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ON WINGS OF THE AUSLANDER–REITEN QUIVERS OF SELFINJECTIVE ALGEBRAS

BY

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Abstract. We give necessary and sufficient conditions for a wing of an Auslander–Reiten quiver of a selfinjective algebra to be the wing of the radical of an indecomposable projective module. Moreover, a characterization of indecomposable Nakayama algebras of Loewy length ≥ 3 is obtained.

Introduction and the main results. Throughout the paper, by an *algebra* we mean a finite-dimensional associative K-algebra with an identity over a fixed (commutative) field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules, by D the standard duality $\operatorname{Hom}_K(-,K)$ on $\operatorname{mod} A$, by Γ_A the *Auslander–Reiten quiver* of A, and by τ_A and τ_A^- the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We identify an indecomposable module from $\operatorname{mod} A$ with the corresponding vertex of Γ_A. For a module M in $\operatorname{mod} A$, we denote by $P_A(M)$ the projective cover of M and by $I_A(M)$ the injective envelope of M.

An algebra A is called *selfinjective* if $A \cong D(A)$ in $\operatorname{mod} A$, that is, the projective A-modules are injective. Further, A is called *symmetric* if A and $D(A)$ are isomorphic as A-A-bimodules. The classical examples of selfinjective algebras are provided by the group algebras KG of finite groups G, or more generally finite-dimensional Hopf algebras. An important class of selfinjective algebras is formed by the orbit algebras $\widehat{B}/G$ where $\widehat{B}$ is the repetitive algebra (locally finite-dimensional, without identity)

$$\widehat{B} = \bigoplus_{m\in\mathbb{Z}} (B_m \oplus D(B)_m)$$

of an algebra B, where $B_m = B$ and $D(B)_m = D(B)$ for all $m \in \mathbb{Z}$, and the multiplication in $\widehat{B}$ is defined by

$$(a_m, f_m)_m \cdot (b_m, g_m)_m = (a_m b_m, a_m g_m + f_m b_{m-1})_m$$

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for $a_m, b_m \in B_m$, $f_m, g_m \in D(B)_m$, and G is an admissible group of automorphisms of $\widehat{B}$. In particular, for the infinite cyclic group $(\nu_{\widehat{B}})$ generated by the Nakayama automorphisms $\nu_{\widehat{B}}$ of $\widehat{B}$, $T(B) = \widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension algebra $B \ltimes D(B)$ of B by the injective cogenerator $D(B)$ of B (see [5], [7], [9], [10], [12], [17] for a theory of orbit algebras of repetitive algebras).

The Auslander–Reiten quiver is an important combinatorial and homological invariant of selfinjective algebras. We are concerned with the distribution of indecomposable projective modules in the Auslander–Reiten quivers of selfinjective algebras. By general theory, for any indecomposable projective module P over a selfinjective algebra A, there is a canonical Auslander–Reiten sequence in $\operatorname{mod} A$ of the form

$$0 \to \operatorname{rad} P \to (\operatorname{rad} P/\operatorname{soc} P) \oplus P \to P/\operatorname{soc} P \to 0$$

(see [3, (V.5.5)]), and $H(P) = \operatorname{rad} P/\operatorname{soc} P$ is called the *heart* of P.

A full translation subquiver $W(M)$ of an Auslander–Reiten quiver Γ_A of the form

$$\begin{tikzcd}[column sep=tiny]
M_{0,0} \arrow[dr] & & M_{1,1} \arrow[dr] & & M_{2,2} \arrow[dr] & & \cdots & & M_{n-2,n-2} \arrow[dr] & & M_{n-1,n-1} \arrow[dr] & & M_{n,n} \\
& M_{0,1} \arrow[ur] \arrow[dr] & & M_{1,2} \arrow[ur] \arrow[dr] & & {} & & {} \arrow[ur] & & M_{n-2,n-1} \arrow[ur] \arrow[dr] & & M_{n-1,n} \arrow[ur] & \\
& & M_{0,2} \arrow[ur] \arrow[dr] & & {} & & & & {} \arrow[ur] & & M_{n-2,n} \arrow[ur] & & \\
& & & \ddots & & {} \arrow[dr] & & {} & & \iddots \arrow[ur] & & & \\
& & & & {} \arrow[dr] & & M_{1,n-1} \arrow[ur] \arrow[dr] & & {} & & & & \\
& & & & & M_{0,n-1} \arrow[ur] \arrow[dr] & & M_{1,n} \arrow[ur] & & & & & \\
& & & & & & M_{0,n} = M \arrow[ur] & & & & & &
\end{tikzcd}$$

with $n \geq 1$ is said to be a *wing* (*of length* $n+1$) induced by an indecomposable A-module M [11, (2.3)]. We note that a wing $W(M)$ of length $n+1$ is isomorphic to the Auslander–Reiten quiver of the path algebra of the equioriented quiver $1 \to 2 \to \cdots \to n \to n+1$ of type $\mathbb{A}_{n+1}$.

A module X over an algebra A is called *uniserial* if the set of submodules of X is linearly ordered by inclusion. An algebra A is said to be a *Nakayama algebra* if both the indecomposable projective and indecomposable injective A-modules are uniserial.

The following theorems are the main results of the paper.

THEOREM A. *Let A be a selfinjective algebra and $W(M)$ a wing of length $n+1 \geq 2$ in Γ_A induced by an indecomposable A-module M. The following statements are equivalent*:

(i) *M is the radical of an indecomposable projective A-module P.*
(ii) *$M_{1,1}, \ldots, M_{n,n}$ are simple modules and $P_A(M_{1,1}), \ldots, P_A(M_{n,n})$ are of Loewy length $n+2$.*
(iii) *$M_{1,1}, \ldots, M_{n,n}$ are simple modules and $I_A(M_{1,1}), \ldots, I_A(M_{n,n})$ are of Loewy length $n+2$.*
(iv) *$M_{1,n}$ is a uniserial module and $P_A(M_{1,1}), \ldots, P_A(M_{n,n})$ are uniserial modules of length $n+2$.*
(v) *$M_{1,n}$ is a uniserial module and $I_A(M_{1,1}), \ldots, I_A(M_{n,n})$ are uniserial modules of length $n+2$.*

Moreover, *if one of the above statements holds*, *then the wing $W(M_{1,n})$ induced by $M_{1,n}$ consists of uniserial modules.*

THEOREM B. *Let A be an indecomposable selfinjective algebra and $m \geq 3$ a natural number. The following statements are equivalent*:

(i) *A is a Nakayama algebra of Loewy length m.*
(ii) *Γ_A admits a wing $W(M)$ of length $m-1$ induced by the radical M of an indecomposable projective A-module P such that the module $M_{0,0}$ is simple.*

As a direct consequence of Theorems A and B we obtain the following fact.

COROLLARY C. *Let A be an indecomposable selfinjective algebra and $W(M)$ a wing of length $n+1 \geq 2$ in Γ_A induced by the radical M of an indecomposable projective A-module P. Moreover*, *assume that A is not a Nakayama algebra. Then $M_{1,1}, \ldots, M_{n,n}$ are pairwise nonisomorphic simple modules*, *and $\tau_A M_{1,1}$*, *$\tau_A^{-} M_{n,n}$ are not simple.*

The paper is organized as follows. In Section 1 we present preliminary results applied in the proofs of Theorems A and B. Sections 2 and 3 are devoted to the proofs of Theorems A and B, respectively. In the final Section 4 we present examples illustrating both theorems.

For basic background in the representation theory applied here we refer to [2], [3] and [18].

