^{\prime,N+jM_N}, \bigcup_{j=0}^{\infty} J_2^{\prime,N+jM_N})$ as the limit of the σ_n. We claimed earlier that $(\mathcal{P}_{I'_2}, \mathcal{P}_{J'_2}, \sigma)$ would be a partnered partition of (I'_2, J'_2), i.e. we have claimed that

$$\Big(\bigcup_{j=0}^{\infty} I_2^{\prime,N+jM_N}, \bigcup_{j=0}^{\infty} J_2^{\prime,N+jM_N}\Big) = (I'_2, J'_2).$$

To see this, note that the containment $\subset$ is clear and that by the remarks in the paragraph containing the definition of d,

$$\mu(I'_2 \setminus I_2^{\prime,n+M_N}) \le \Big(1 - \frac{1}{2 \cdot 4^{M_N}}\Big)\mu(I'_2 \setminus I_2^{\prime,n}),$$

whence

$$\mu(I'_2 \setminus I_2^{\prime,N+jM_N}) \le \Big(1 - \frac{1}{2 \cdot 4^{M_N}}\Big)^j \mu(I'_2),$$

which establishes the desired equality. Thus we have constructed the partnered partition $(\mathcal{P}_{I'_2}, \mathcal{P}_{J'_2}, \sigma)$ of (I'_2, J'_2).

Furthermore, by (6), for all $\tilde{I} \in \mathcal{P}_{I'_2}$, there exists a positive integer n such that

$$\mu(T^{s_n}\tilde{I} \cap \sigma(\tilde{I})) \ge \frac{1}{2}\,(1-\varepsilon)\mu(\tilde{I}). \tag{7}$$

Indeed, assume to the contrary that A and B do not intersect at any time s_n. Take $\tilde{I}_2 \in \mathcal{P}_{I'_2}$ and let $\widetilde{J}_2 = \sigma(\tilde{I}_2)$. Let n be as in (7). Define $\widetilde{K} = (T \times T)^{s_n} I_1 \times \tilde{I}_2 \cap J_1 \times \widetilde{J}_2$. As $\mu(T^{s_n} I_1 \cap J_1) \ge \frac{1}{2}\mu(\widetilde{J}_1)$ and $\mu(T^{s_n}\tilde{I}_2 \cap \widetilde{J}_2) \ge ((1-\varepsilon)/2)\mu(\widetilde{J}_2)$, we have $\mu \times \mu(\widetilde{K}) \ge \mu \times \mu(\widetilde{J}_1 \times \widetilde{J}_2)(1-\varepsilon)/4$. Since A and

B do not intersect in a nonnull set, $\mathrm{Full}((T\times T)^{s_n}A,\widetilde{K})+\mathrm{Full}(B,\widetilde{K})\le 1$. This implies

$$\begin{aligned}&\mathrm{Full}(A, I_1\times \tilde{I}_2)+\mathrm{Full}(B, J_1\times \widetilde{J}_2)\\&\qquad\le \frac{1-\varepsilon}{4}+\left(1-\frac{1-\varepsilon}{4}\right)+\left(1-\frac{1-\varepsilon}{4}\right)=2-\frac{1-\varepsilon}{4}.\end{aligned}$$

Since the union over all $\tilde{I}_2$ is I_2' and the union over all $\widetilde{J}_2$ is J_2' (and since the $\tilde{I}_2$ are disjoint as are the $\widetilde{J}_2$), this implies that $\mathrm{Full}(A, I_1\times I_2')+\mathrm{Full}(B, J_1\times J_2')\le 2-(1-\varepsilon)/4$. By (5), this implies that

$$\begin{aligned}&\mathrm{Full}(A, I_1\times I_2)+\mathrm{Full}(B, J_1\times J_2)\\&\qquad\le 2-\frac{1-\varepsilon}{4}+\delta<2-\frac{1-(1-12\delta)}{4}+\delta=2-2\delta.\end{aligned}$$

Since $\mathrm{Full}(A, I_1\times I_2)+\mathrm{Full}(B, J_1\times J_2)=2-2\delta$, we have a contradiction, so there exists n such that $\mu\times\mu((T\times T)^{s_n}A\cap B)>0$. ∎

We conclude this section with a construction of measure-preserving rank-one $\mathbb{R}^d$-actions that can be shown to be doubly ergodic. Our initial construction is finite measure-preserving, but we show how it can be easily modified to obtain infinite measure-preserving examples. We omit the proof that these actions are doubly ergodic as our interest is to show how the previous constructions can be generalized to the case of $\mathbb{R}^d$-actions. It is clear that these constructions have some partial rigidity and therefore are not mixing.

