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EQUIVALENCE RELATIONS INDUCED BY SOME LOCALLY COMPACT GROUPS OF HOMEOMORPHISMS OF $2^{\mathbb{N}}$

BY

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Abstract. Let T be a locally finite rooted tree and $B(T)$ be the boundary space of T. We study locally compact subgroups of the group $\mathrm{TH}(B(T)) = \langle \mathrm{Iso}(T), V\rangle$ generated by the group $\mathrm{Iso}(T)$ of all isometries of $B(T)$ and the group V of Richard Thompson. We describe orbit equivalence relations arising from actions of these groups on $B(T)$.

0. Preliminaries

0.1. *Introduction.* Given two Borel equivalence relations E_1, E_2 on X_1, X_2 respectively, we say E_1, E_2 are *Borel isomorphic* if there is a Borel bijection $f : X_1 \to X_2$ such that $x\, E_1\, y \Leftrightarrow f(x)\, E_2\, f(y)$, for all $x, y \in X_1$. In [6] A. Kechris gives the following characterization of orbit equivalence relations induced by Borel actions of locally compact groups on a standard Borel space (some converse versions of this theorem have been found in [7]).

Let G be a second countable locally compact group acting in a Borel way on a standard Borel space X. Then there is a unique decomposition $X = C \cup U$ into invariant Borel sets satisfying the following conditions:

(1) *$E_G|C$ is countable, i.e. each $E_G|C$-class is countable*;
(2) *there is a Borel set $Z \subseteq U$, meeting each $E_G|U$-class in a countable set, such that $E_G|U$ is Borel isomorphic to the equivalence relation defined on $Z \times \mathbb{R}$ as follows*: $(z, r) \sim (z', r') \Leftrightarrow (z, z') \in E_G|Z$ (*in symbols* $((z, r), (z', r')) \in (E_G|Z) \times I_{\mathbb{R}}$).

This theorem is the starting point of the paper. It is natural to conjecture that in many particular situations the theorem can be improved by description of Borel complexity of U, Z and the isomorphism arising in the formulation. We study this for actions of some locally compact groups of homeomorphisms of the boundary space $B(T)$ (of all branches) of a locally finite rooted tree T. We consider all locally compact subgroups of the group

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$\mathrm{TH}(B(T)) = \langle \mathrm{Iso}(T), V\rangle$ generated by the group $\mathrm{Iso}(T)$ of all isometries of $B(T)$ and the group V of Richard Thompson (see [3]; elements of $\mathrm{TH}(B(T))$ will be called *Thompson's type homeomorphisms* of $B(T)$). In particular our results describe the case of all locally compact groups of *local isometries* of $B(T)$, i.e. homeomorphisms $g : B(T) \to B(T)$ such that any $x \in B(T)$ has a neighbourhood U where g is an isometry $U \to g(U)$.

It is worth noting that both Thompson's group and the group of (local) isometries of a rooted tree have become quite important in mathematics. On the one hand, they naturally arise in classification problems of group theory [11] (moreover any profinite group can be realized as a closed subgroup of the group $\mathrm{Iso}(T)$ of all isometries of $B(T)$ [4]). On the other hand, they have become a source of important examples (Burnside groups [4]) and applications in discrete mathematics [1], [5] and geometry [3]. From the viewpoint of classification of Borel equivalence relations, actions of (local) isometry groups on the space of tree branches look very typical.

Our main result provides a precise formulation of the theorem of Kechris in the situation when T is a locally finite tree and G is a locally compact group continuously embedded into the group $\mathrm{TH}(B(T))$ of all Thompson's type homeomorphisms of $B(T)$. In particular, we show that the Borel isomorphism from part (2) can be realized by a homeomorphism.

The paper contains several examples which show that some statements of the paper cannot be further improved. We believe that these examples can be useful for some other questions.

One could think that the equivalence relations studied in this paper are casual and for example there are actions of profinite groups (not necessarily isometric) which induce much more complicated equivalence relations. In the final part of the paper we show that this is not the case. We prove that any profinite group G can be realized as a closed subgroup of the group of all isometries of a locally finite tree, so that the space $B(T)$ with the corresponding G-action is a universal Borel G-space. In a sense this can be considered as an improvement of the fact of universality of $\mathrm{Iso}(T)$ mentioned above.