1. Preliminary results. Let A be a selfinjective algebra. We denote by $\underline{\mathrm{mod}}\, A$ the *stable category* of $\mathrm{mod}\, A$. Recall that the objects of $\underline{\mathrm{mod}}\, A$ are the objects of $\mathrm{mod}\, A$ without nonzero projective direct summands, and for any two objects M and N of $\underline{\mathrm{mod}}\, A$ the space of morphisms from M to N in $\underline{\mathrm{mod}}\, A$ is the quotient $\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/P(M,N)$, where $P(M,N)$ is the subspace of $\mathrm{Hom}_A(M,N)$ consisting of all morphisms which factor through projective A-modules. Then the *Auslander–Reiten translations* induce two mutually inverse functors

$$\tau_A, \tau_A^- : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A.$$

We also have two mutually inverse *Heller's syzygy functors* [3, (IV.3.5)]

$$\Omega_A, \Omega_A^- : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A,$$

which assign to any object M of $\underline{\mathrm{mod}}\, A$ respectively the kernel $\Omega_A(M)$ of the projective cover $P_A(M) \to M$ and the cokernel $\Omega_A^-(M)$ of the injective envelope $M \to I_A(M)$. Consider also two mutually inverse *Nakayama functors*

$$\nu_A, \nu_A^- : \mathrm{mod}\, A \to \mathrm{mod}\, A$$

defined as the compositions of functors $\nu_A = D\,\mathrm{Hom}_A(-, A)$ and $\nu_A^- = \mathrm{Hom}_{A^{\mathrm{op}}}(-, A)D$. We note that for any simple A-module T, the modules $\nu_A(T)$ and $\nu_A^-(T)$ are simple, and $\nu_A(P_A(T)) \cong I_A(T) \cong P_A(\nu_A(T))$, $\nu_A^- I_A(T) \cong P_A(T) \cong I_A(\nu_A^-(T))$.

Proposition 1.1. *Let A be a selfinjective algebra. Then*

(i) *The functors $\tau_A, \Omega_A^2 \nu_A, \nu_A \Omega_A^2 : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ are isomorphic.*
(ii) *The functors $\tau_A^-, \Omega_A^{-2} \nu_A^-, \nu_A^- \Omega_A^{-2} : \underline{\mathrm{mod}}\, A \to \underline{\mathrm{mod}}\, A$ are isomorphic.*

Proof. See [3, (IV.3.7)]. ■

For a selfinjective algebra A, we denote by Γ_A^{s} the *stable Auslander–Reiten quiver* of A, obtained from Γ_A by removing the projective vertices and the arrows attached to them.

Proposition 1.2. *Let A be an indecomposable nonsimple selfinjective algebra. Then the syzygy functors Ω_A and Ω_A^- induce two mutually inverse automorphisms of the stable Auslander–Reiten quiver Γ_A^{s}.*

Proof. See [3, (X.1.10)].

Let A be an algebra and M a module in $\mathrm{mod}\, A$. We denote by $\ell(M)$ the *length* of M (the number of simple composition factors of M) and by $\ell\ell(M)$ the *Loewy length* of M (the length of the radical series of M). We note that a module M is uniserial if and only if the radical series of M is the unique composition series of M [3, (IV.2.1)].

An important role in the proof of our main theorems will be played by the following known lemma.

Lemma 1.3. *Let A be an algebra and*

$$0 \longrightarrow X \xrightarrow{\binom{\alpha}{\beta}} Y \oplus Z \xrightarrow{(\gamma,\delta)} U \longrightarrow 0$$

be an Auslander–Reiten sequence in $\mathrm{mod}\, A$. *Then the following statements hold.*

(i) *If the irreducible morphism* $\gamma : Y \to U$ *is a monomorphism then the irreducible morphism* $\beta : X \to Z$ *is a monomorphism and* $\operatorname{Coker}\gamma \cong \operatorname{Coker}\beta$.

(ii) *If the irreducible morphism* $\delta : Z \to U$ *is an epimorphism then the irreducible morphism* $\alpha : X \to Y$ *is an epimorphism and* $\operatorname{Ker}\delta \cong \operatorname{Ker}\alpha$.

Proof. For the convenience of the reader we present the proof of (i). The proof of (ii) is dual.

Assume $\gamma : Y \to U$ is a monomorphism. Since $\beta : X \to Z$ is an irreducible morphism, it follows from [3, (V.5.1)] that β is either a monomorphism or an epimorphism. Then $\ell(X)+\ell(U) = \ell(Y)+\ell(Z)$ and $\ell(Y) < \ell(U)$ force $\ell(X) < \ell(Z)$, and consequently β is a monomorphism. Moreover, we have the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\beta"] \arrow[d, "-\alpha"'] & Z \arrow[r, "\xi"] \arrow[d, "\delta"] & \operatorname{Coker}\beta \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & Y \arrow[r, "\gamma"] & U \arrow[r, "\eta"] & \operatorname{Coker}\gamma \arrow[r] & 0 \end{tikzcd}$$

where $\xi : Z \to Z/\beta(X) = \operatorname{Coker}\beta$ and $\eta : U \to U/\gamma(Y) = \operatorname{Coker}\gamma$ are the canonical epimorphisms. Further, $U = \gamma(Y) + \delta(Z)$ implies $U + \gamma(Y) = \delta(Z) + \gamma(Y)$, and hence f is an epimorphism. Finally, $\ell(\operatorname{Coker}\beta) = \ell(Z) - \ell(X) = \ell(U) - \ell(Y) = \ell(\operatorname{Coker}\gamma)$ implies that f is an isomorphism. ∎

Recall that a path of irreducible morphisms

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \to X_{r-1} \xrightarrow{f_{r-1}} X_r \xrightarrow{f_r} X_{r+1}$$

between indecomposable modules in $\operatorname{mod} A$ is called *sectional* if $\tau_A X_{i+2} \ncong X_i$ for $i \in \{1, \ldots, r-1\}$. Then we have the following useful fact.

LEMMA 1.4. *Let* A *be an algebra and*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \to X_{r-1} \xrightarrow{f_{r-1}} X_r \xrightarrow{f_r} X_{r+1}$$

be a sectional path of irreducible morphisms in $\operatorname{mod} A$. *Then the composition* $f_r \ldots f_2 f_1$ *is nonzero.*

Proof. See [4] or [3, (VII.2.4)]. ∎

We also need the following characterization of indecomposable Nakayama algebras.

PROPOSITION 1.5. *Let* A *be an indecomposable algebra. Then* A *is a Nakayama algebra if and only if* Γ_A *contains a* τ_A*-orbit consisting entirely of simple* A*-modules.*

Proof. See [3, (IV.2)]. ∎

2. Proof of Theorem A. Let A be a selfinjective algebra and $W(M)$ a wing of length $n \geq 2$ in the Auslander–Reiten quiver Γ_A of A, induced by an indecomposable module M from $\operatorname{mod} A$. Without loss of generality we may assume that A is indecomposable. Then A is of Loewy length at least 3, because Γ_A admits the wing $W(M)$ of length at least 2 (see [3, (X.1.8)]).

Observe first that the equivalences (ii)⇔(iii) and (iv)⇔(v) follow from the fact that the Nakayama functors ν_A and ν_A^- are mutually inverse autoequivalences of the category $\operatorname{mod} A$ and $\nu_A P_A(T) \cong I_A(T) \cong P_A(\nu_A(T))$ for any simple A-module T.