Let $e_1,\dots,e_{d+1}$ be the standard basis of $\mathbb{R}^{d+1}$. We define recursively a sequence of $(d+1)$-dimensional rectangular prisms G_n for $n\ge 0$. Let $G_0=[0,1)\times\cdots\times[0,1)$. Let

$$\alpha_0=\frac{1}{2},$$
$$\alpha_1=\frac{1}{2},\alpha_2=\frac{1}{2},\alpha_3=\frac{1}{4},\alpha_4=\frac{1}{4},$$
$$\alpha_5=\frac{1}{2},\alpha_6=\frac{1}{2},\alpha_7=\frac{1}{2},\alpha_8=\frac{1}{4},\alpha_9=\frac{1}{4},\alpha_{10}=\frac{1}{4},\alpha_{11}=\frac{1}{8},\alpha_{12}=\frac{1}{8},\alpha_{13}=\frac{1}{8},$$
$$\vdots$$

and $l_0=1$. Note that $\alpha_n=2^{-s}$ for some integer s and that for a given k and s there are infinitely many n such that $\alpha_n=\alpha_{n+1}=\cdots=\alpha_{n+k}=2^{-s}$. Assume $l_n\in\mathbb{R}_{>0}$ and $G_n\subset\mathbb{R}^{d+1}$ have been defined and

$$G_n=[0,l_n)\times\cdots\times[0,l_n)\times[0,(2^d)^{-n}).$$

We think of the $(d+1)$st dimension as the height (sometimes we may write a vector in $\mathbb{R}^{d+1}$ as (v,x) where v is in $\mathbb{R}^d$ and $x\in\mathbb{R}$). Then to define G_{n+1}, cut G_n along the $(d+1)$st dimension with 2^d-1 cuts, into 2^d pieces. Use the pieces to tile the section of the $e_1,\dots,e_d$ plane that is a d-dimensional cube of side length $2l_n$. Now we have a rectangular prism twice as long in the $e_1, \ldots, e_d$ dimensions and 2^{-d} as long in the e_{d+1} dimension. Then add a spacer of length α_n around the outside of the generalized "quadrant" in which G_n is sitting, to create a $(2l_n + \alpha_n) \times \cdots \times (2l_n + \alpha_n) \times (2^d)^{-n-1}$ rectangular prism. This is shown for $d = 2$ and $n = 1$ in Figure 6. Note $l_{n+1} = 2l_n + \alpha_n$ and G_{n+1} has height of $(2^d)^{-n}/2^d = (2^d)^{-n-1}$.

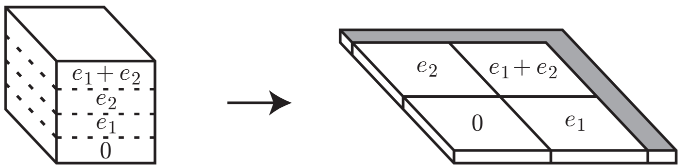

Fig. 6. Construction of G_1 out of G_0

Each G_n defines a partial flow $T^g_{G_n}$ for $g \in \mathbb{R}^d$ by the translation that maps $(x_1, \ldots, x_d, x_{d+1}) \in G_n$ to $((x_1, \ldots, x_d) + g, x_{d+1})$ if the latter is in G_n. Otherwise $T^g_{G_n}$ remains undefined in G_n. Define $X = \bigcup_{n \geq 0} G_n$ and the action as

$$T^g_d = \lim_{n \to \infty} T^g_{G_n}.$$

It can be shown that one can vary the sequence α_n and still obtain double ergodicity provided α_n has the property that for any positive integers s and k there are infinitely many n such that $\alpha_n = \alpha_{n+1} = \cdots = \alpha_{n+k} = 2^{-s}$. In particular, if we insert a spacer of length l_n every time $\alpha_n \neq \alpha_{n+1}$, we obtain an infinite measure space with a doubly ergodic $\mathbb{R}^d$-action.

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