The structure of the paper is as follows. In Section 1 we find a version of Kechris' theorem for closed subgroups of the group $\mathrm{Iso}(T)$ of all isometries of T. The fact that these groups are compact implies that there is a Borel transversal for the equivalence relation induced by G on $B(T)$. This gives a standard method of obtaining versions of Kechris' theorem. In our case the existence of a tree structure allows making the corresponding statements more precise and straightforward. This will be applied in Section 2 to groups of local isometries and Thompson's type groups. In Section 3 we discuss universality properties of $\mathrm{Iso}(T)$.

0.2. *Locally finite rooted trees.* In this subsection we present necessary information concerning trees. We also prove a technical result (Lemma 3) which will be applied below.

A tree T is *locally finite* if any vertex has finite *valency* (= the number of adjacent edges). Distinguishing a point we obtain a rooted tree. A vertex v of a rooted tree is identified with the path from the root to v. If this path consists of n edges, then we say that v belongs to *level* n. Thus the root $\emptyset$ forms level 0. We will write $s \subseteq s'$ if the path s' extends s. We say that $s, s' \in T$ are *incomparable* if neither $s \subseteq s'$ nor $s' \subseteq s$.

The elements of a locally finite tree will be represented by (initial) finite sequences of natural numbers in the following way. The root corresponds to the empty sequence $\emptyset$. For $s \in T$ let $\mathrm{lh}(s) = n$ be the distance from the root. If the valency of s is $k+1$, then we fix an enumeration by $\{0, 1, \ldots, k-1\}$ of all edges incident with s excluding one which is between s and the root. Now for any $s \in T$, the path from the root to s uniquely defines a $\mathrm{lh}(s)$-sequence of natural numbers consisting of the numbers enumerating the edges of the path. Below we shall frequently identify elements of the tree T with the corresponding sequences. For given sequences s, u, we denote by $s^\frown u$ the concatenation of s and u. Let T_n be the set of all elements of T represented by sequences of length $\leq n$.

The *boundary* of a locally finite rooted tree T is the set of all branches of T (denoted by $B(T)$). For given $s \in T$, put $(s) = \{\alpha \in B(T) : s \subseteq \alpha\}$. The family of all such (s), where $s \in T$, forms a (countable) base of a topology on $B(T)$. Then $B(T)$ becomes a compact space where the base above consists of clopen sets. We consider this space under the standard metric defined by $d(\gamma, \delta) = 2^{-n}$, where n is the minimal number m satisfying $\gamma|_m \neq \delta|_m$.

The group $H(B(T))$ of all homeomorphisms of $B(T)$ is equipped with the (standard) metric $d(f, g) = 2^{-n}$, where for $f \neq g$, $n = \min\{l \in \omega : (\exists \alpha \in B(T))(f(\alpha)|_l \neq g(\alpha)|_l)\}$. Then $H(B(T))$ is a separable metric group. For a bijection $f : B(T) \to B(T)$ and natural number n, let $f|_n$ denote the relation on the set T_n defined by

$$(s,t) \in f|_n \Leftrightarrow (s, t \in T_n) \wedge (\exists \alpha, \beta \in B(T))((s \text{ is an initial segment of } \alpha) \wedge (t \text{ is an initial segment of } \beta) \wedge f(\alpha) = \beta).$$

Now for any $n \in \omega$ and any relation $R \subseteq T_n \times T_n$ with $\mathrm{dom}(R) = \mathrm{rng}(R) = T_n$, define (R) as the set of all homeomorphisms $f : B(T) \to B(T)$ such that $f|_n = R$. The family of all sets of this kind forms a countable base of the topology given by the metric above. We will call this topology the *tree topology*.

Definition 1. Let $f : B(T) \to B(T)$ be a homeomorphism. We say that f is a *Thompson's type homeomorphism* if there is a natural number $l > 0$ and two sequences $(s_i)_{i<l}$, $(t_i)_{i<l}$ of vertices of the tree T such that:

(i) $\bigcup_{i<l}(s_i) = \bigcup_{i<l}(t_i) = B(T)$;
(ii) s_i, s_j are incomparable for any distinct $i, j < l$;
(iii) t_i, t_j are incomparable for any distinct $i, j < l$;
(iv) $\alpha \in (s_i) \Leftrightarrow f(\alpha) \in (t_i)$, for every $i < l$;
(v) $2^{\mathrm{lh}(s_i)} d(\alpha,\beta) = 2^{\mathrm{lh}(t_i)} d(f(\alpha), f(\beta))$, for any $i < l$ and $\alpha, \beta \in (s_i)$.