We prove now that (i) implies (iii) and (v). Assume that M is the radical of an indecomposable projective A-module P. Denote by S the top of P. Then we have an Auslander–Reiten sequence

$$0 \to \operatorname{rad} P \to H(P) \oplus P \to P/\nu_A^-(S) \to 0.$$

Observe that the modules $M_{i,j}$, $0 \leq i \leq j \leq n$, occurring in the wing $W(M)$ are neither projective nor injective, because A is selfinjective, and $W(M)$ is a full translation subquiver of Γ_A, $M_{1,n}$ is a direct summand of $H(P)$, and the irreducible monomorphism $H(P) \to P/\nu_A^-(S)$ is not splittable. Moreover, invoking Lemma 1.3 and the fact that $W(M)$ is a full translation subquiver of Γ_A, we conclude that all irreducible morphisms $M_{i,j} \to M_{i,j+1}$ (respectively, $M_{i,j} \to M_{i+1,j}$) in $\operatorname{mod} A$ are monomorphisms (respectively, epimorphisms). Applying the functor Ω_A^- to $W(M)$ we obtain, by Proposition 1.2, a wing $W(S)$ in the stable Auslander–Reiten quiver Γ_A^{s} of A

$$\begin{tikzcd}[column sep=small]
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dr] & & X_{2,2} \arrow[dr] & & & & X_{n-2,n-2} \arrow[dr] & & X_{n-1,n-1} \arrow[dr] & & X_{n,n} \\
& X_{0,1} \arrow[ur] \arrow[dr] & & X_{1,2} \arrow[ur] \arrow[dr] & & {} & \cdots & {} \arrow[ur] & & X_{n-2,n-1} \arrow[ur] \arrow[dr] & & X_{n-1,n} \arrow[ur] & \\
& & X_{0,2} \arrow[dr] & & {} & & & & {} \arrow[ur] & & X_{n-2,n} \arrow[ur] & & \\
& & & {} & \ddots & {} \arrow[dr] & & {} & \iddots & {} \arrow[ur] & & & \\
& & & \ddots & {} \arrow[dr] & & X_{1,n-1} \arrow[ur] \arrow[dr] & & {} & \iddots & & & \\
& & & & & X_{0,n-1} \arrow[ur] \arrow[dr] & & X_{1,n} \arrow[ur] & & & & & \\
& & & & & & X_{0,n} = S \arrow[ur] & & & & & &
\end{tikzcd}$$

where $X_{i,j} = \Omega_A^-(M_{i,j})$, $0 \leq i \leq j \leq n$. Moreover, we have in $\operatorname{mod} A$ the Auslander–Reiten sequences

$$0 \to X_{i,j} \to X_{i+1,j} \oplus X_{i,j+1} \to X_{i+1,j+1} \to 0$$

for all $0 \le i < j \le n-1$. Indeed, if $X_{i+1,j+1} \cong P_A(T)/\nu_A^-(T)$ for some indecomposable projective A-module $P_A(T)$ and $0 \le i < j \le n-1$, then $M_{i+1,j+1}$ is a simple module isomorphic to $\nu_A^-(T)$. On the other hand, we have in $\operatorname{mod} A$ a proper monomorphism $M_{i+1,i+1} \to M_{i+1,j+1}$ which is a composition of irreducible monomorphisms corresponding to the arrows

$$M_{i+1,i+1} \to M_{i+1,i+2} \to \cdots \to M_{i+1,j+1}$$

in $W(M)$, a contradiction.

We prove now that there exist simple A-modules $T_1, \ldots, T_n$ such that the indecomposable projective A-modules

$$P_1 = P_A(T_1),\ P_2 = P_A(T_2),\ \ldots,\ P_n = P_A(T_n)$$

are uniserial and $X_{i-1,i-1} = \operatorname{rad} P_i$, $X_{i,i} = P_i/\operatorname{soc} P_i$ for $1 \le i \le n$. We modify arguments from [8, Section 1], applied there for the symmetric algebras. Choose irreducible morphisms $\alpha_{i,j} : X_{i,j} \to X_{i+1,j}$ and $\beta_{i,j} : X_{i,j} \to X_{i,j+1}$ in $\operatorname{mod} A$ corresponding to the arrows of the wing $W(S)$. Observe that $\alpha_{0,n} : X_{0,n} \to X_{1,n}$ is a monomorphism, because S is simple. Applying Lemma 1.3, we infer that there is an irreducible monomorphism $X_{0,1} \to X_{1,1}$, and consequently there is an indecomposable projective module P_1 with $X_{0,0} = \operatorname{rad} P_1$ and $X_{1,1} = P_1/\operatorname{soc} P_1$. Let T_1 be a simple A-module with $P_1 = P_A(T_1)$. Then $\operatorname{soc} P_1 = \nu_A^-(T_1)$ and $X_{0,1} = \operatorname{rad} P_1/\nu_A^-(T_1)$. Observe that, for $n = 1$, $P_1 = P_A(T_1)$ is a uniserial module of length 3, and so $I_A(\nu_A^-(T_1)) = P_A(T_1)$ is a uniserial module of length 3. Moreover, $M_{1,1} = \Omega_A(X_{1,1}) = \nu_A^-(T_1)$ is simple. Thus (iii) and (v) hold.

Assume $n \ge 2$. Invoking Lemma 1.3 again, we conclude that $\operatorname{Coker} \alpha_{0,n}$ is isomorphic to the cokernel $\operatorname{Coker} \alpha_{0,1} = T_1$ of the canonical irreducible monomorphism $\alpha_{0,1} : \operatorname{rad} P_1/\operatorname{soc} P_1 \to P_1/\operatorname{soc} P_1$. Dually, the irreducible morphism $\beta_{0,n-1} : X_{0,n-1} \to X_{0,n} = S$ is an epimorphism, and then it follows from Lemma 1.3 that we have an irreducible epimorphism $X_{n-1,n-1} \to X_{n-1,n}$. Hence, there exists an indcomposable projective A-module P_n such that $X_{n-1,n-1} = \operatorname{rad} P_n$ and $X_{n,n} = P_n/\operatorname{soc} P_n$. Let T_n be a simple A-module with $P_n = P_A(T_n)$. Thus

$$\operatorname{soc} P_n = \nu_A^-(T_n) \quad \text{and} \quad X_{n-1,n} = \operatorname{rad} P_n/\nu_A^-(T_n).$$

Moreover $\operatorname{Ker} \beta_{0,n-1}$ is isomorphic to the kernel $\operatorname{Ker} \beta_{n-1,n-1} = \nu_A^-(T_n)$ of the canonical epimorphism $\beta_{n-1,n-1} : \operatorname{rad} P_n \to \operatorname{rad} P_n/\nu_A^-(T_n)$. Therefore, $X_{0,n-1}$ and $X_{1,n}$ are uniserial modules of length 2 with the simple composition factors

$$X_{0,n-1} = \begin{pmatrix} S \\ \nu_A^-(T_n) \end{pmatrix} \quad \text{and} \quad X_{1,n} = \begin{pmatrix} T_1 \\ S \end{pmatrix}.$$

Moreover, $\alpha_{0,n-1} : X_{0,n-1} \to X_{1,n-1}$ is an irreducible monomorphism with $\operatorname{Coker}\alpha_{0,n-1} \cong \operatorname{Coker}\alpha_{0,1} = T_1$, and so $X_{1,n-1}$ is a uniserial module of length 3 with the simple composition factors

$$X_{1,n-1} = \begin{pmatrix} T_1 \\ S \\ \nu_A^-(T_n) \end{pmatrix}.$$

Assume now that, for some i with $1 \le i \le n-1$, there exist indecomposable projective A-modules $P_1 = P_A(T_1), \ldots, P_i = P_A(T_i)$, with $T_1, \ldots, T_i$ simple modules, such that the following statements hold:

(a) $X_{j-1,j-1} = \operatorname{rad} P_j$ and $X_{j,j} = P_j/\operatorname{soc} P_j$ for $1 \le j \le i$;
(b) $X_{j,n-1}$ and $X_{j,n}$, $1 \le j \le i$, are uniserial modules with the simple composition factors

$$X_{j,n-1} = \begin{pmatrix} T_j \\ \vdots \\ T_1 \\ S \\ \nu_A^-(T_n) \end{pmatrix} \quad \text{and} \quad X_{j,n} = \begin{pmatrix} T_j \\ T_{j-1} \\ \vdots \\ T_1 \\ S \end{pmatrix}.$$

We claim now that the irreducible morphism $\alpha_{i,n} : X_{i,n} \to X_{i+1,n}$ is also a monomorphism. Suppose it is not the case. Then $\alpha_{i,n}$ is an irreducible epimorphism, and consequently $\operatorname{Ker}\alpha_{i,n}$ contains the simple socle of $X_{i,n}$, which is the image of $\alpha_{i-1,n} \cdots \alpha_{0,n}$. On the other hand, by Lemma 1.4, the composition $\alpha_{i,n}\alpha_{i-1,n} \cdots \alpha_{0,n} : S \to X_{i+1,n}$ of irreducible morphisms corresponding to the sectional path

$$S = X_{0,n} \to X_{1,n} \to \cdots \to X_{i-1,n} \to X_{i,n} \to X_{i+1,n}$$

of Γ_A is nonzero, a contradiction. Thus $\alpha_{i,n}$ is indeed a monomorphism. Applying Lemma 1.3 we deduce that $\alpha_{i,i+1} : X_{i,i+1} \to X_{i+1,i+1}$ is an irreducible monomorphism, and hence there exists an indecomposable projective module $P_{i+1} = P_A(T_{i+1})$ such that $X_{i,i} = \operatorname{rad} P_{i+1}$ and $X_{i+1,i+1} = P_{i+1}/\operatorname{soc} P_{i+1}$. Moreover, $\operatorname{Coker}\alpha_{i,n} \cong \operatorname{Coker}\alpha_{i,i+1} = T_{i+1}$. Further, $\alpha_{i,n-1} : X_{i,n-1} \to X_{i+1,n-1}$ is also an irreducible monomorphism with $\operatorname{Coker}\alpha_{i,n-1} \cong \operatorname{Coker}\alpha_{i,i+1} = T_{i+1}$. In particular, $X_{i+1,n-1}$ and $X_{i+1,n}$ are indecomposable modules with simple top isomorphic to T_{i+1}, and hence are factor modules of $P_{i+1} = P_A(T_{i+1})$. As a consequence, $\operatorname{rad} X_{i+1,n-1}$ is a unique maximal submodule of $X_{i+1,n-1}$ and $\operatorname{rad} X_{i+1,n}$ is a unique maximal submodule of $X_{i+1,n}$. Therefore, $X_{i+1,n-1}$ and $X_{i+1,n}$ are uniserial modules with the simple composition factors

$$X_{i+1,n-1} = \begin{pmatrix} T_{i+1} \\ T_i \\ \vdots \\ T_1 \\ S \\ \nu_A^-(T_n) \end{pmatrix} \quad \text{and} \quad X_{i+1,n} = \begin{pmatrix} T_{i+1} \\ T_i \\ \vdots \\ T_1 \\ S \end{pmatrix}.$$

Therefore, it follows by induction that there exist simple modules $T_1, \ldots, T_n$ such that

$$X_{i-1,i-1} = \operatorname{rad} P(T_i)$$

and

$$X_{i,i} = P_A(T_i)/\nu_A^-(T_i)$$

for all $i \in \{1, \ldots, n\}$, and additionally $X_{n,n}$ is a uniserial module with the simple composition factors

$$X_{n,n} = \begin{pmatrix} T_n \\ T_{n-1} \\ \vdots \\ T_1 \\ S \end{pmatrix}.$$

Then we conclude that $P_1 = P_A(T_1), \ldots, P_n = P_A(T_n)$ are uniserial modules with the Loewy series as follows:

$$P_1 = \begin{pmatrix} T_1 \\ S \\ \nu_A^-(T_n) \\ \nu_A^-(T_{n-1}) \\ \vdots \\ \nu_A^-(T_2) \\ \nu_A^-(T_1) \end{pmatrix}, \quad P_2 = \begin{pmatrix} T_2 \\ T_1 \\ S \\ \nu_A^-(T_n) \\ \nu_A^-(T_{n-1}) \\ \vdots \\ \nu_A^-(T_3) \\ \nu_A^-(T_2) \end{pmatrix}, \quad \ldots,$$

$$P_i = \begin{pmatrix} T_i \\ T_{i-1} \\ \vdots \\ T_1 \\ S \\ \nu_A^-(T_n) \\ \nu_A^-(T_{n-1}) \\ \vdots \\ \nu_A^-(T_{i+1}) \\ \nu_A^-(T_i) \end{pmatrix}, \quad \dots, \quad P_n = \begin{pmatrix} T_n \\ T_{n-1} \\ \vdots \\ \vdots \\ T_2 \\ T_1 \\ S \\ \nu_A^-(T_n) \end{pmatrix}.$$

In particular, we find that

$$M_{i,i} = \Omega_A(P_A(T_i)/\nu_A^-(T_i)) = \nu_A^-(T_i), \quad 1 \le i \le n,$$

are simple modules. Further, we conclude that the injective envelopes

$$I_A(M_{i,i}) = I_A(\nu_A^-(T_i)) = P_A(T_i) = P_i, \quad 1 \le i \le n,$$

of the modules $M_{1,1}, \dots, M_{n,n}$ are uniserial modules of length $n+2$, and so also of Loewy length $n+2$. Finally, observe that we have in $\operatorname{mod} A$ a sequence of irreducible monomorphisms

$$\nu_A^-(T_1) = M_{1,1} \xrightarrow{f_1} M_{1,2} \xrightarrow{f_2} M_{1,3} \to \dots \to M_{1,n-1} \xrightarrow{f_{n-1}} M_{1,n}$$

with $\operatorname{Coker} f_i \cong \nu_A^-(T_{i+1})$ for $i \in \{1, \dots, n-1\}$ and hence $M_{1,n}$ has a simple socle, isomorphic to $M_{1,1}$. Then $I_A(M_{1,n}) \cong I_A(M_{1,1})$, and consequently $M_{1,n}$ is a uniserial module of length n with the simple composition factors

$$M_{1,n} = \begin{pmatrix} \nu_A^-(T_n) \\ \nu_A^-(T_{n-1}) \\ \vdots \\ \nu_A^-(T_2) \\ \nu_A^-(T_1) \end{pmatrix}.$$

This finishes the proof of the implications (i)⇒(iii) and (i)⇒(v).

We prove now that (v) implies (iii). Assume that $M_{1,n}$ is a uniserial module and $I_A(M_{1,1}), \dots, I_A(M_{n,n})$ are uniserial modules of length $n+2$. Obviously then $I_A(M_{1,1}), \dots, I_A(M_{n,n})$ are of Loewy length $n+2$. We claim that the modules $M_{1,1}, \dots, M_{n,n}$ are simple. Since $W(M)$ is a full translation subquiver of Γ_A, we know that all irreducible morphisms $M_{i,j} \to M_{i,j+1}$ (respectively, $M_{i,j} \to M_{i+1,j}$) in $\operatorname{mod} A$ are irreducible monomorphisms (respectively, irreducible epimorphisms). We also note that if $U \to V$ is an irreducible morphism between two uniserial modules in $\operatorname{mod} A$ then $|\ell(U) - \ell(V)| = 1$. In particular, we have in $\operatorname{mod} A$ a chain of irreducible monomorphisms

$$M_{1,1} \to M_{1,2} \to M_{1,3} \to \cdots \to M_{1,n-1} \to M_{1,n}$$

and $M_{1,1}, M_{1,2}, \ldots, M_{1,n-1}, M_{1,n}$ are uniserial modules with

$$\ell(M_{1,2}) - \ell(M_{1,1}) = 1, \quad \ldots, \quad \ell(M_{1,n}) - \ell(M_{1,n-1}) = 1,$$