The last condition says that to every $i < l$ we can assign an isometry f_i from the subtree defined by (s_i) to the subtree defined by (t_i) so that $f(s_i{}^\frown\alpha) = t_i{}^\frown f_i(\alpha)$. (It is clear that the definition implies that these subtrees are isomorphic, in particular s_i and t_i have the same valency.) It is routine to check that the set of all Thompson's type homeomorphisms is a group; we denote it by $\mathrm{TH}(B(T))$.

Richard Thompson's original group V consists of all Thompson's type homeomorphisms which satisfy a version of condition (v) where we additionally demand that all appropriate isometries f_i are identities of the corresponding $\{0,1\}$-labelled subtrees. It is easy to see that $\mathrm{TH}(B(T)) = \langle \mathrm{Iso}(T), V\rangle$.

A locally finite tree T will be considered with the lexicogrphical ordering $\prec$ defined as follows. For two sequences $s, s' \in T$,

$s \prec s'$ iff

$$((s \subseteq s') \vee (\exists n \le \min\{\mathrm{lh}(s), \mathrm{lh}(s')\})((\forall i < n)(s(i) = s'(i)) \wedge s(n) < s'(n))).$$

We shall write $s \preceq s'$ whenever $s \prec s' \vee s = s'$. It is clear that the order $\preceq$ extends $\subseteq$.

The ordering $\preceq$ induces a natural linear ordering $\preceq_B$ on $B(T)$ in the following way. For $\alpha, \beta \in B(T)$, $\alpha \preceq_B \beta$ iff $(\forall n \in \mathbb{N})\,(\alpha|_n \preceq \beta|_n)$. Below we shall use the same symbols $\prec$ and $\preceq$ for both the orderings on T and $B(T)$. It is easily seen that $\prec$ and $\preceq$ are open and closed subsets of $T \times T$ and $B(T) \times B(T)$ respectively.

We say that T is *spherically homogeneous* if any two points of the same distance from the root have the same valency. In the case of spherically homogeneous trees $B(T)$ can be represented by $\prod_{i\in\mathbb{N}}\{0, 1, \ldots, k_i - 1\}$ (here $k_i + 1$ is the valency of vertices of level i) and the topology becomes the usual product topology. Since the boundary of the binary tree $2^{<\mathbb{N}}$ is just the Cantor space, we will use $2^{\mathbb{N}}$ instead of $B(2^{<\mathbb{N}})$.

We now define a procedure which codes any spherically homogeneous locally finite tree in the binary one. This will be one of the basic tools in Section 1.

LEMMA 2. *For every natural number $k \ge 1$, there exists a sequence $u_k(0) \prec u_k(1) \prec \cdots \prec u_k(k-1)$ of pairwise incompatible elements from $2^{<\mathbb{N}}$ such that $\bigcup_{i<k}(u_k(i)) = 2^{\mathbb{N}}$.*

Proof. Put $u_1(0) = \emptyset$ and, for $k > 1$,

$$u_k(i) = \underbrace{11\ldots1}_{i\,\text{times}}0 \quad \text{for } i < k-1, \qquad u_k(k-1) = \underbrace{11\ldots1}_{k-1\,\text{times}}. \ \blacksquare$$

LEMMA 3. *For every spherically homogeneous tree T, there is a $\prec$-preserving homeomorphism $\psi_T : B(T) \to 2^{\mathbb{N}}$.*

Proof. Let k_i+1 be the valency of T at level i, $i \geq 0$. Define $\psi_T : B(T) \to 2^{\mathbb{N}}$ as follows (under the notation of Lemma 2):

$$\psi_k(\alpha) = \lim_{n\to\infty} u_{k_1}(\alpha(1))^\frown u_{k_2}(\alpha(2))^\frown \ldots {}^\frown u_{k_n}(\alpha(n)) \quad \text{for } \alpha \in B(T_k).$$

Note that when $k_i = 1$, $u_{k_i}(0)$ becomes $\emptyset$ and does not appear in the sequences. From the definition of the sequences $(u_k(j))_{0\leq j<k}$ we conclude that ψ_T is a continuous, $\prec$-preserving bijection. Then the inverse function ψ_T^{-1} is also continuous. $\blacksquare$

1. Actions of closed isometry groups on a rooted tree. Let T be a locally finite rooted tree. The group $\mathrm{Iso}(T)$ of all isometries of T (with respect to the natural length function) is a profinite group with respect to the canonical homomorphisms $\pi_n : \mathrm{Iso}(T) \to \mathrm{Iso}(T_n)$. Thus $\mathrm{Iso}(T)$ and all its closed subgroups are compact. We will see later that any locally compact group G of Thompson's type homeomorphisms is somehow determined by the subgroup of all isometries from G. This suggests that we should start with the case of closed subgroups of $\mathrm{Iso}(T)$. In this case we can apply some standard methods together with the existence of a tree structure.