because $M_{1,n}$ is uniserial. We claim that in fact $M_{1,1}$ is a simple module, and consequently $M_{1,n}$ is of length n. Suppose $M_{1,1}$ is not simple. Then we have a nonsplittable epimorphism $h : M_{1,1} \to M_{1,1}/\operatorname{rad} M_{1,1}$ from $M_{1,1}$ onto its simple top $M_{1,1}/\operatorname{rad} M_{1,1}$. On the other hand, we have in $\operatorname{mod} A$ an Auslander–Reiten sequence of the form

$$0 \to M_{1,1} \xrightarrow{f} M_{1,2} \to M_{2,2} \to 0,$$

and hence $h = gf$ for some $g : M_{1,2} \to M_{1,1}/\operatorname{rad} M_{1,1}$. But this is a contradiction since $f(M_{1,1}) = \operatorname{rad} M_{1,2}$. Therefore, for each $i \in \{1, \ldots, n\}$, $M_{1,i}$ is a uniserial module of length i. Finally, for each $i \in \{2, \ldots, n\}$, we have in $\operatorname{mod} A$ a chain of irreducible epimorphisms $M_{1,i} \to M_{2,i} \to \cdots \to M_{i-1,i} \to M_{i,i}$, and consequently $M_{i,i}$ is a simple module. Observe also that the wing $W(M_{1,n})$ consists entirely of uniserial modules.

We prove now that (iii) implies (i). Assume that $M_{1,1}, \ldots, M_{1,n}$ are simple modules and $I_A(M_{1,1}), \ldots, I_A(M_{n,n})$ are of Loewy length $n+2$. Applying Ω_A^- to $W(M)$ we obtain a full translation subquiver of Γ_A of the form

$$\begin{tikzcd}[column sep=small]
 & I_A(M_{1,1}) \arrow[dr] & & I_A(M_{2,2}) \arrow[dr] & & & & & & {} \arrow[dr] & & I_A(M_{n-1,n-1}) \arrow[dr] & & I_A(M_{n,n}) \arrow[dr] & \\
X_{0,0} \arrow[ur] \arrow[dr] & & X_{1,1} \arrow[ur] \arrow[dr] & & X_{2,2} \arrow[ur] \arrow[dr] & & & & & & X_{n-2,n-2} \arrow[ur] \arrow[dr] & & X_{n-1,n-1} \arrow[ur] \arrow[dr] & & X_{n,n} \\
 & X_{0,1} \arrow[ur] \arrow[dr] & & X_{1,2} \arrow[ur] \arrow[dr] & & & & \cdots & & {} \arrow[ur] & & X_{n-2,n-1} \arrow[ur] \arrow[dr] & & X_{n-1,n} \arrow[ur] & \\
 & & X_{0,2} \arrow[dr] & & & & & & & & {} \arrow[ur] & & X_{n-2,n} \arrow[ur] & & \\
 & & & {} & \ddots & & {} \arrow[dr] & & {} & & \iddots & {} \arrow[ur] & & & \\
 & & & & \ddots & {} \arrow[dr] & & X_{1,n-1} \arrow[ur] \arrow[dr] & & {} & \iddots & & & & \\
 & & & & & & X_{0,n-1} \arrow[ur] \arrow[dr] & & X_{1,n} \arrow[ur] & & & & & & \\
 & & & & & & & X_{0,n} \arrow[ur] & & & & & & &
\end{tikzcd}$$

where $X_{i,j} = \Omega_A^-(M_{i,j})$, $0 \le i \le j \le n$. Further, we have $X_{i-1,i-1} = \operatorname{rad} I_A(M_{i,i})$ and $X_{i,i} = I_A(M_{i,i})/\operatorname{soc} I_A(M_{i,i})$ for $i \in \{1,\dots,n\}$. Choose irreducible morphisms $\alpha_{i,j} : X_{i,j} \to X_{i+1,j}$, $\beta_{i,j} : X_{i,j} \to X_{i,j+1}$ and $\gamma_{i-1} : X_{i-1,i-1} \to I_A(M_{i,i})$, $\delta_i : I_A(M_{i,i}) \to X_{i,i}$ in $\operatorname{mod} A$, corresponding to the arrows of the above translation subquiver of Γ_A. Since γ_{i-1} are monomorphisms and δ_i are epimorphisms, applying Lemma 1.3, we conclude that all $\alpha_{i,j}$ are monomorphisms and all $\beta_{i,j}$ are epimorphisms. Consider the chain of irreducible monomorphisms

$$X_{0,n-1} \xrightarrow{\alpha_0} X_{1,n-1} \xrightarrow{\alpha_1} X_{2,n-1} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-3}} X_{n-2,n-1} \xrightarrow{\alpha_{n-2}} X_{n-1,n-1} \xrightarrow{\gamma_{n-1}} I_A(M_{n,n})$$

where $\alpha_i = \alpha_{i,n-1}$, $0 \le i \le n-2$. Fix $i \in \{0,1,\dots,n-2\}$. Applying Lemma 1.3, we have $\operatorname{Coker} \alpha_i \cong \operatorname{Coker} \gamma_i \cong M_{i+1,i+1}$, and hence $\alpha_i(X_{i,n-1})$ is a maximal submodule of $X_{i+1,n-1}$. Moreover, we have the epimorphism $\beta_{i+1,n-2} \cdots \beta_{i+1,i+1}\delta_{i+1} : I_A(M_{i+1,i+1}) \to X_{i+1,n-1}$, and consequently $\operatorname{rad} X_{i+1,n-1}$ is a unique maximal submodule of $X_{i+1,n-1}$. Hence $\alpha_i(X_{i,n-1}) = \operatorname{rad} X_{i+1,n-1}$. Obviously, $\gamma_{n-1}(X_{n-1,n-1}) = \operatorname{rad} I_A(M_{n,n})$. Therefore, we conclude that $X_{i,n-1} \cong \operatorname{rad}^{n-i} I_A(M_{n,n})$ for $i \in \{0,\dots,n-1\}$. In particular, we have $X_{0,n-1} \cong \operatorname{rad}^n I_A(M_{n,n})$. Further, since $I_A(M_{n,n})$ is of Loewy length $n+2$, we also obtain

$$\operatorname{rad} X_{0,n-1} = \operatorname{soc} X_{0,n-1} \cong \operatorname{soc} I_A(M_{n,n}) = \operatorname{rad}^{n+1} I_A(M_{n,n}).$$

Finally, $\beta_{0,n-1} : X_{0,n-1} \to X_{0,n}$ is an irreducible epimorphism with simple kernel $\operatorname{Ker} \beta_{0,n-1} \cong \operatorname{Ker} \delta_n \cong \operatorname{soc} I_A(M_{n,n}) = M_{n,n}$. Hence $X_{0,n} \cong X_{0,n-1}/\operatorname{rad} X_{0,n-1}$. Therefore, $X_{0,n}$ is a simple module S, because $X_{0,n}$ is semisimple and indecomposable. Therefore, $M \cong \Omega_A \Omega_A^-(M) = \Omega_A(S) = \operatorname{rad} P(S)$ is the radical of an indecomposable projective A-module, as required in (i). This finishes the proof of Theorem A. ∎

3. Proof of Theorem B. Let A be an indecomposable selfinjective algebra and $m \ge 3$ a natural number.

(i)⇒(ii). Assume A is a Nakayama algebra of Loewy length m. Then all indecomposable A-modules are uniserial of length at most m, and the indecomposable modules of length m are the indecomposable projective A-modules (see [3, (IV.2)]). Take a simple A-module T and its injective envelope $I(T) = P(S)$, where $S = \nu_A(T)$. Then we have in $\operatorname{mod} A$ a chain of irreducible monomorphisms

$$T = M_{0,0} \to M_{0,1} \to \cdots \to M_{0,m-2} \to M_{0,m-1} = P(S)$$

where $M_{0,m-1-i} = \operatorname{rad}^i P(S)$ for $i \in \{1,\dots,m-1\}$. Further, it follows from [3, (IV.2.6)] that, for each $i \in \{0,\dots,m-2\}$, the τ_A-orbit of $M_{0,i}$ consists entirely of uniserial modules of length $i = \ell(M_{0,i})$. Therefore, Γ_A contains (as a full translation subquiver) a wing $W(M)$ of length $m-1$

$$\begin{tikzcd}[column sep=small]
M_{0,0} \arrow[dr] & & M_{1,1} \arrow[dr] & & M_{2,2} & & & \cdots & & M_{m-3,m-3} \arrow[dr] & & M_{m-2,m-2} \\
& M_{0,1} \arrow[ur] \arrow[dr] & & M_{1,2} \arrow[ur] & & & & & & & M_{m-3,m-2} \arrow[ur] & \\
& & M_{0,2} \arrow[ur] & & & \ddots & & & \iddots & & & \\
& & & \ddots & & & & & & \iddots & & \\
& & & & & M_{1,m-3} \arrow[dr] & & & & & & \\
& & & & M_{0,m-3} \arrow[ur] \arrow[dr] & & M_{1,m-2} & & & & & \\
& & & & & M_{0,m-2} = M \arrow[ur] & & & & & &
\end{tikzcd}$$

induced by the radical $M = M_{0,m-2}$ of the indecomposable projective A-module $P(S)$, and with $M_{0,0} = T$ simple.

(ii)$\Rightarrow$(i). Assume Γ_A admits a wing $W(M)$ of length $m-1$ induced by the radical M of an indecomposable projective A-module P and with $M_{0,0}$ simple. Then it follows from Theorem A that $M_{1,1}, \dots, M_{m-2,m-2}$ are simple modules and their injective envelopes $I_A(M_{1,1}) = P_A(\nu_A(M_{1,1})), \dots, I_A(M_{m-2,m-2}) = P_A(\nu_A(M_{m-2,m-2}))$ are uniserial modules of length m. Let $S = \nu_A(M_{0,0})$. Then $M_{0,0} = \operatorname{soc} M$ and $M = \operatorname{rad} P_A(S)$. Further, since $M_{0,0} = \tau_A M_{1,1}$, we also see that $P_A(S)/\operatorname{soc} P_A(S) = P_A(\nu_A(M_{0,0}))/M_{0,0} \cong \operatorname{rad} P_A(\nu_A(M_{1,1}))$, and consequently $P_A(S)$ is a uniserial module of length m. We claim that $\tau_A S = M_{m-2,m-2}$. Since $M_{1,1}, \dots, M_{m-2,m-2}$ are simple, we have in $\operatorname{mod} A$ irreducible epimorphisms $M_{i-1,i} \to M_{i,i}$, $1 \le i \le m-2$. Applying now Lemma 1.3, we conclude that there is in $\operatorname{mod} A$ a chain of irreducible epimorphisms

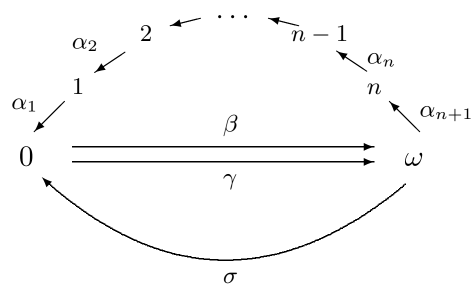
$$\tau_A^- M_{0,m-2} \to \tau_A^- M_{1,m-2} \to \cdots \to \tau_A^- M_{m-3,m-2} \to \tau_A^- M_{m-2,m-2}.$$

Further, $\tau_A^- M_{0,m-2} = \tau_A^- M = P_A(S)/\operatorname{soc} P_A(S)$. Since $P_A(S)$ is uniserial of length m, we infer that $\tau_A^-(M_{i,m-2}) \cong P_A(S)/\operatorname{rad}^{m-i-1} P_A(S)$ for any $i \in \{0, \dots, m-2\}$. In particular, $\tau_A^-(M_{m-2,m-2}) \cong P_A(S)/\operatorname{rad} P_A(S) \cong S$, and so $M_{m-2,m-2} = \tau_A(S)$. We also know that ν_A is an autoequivalence of the category $\operatorname{mod} A$. Summing up, we conclude that Γ_A contains a τ_A-orbit consisting entirely of the simple modules of the form $\nu_A^r(M_{i,i})$, $r \in \mathbb{Z}$, $i \in \{0, 1, \dots, m-2\}$. Therefore, applying Proposition 1.5, we infer that A is a Nakayama algebra of Loewy length m. ■

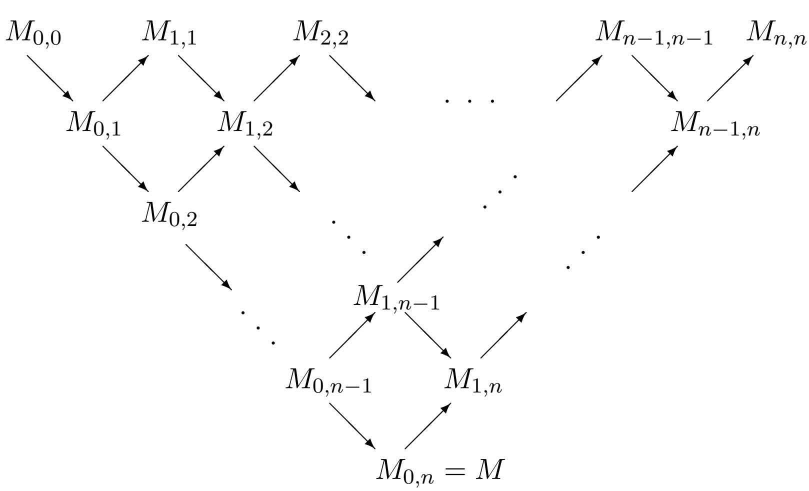
4. Examples. The aim of this section is to present some examples illustrating Theorems A and B. We refer to [6, Section 1] for the description of the Auslander–Reiten sequences over special biserial selfinjective algebras.

The first example shows that our assumptions on the (Loewy) length of the projective covers and injective envelopes of the simple modules lying on the top part of a wing are necessary for the validity of Theorem A.

EXAMPLE 4.1. Let $n \geq 1$ be a natural number and A the bound quiver algebra KQ/I, where Q is the quiver



and I is the ideal in the path algebra KQ of Q generated by the elements $\beta\sigma, \sigma\beta$, $\gamma\alpha_{n+1}$, $\alpha_1\gamma$, $\alpha_{n+1}\ldots\alpha_2\alpha_1\beta - \sigma\gamma$, $\beta\alpha_{n+1}\ldots\alpha_2\alpha_1 - \gamma\sigma$ and $\alpha_i\ldots\alpha_1\beta\alpha_{n+1}\ldots\alpha_i$ for $i \in \{1,\ldots,n\}$. Then A is a special biserial algebra (in the sense of [16]) isomorphic to the trivial extension algebra $T(H) = H \ltimes D(H)$ of the hereditary algebra $H = K\Delta$ of Euclidean type $\widetilde{\mathbb{A}}_{n+1}$, given by the quiver Δ obtained from Q by removing the arrows β and γ, by the minimal injective cogenerator $\mathrm{Hom}_K(H,K)$. The Auslander–Reiten quiver Γ_H of H admits a unique large stable tube $\mathcal{T}$ of rank $n+1$ containing a wing $W(M)$ of the form



of length $n+1$, where $M_{i,i} = S_i$ is the simple H-module at the vertex i, $1 \le i \le n$, $M_{0,0}$ is the unique uniserial H-module of dimension 2 having as socle the simple module S_0 at the vertex 0 and as top the simple module S_ω at the vertex ω, and $M_{0,n} = M$ is a uniserial H-module with the simple composition factors

$$M = \begin{pmatrix} S_n \\ S_{n-1} \\ \vdots \\ S_2 \\ S_1 \\ S_\omega \\ S_0 \end{pmatrix}.$$