Let G be a closed subgroup of $\mathrm{Iso}(T)$. Consider the action of G on the space $B(T)$. The action is obviously continuous. Let E_G denote the corresponding equivalence relation on $B(T)$. For $\alpha \in B(T)$ let $[\alpha]$ denote the E_G-orbit of α. In the following lemma we collect some folklore facts concerning compact groups ([1]).

LEMMA 4. *Let G be a closed subgroup of* $\mathrm{Iso}(T)$ *and E_G the corresponding equivalence relation on $B(T)$.*

(a) *Each orbit of G is a closed subset of $B(T)$.*
(b) *E_G is a closed subset of $B(T) \times B(T)$.*
(c) *The function picking up the leftmost branch in each orbit, that is, the function $S : B(T) \to B(T)$ defined by*

$$S(\alpha) = \beta \textit{ iff } ((\alpha,\beta) \in E_G \wedge (\forall \gamma \in B(T))((\alpha,\gamma) \in E_G \Rightarrow \beta \preceq \gamma)),$$

is a continuous selector for E_G and the image of S is a closed transversal of this relation.

([1]) Our lemma also resembles Theorem 5.4.3 of [8].

Proof. To prove (a) and (b) notice that each orbit is a continuous image of a compact space G. Hence it is a compact subset of the compact space $B(T)$ and thus it is closed.

On the other hand, E_G is the continuous image of the compact space $B(T)\times G$ under the function $B(T)\times G\to B(T)\times B(T)$ given by $(\delta,g)\mapsto(\delta,g(\delta))$.

(c) Suppose that (α_n) is a sequence of elements of $B(T)$ convergent to some $\alpha\in T$. We shall prove that $S(\alpha_n)\to S(\alpha)$. Since $B(T)$ is a compact space, it suffices to show that the limit of each convergent subsequence of $(S(\alpha_n))$ is exactly $S(\alpha)$. Passing to a subsequence if necessary, we may assume that the sequence $(S(\alpha_n))$ is already convergent and let $\lim_{n\to\infty}S(\alpha_n)=\beta$. For every $n\in\mathbb{N}$, we have $(\alpha_n,S(\alpha_n))\in E_G$ and then $(\alpha,\beta)\in E_G$, since E_G is closed. Hence $S(\alpha)\preceq\beta$ and there is some $g\in G$ such that $g(\beta)=S(\alpha)$. Since g is continuous we have $\lim_{n\to\infty}g(S(\alpha_n))=g(\beta)$. Since $S(\alpha_n)\preceq g(S(\alpha_n))$ for every $n\in\mathbb{N}$, we have $\beta\preceq S(\alpha)$. Thus $\beta=S(\alpha)$, which completes the proof of the first part.

To prove the second part, notice that the image of S is the image of a compact space under a continuous function. ∎

Given $n\in\mathbb{N}$ and $\alpha\in B(T)$, we say that n is a *branching point* of $\alpha\in B(T)$ if there is some $\delta\in[\alpha]$ such that $\alpha|_n=\delta|_n$ but $\alpha(n)\neq\delta(n)$. Obviously, $\alpha E_G\beta$ implies that $n\in\mathbb{N}$ is a branching point of α if and only if it is a branching point of β. So we will say that $n\in\mathbb{N}$ is a *branching point* of an orbit if it is a branching point of some (any) of its elements.

The E_G-orbit of $\alpha\in B(T)$ has cardinality $<2^{\aleph_0}$ if and only if the set of its branching points is finite. Now the following formula describes the union of all E_G-classes of cardinality $<2^{\aleph_0}$:

$$(\exists n\in\mathbb{N})(\forall g,g'\in G)(g(\alpha)\neq g'(\alpha)\Rightarrow(\exists m\le n)(g(\alpha(m))\neq g'(\alpha(m)))).$$

As a result we have the following lemma.

LEMMA 5. (a) *Any class of E_G of cardinality $<2^{\aleph_0}$ is finite.*

(b) *The union of all E_G-classes of cardinality $<2^{\aleph_0}$ is an invariant F_σ-set.*

(c) *Let $\alpha\in B(T)$. The orbit of α is infinite if and only if the set of its branching points is infinite. The union of all infinite orbits is an invariant G_δ-set.*

Following Kechris [6], we call the set from part (b) of the lemma the *countable part* of E_G and the set from part (c) the *continuous part* of E_G.

The following example shows that we cannot claim that the countable part is a G_δ-set.