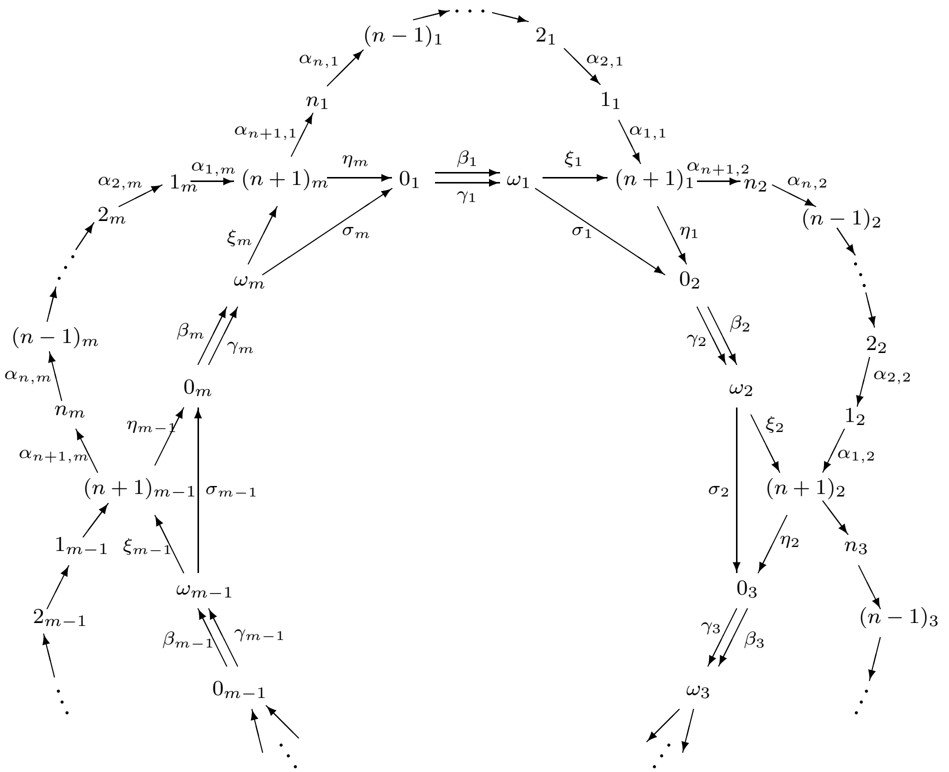
Further, the stable tube $\mathcal{T}$ of Γ_H is a stable tube of the Auslander–Reiten quiver Γ_A of A (see [15, Corollary 1.2]), and hence $W(M)$ is a wing of Γ_A. Since A is a symmetric algebra, the Nakayama functor ν_A is the identity functor of $\operatorname{mod} A$, and consequently $I_A(S_i) = P_A(S_i)$ for any $i \in \{1, \dots, n\}$. In fact, for any $i \in \{1, \dots, n\}$, $P_A(S_i)$ is a uniserial module of length $n+3$ with the simple composition factors

$$P_A(S_i) = \begin{pmatrix} S_i \\ \vdots \\ S_1 \\ S_0 \\ S_\omega \\ S_n \\ \vdots \\ S_{i+1} \\ S_i \end{pmatrix}.$$

Finally, note that M is not the radical of an indecomposable projective A-module, because M lies in the stable tube $\mathcal{T}$ of Γ_A.

The second example shows that there are selfinjective algebras with arbitrarily large finite order of the Nakayama functor and the Auslander–Reiten quivers having wings described by Theorem A.

Example 4.2. Let $m, n \ge 1$ be natural numbers. Denote by $Q = Q(m,n)$ the quiver of the form



and by $I = I(m,n)$ the ideal in the path algebra $KQ(m,n)$ of $Q(m,n)$ generated by the elements of the form

$$\alpha_{1,j}\eta_j,\ \xi_j\alpha_{n+1,j+1},\ \sigma_j\beta_{j+1},\ \eta_j\gamma_{j+1},\ \beta_j\sigma_j,\ \gamma_j\xi_j,$$
$$\alpha_{i,j}\alpha_{i-1,j}\ldots\alpha_{1,j}\alpha_{n+1,j+1}\ldots\alpha_{i+1,j+1}\alpha_{i,j+1},$$
$$\alpha_{n+1,j}\alpha_{n,j}\ldots\alpha_{2,j}\alpha_{1,j}-\eta_{j-1}\beta_j\xi_j,\ \beta_j\xi_j\eta_j-\gamma_j\sigma_j,\ \xi_j\eta_j\beta_j-\sigma_j\gamma_{j+1},$$

for $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,m\}$, where $\alpha_{n+1,m+1} = \alpha_{n+1,1}$, $\beta_{m+1} = \beta_1$, $\gamma_{m+1} = \gamma_1$, $\eta_0 = \eta_m$. Let $A = A(m,n)$ be the associated bound quiver algebra $KQ(m,n)/I(m,n)$. Consider also the bound quiver algebra $B = K\varDelta_n/J_n$ given by the quiver $\varDelta_n$ of the form

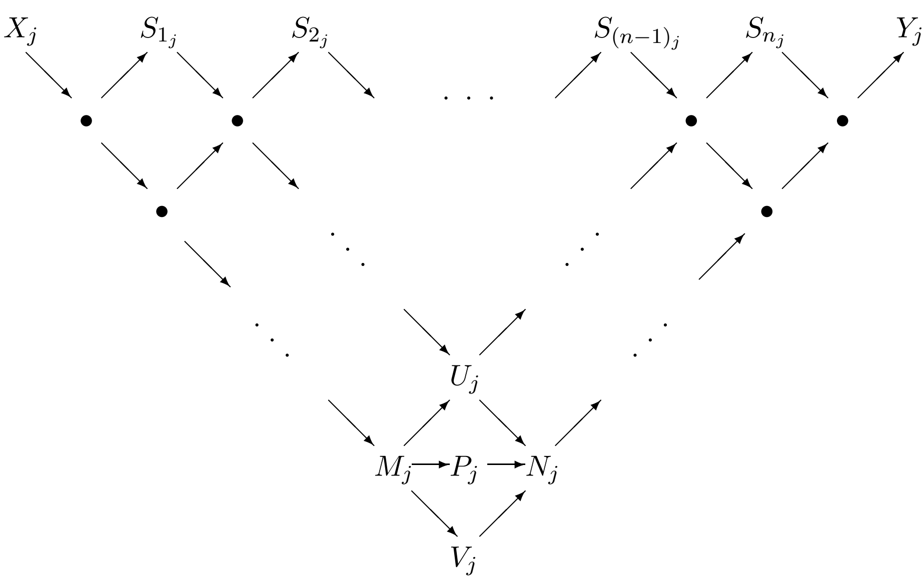
$$\varDelta_n:\quad \begin{tikzcd} & n+1 \arrow[dl, "\eta"'] & 1 \arrow[l, "\alpha_1"'] & 2 \arrow[l, "\alpha_2"'] & \cdots \arrow[l] & n-1 \arrow[l] & n \arrow[l, "\alpha_n"'] \\ 0 & & \omega \arrow[ll, "\sigma"] \arrow[ul, "\xi"'] \end{tikzcd}$$

and the ideal J_n in the path algebra $K\varDelta_n$ of $\varDelta_n$ generated by $\alpha_1\eta$. Then B_n is a tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_{n+2}$ which is a tubular extension of the hereditary algebra H of Euclidean type $\widetilde{\mathbb{A}}_2$ (given by the vertices 0, ω, $n+1$) using the simple (regular) module at the vertex $n+1$ [11, (4.9)]. Then

$A = A(m,n)$ is the m-fold trivial extension algebra $\widehat{B}_n/(\nu^m_{\widehat{B}_n})$, where $\widehat{B}_n$ is the repetitive algebra of B_n and $\nu_{\widehat{B}_n}$ is the Nakayama automorphism of $\widehat{B}_n$ (see [1], [12]). In particular, the Nakayama automorphism ν_A of $\operatorname{mod} A$ has order m. Observe also that A is a special biserial algebra. For each vertex a of Q, denote by S_a the simple A-module at a. Then, for each $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m\}$, the indecomposable projective A-module $P_A(S_{i_j})$ is uniserial of length $n+2$ with the simple composition factors

$$P_A(S_{i_j}) = \begin{pmatrix} S_{i_j} \\ S_{(i-1)_j} \\ \vdots \\ S_{1_j} \\ S_{(n+1)_{j+1}} \\ S_{n_{j+1}} \\ \vdots \\ S_{(i+1)_{j+1}} \\ S_{i_{j+1}} \end{pmatrix}.$$

Hence $I_A(S_{i_{j+1}}) = P_A(S_{i_j})$, and consequently $\nu_A(S_{i_{j+1}}) = S_{i_j}$. Invoking now the rules for the Auslander–Reiten sequences over special biserial selfinjective algebras (see [6, Section 1]) we conclude that, for each $j \in \{1,\dots,m\}$, the Auslander–Reiten quiver Γ_A admits a full translation subquiver of the form



where

$$P_j = P_A(S_{(n+1)_{j-1}}), \qquad M_j = \operatorname{rad} P_j,$$
$$N_j = P_j/S_{(n+1)_j}, \qquad M_j/S_{(n+1)_j} = U_j \oplus V_j.$$

Observe that P_j has the simple composition factors

$$P_j = \left(\begin{array}{ccc} & S_{(n+1)j-1} & \\ & & S_{n_j} \\ & & S_{(n-1)_j} \\ S_{0_j} & & \\ & & \\ S_{\omega_j} & & S_{2_j} \\ & & S_{1_j} \\ & S_{(n+1)_j} & \end{array} \right).$$

Therefore, we have in Γ_A the wings $W(M_1), W(M_2), \dots, W(M_m)$ of length $n+1$ induced by the radicals of the indecomposable projective A-modules $P_A(S_{(n+1)_m})$, $P_A(S_{(n+1)_1})$, $\dots$, $P_A(S_{(n+1)_{m-1}})$, respectively. Observe also that

$$\nu_A(W(M_{j+1})) = W(M_j) \quad \text{for any } j \in \{1, \dots, m\}.$$

Our final example shows that every finite-dimensional module over an arbitrary algebra can be a subfactor of the radical of an indecomposable projective module over a selfinjective algebra, determining the wing described in Theorem A.

EXAMPLE 4.3. Let Λ be a finite-dimensional algebra over a field K and X a finite-dimensional right Λ-module. Consider the faithful Λ-module $Y = \Lambda \oplus X$ and the matrix algebra B of the form

$$B = \begin{bmatrix} K & Y & K \\ 0 & \Lambda & D(Y) \\ 0 & 0 & K \end{bmatrix}$$

where the multiplication is given by the K-Λ-bimodule structure of Y, the Λ-K-bimodule structure of $D(Y) = \operatorname{Hom}_K(Y, K)$, the canonical K-K-bimodule structure of K, and the K-linear map $\varphi : Y \otimes_K D(Y) \to K$ given by $\varphi(y \otimes f) = f(y)$ for $y \in Y$, $f \in D(Y)$. Observe that B admits a unique indecomposable projective-injective faithful module Q whose heart $\operatorname{rad} Q/\operatorname{soc} Q$ is isomorphic to Y. Next consider the generalized canonical algebra (see [14], [15])

$$C = \begin{bmatrix} K & Y & K^2 \\ 0 & \Lambda & D(Y) \\ 0 & 0 & K \end{bmatrix}$$

where the multiplication is given by the algebra structure of B, with

$$\varphi : Y \otimes_K D(Y) \to K = K \times 0,$$

and the canonical K-K-bimodule structure of K^2. Then it follows from [14, Theorem 2.1] that Γ_C admits a stable tube $\mathcal{T}$ of rank 1 whose module R lying on the mouth has simple injective top, simple projective socle, and the heart $\operatorname{rad} R/\operatorname{soc} R$ isomorphic to Y. Moreover, $\mathcal{T}$ is a faithful generalized standard stable tube (in the sense of [13]). Further, take a positive integer n and consider the $(n+2) \times (n+2)$ matrix algebra

$$E = \begin{bmatrix} K & R & K & K & \dots & K & K \\ 0 & C & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & K & K & \dots & K & K \\ 0 & 0 & 0 & K & \dots & K & K \\ & & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & K & K \\ 0 & 0 & 0 & 0 & & 0 & K \end{bmatrix}$$

where the multiplication is given by the algebra structure of C, the K-C-bimodule structure of R, and the canonical K-K-bimodule structure of K. Then E is the tubular extension of C (in the sense of [11, (4.7)]) using the simple regular module R and the linear branch

$$n+1 \to n \to n-1 \to \cdots \to 2 \to 1,$$

with the extension vertex $n+1$. Finally, consider the trivial extension algebra

$$F = T(E) = E \ltimes D(E).$$

Then F is a symmetric algebra and, applying [1, Section 3], we conclude that the Auslander–Reiten quiver Γ_F admits a quasi-tube $\mathcal{C}$ whose upper part contains a subquiver of the form

$$
\begin{tikzcd}
M_{0,0} \arrow[dr] & & M_{1,1} \arrow[dr] & & M_{2,2} \arrow[dr] & & \cdots & & M_{n-2,n-2} \arrow[dr] & & M_{n-1,n-1} \arrow[dr] & & M_{n,n} \\
& M_{0,1} \arrow[ur] \arrow[dr] & & M_{1,2} \arrow[ur] \arrow[dr] & & {} & & {} \arrow[ur] & & M_{n-2,n-1} \arrow[ur] \arrow[dr] & & M_{n-1,n} \arrow[ur] & \\
& & M_{0,2} \arrow[dr] & & {} & & & & {} \arrow[ur] & & M_{n-2,n} \arrow[ur] & & \\
& & & {} & \ddots & {} \arrow[dr] & & {} & \iddots & {} \arrow[ur] & & & \\
& & & \ddots & {} \arrow[dr] & & M_{1,n-1} \arrow[ur] \arrow[dr] & & {} & \iddots & & & \\
& & & & & M_{0,n-1} \arrow[ur] \arrow[dr] & & M_{1,n} \arrow[ur] \arrow[dr] & & & & & \\
& & & & & & M = M_{0,n} \arrow[ur] \arrow[r] \arrow[dr] & P \arrow[r] & P/\operatorname{soc} P & & & & \\
& & & & & & & R \arrow[ur] & & & & &
\end{tikzcd}
$$

where M is the radical of the indecomposable projective-injective F-module P with top and socle isomorphic to the simple module $S(n+1)$ at the vertex $n+1$, and $M_{1,1} = S(1), \ldots, M_{n,n} = S(n)$ are the simple modules at the vertices $1, \ldots, n$, respectively. Observe also that $M/\operatorname{soc} M \cong R \oplus M_{1,n}$ and $\operatorname{rad} R/\operatorname{soc} R \cong Y = \Lambda \oplus X$. Therefore, X is a subfactor of M.

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