EXAMPLE. Consider $2^{<\mathbb{N}}$. Let $g\in\mathrm{Iso}(2^{<\mathbb{N}})$ be defined as follows. At level 2 let g act as an adding machine: $g(ab)=10+ab$ (from left to right), $a,b\in$

$\{0,1\}$. At level 3 let g define two cycles corresponding to the rule $g(abc) = (10+ab)c$. Moreover one of them extends to a g-cycle on $2^{\mathbb{N}}$ consisting of four elements: $g(ab000\ldots) = (10+ab)000\ldots$.

For any sequence $n_1,\ldots,n_k$ of numbers from $\mathbb{N}$ the g-image of the element

$$ba_0 0\ldots 01a_1 0\ldots 01a_2 0\ldots 01\ldots 01a_k,$$

where n_i is the number of zeros in the block of zeros following a_{i-1}, is defined as follows. Let $b'a_0'a_1'\ldots a_k'$ be the 2-adic sum (from left to right) of $100\ldots 0$ and $ba_0a_1\ldots a_k$ (restricted to sequences of length $k+2$). Then let

$$b'a_0' 0\ldots 01a_1' 0\ldots 01a_2' 0\ldots 01\ldots 01a_k'$$

be the g-image of

$$ba_0 0\ldots 01a_1 0\ldots 01a_2 0\ldots 01\ldots 01a_k.$$

We assume that g naturally extends to the cycle of length 2^{k+2} on $2^{\mathbb{N}}$ by

$$g(ba_0 0\ldots 01a_1 0\ldots 01a_2 0\ldots 01\ldots 01a_k 000\ldots) \\ = b'a_0' 0\ldots 01a_1' 0\ldots 01a_2' 0\ldots 01\ldots 01a_k' 000\ldots$$

By this procedure we obtain an action of $\langle g\rangle$ on $2^{\mathbb{N}}$ such that the union of all finite orbits coincides with $Z = \{\varrho \in 2^{\mathbb{N}} : \exists n \forall i(\varrho(n+i) = 0)\}$. It is clear that the set Z is the union of all finite orbits of the profinite completion $\langle g\rangle^*$. On the other hand, Z as well as its complement $2^{\mathbb{N}} \setminus Z$ are dense subsets of $2^{\mathbb{N}}$; thus by the Baire Category Theorem, Z is not G_δ. ∎

THEOREM 6. *Let T be a locally finite rooted tree. Let G be a closed subgroup of* $\mathrm{Iso}(T)$, *E_G be the corresponding orbit equivalence relation on $B(T)$ and $U \subseteq B(T)$ be the continuous part of that relation. Let Z be the intersection of U with the closed transversal $S(B(T))$ of E_G where S is defined as in Lemma* 4. *Then Z is a G_δ transversal of $E_G|_U$ such that there is a homeomorphism $\phi_G : Z \times 2^{\mathbb{N}} \to U$ satisfying*

$$(\phi_G((z,\delta)), \phi_G((z',\delta'))) \in E_G|_U \;\Leftrightarrow\; z = z'.$$

Proof. We use the strategy of [6], although our proof does not use any involved material.

It follows from Lemma 5 that the continuous part U of E_G is a G_δ-set.

For given $z \in Z$, let T_z be the tree consisting of all $\alpha|_n$ with $\alpha \in [z]$ and $n \in \mathbb{N}$. Observe that the elements of T_z of level n form the G-orbit of $\alpha|_{n+1}$. Then it is clear that T_z is spherically homogeneous. Let $\psi_{T_z} : B(T_z) \to 2^{\mathbb{N}}$ be the corresponding coding function defined in Lemma 3. We now define the required function $\phi_G : Z \times 2^{\mathbb{N}} \to U$ by $\phi_G(z,\delta) := \psi_{T_z}^{-1}(\delta)$.

By Lemmas 2 and 3, ψ_{T_z} can be considered as a 1-1 function on T_z satisfying the following conditions:

(1) $(\forall s,s' \in T_z)((s \subseteq s' \Leftrightarrow \psi_{T_z}(s) \subseteq \psi_{T_z}(s')) \wedge (s \prec s' \Leftrightarrow \psi_{T_z}(s) \prec \psi_{T_z}(s')))$;

(2) $(\forall n)(\forall \delta \in 2^{\mathbb{N}})(\exists s \in T_z)(\psi_{T_z}(s)|_n = \delta|_n)$.

Then ϕ_G can be equivalently defined (for an appropriate sequence (n_i)) by

$$\phi_G(z,\delta) = \lim_{n_i\to\infty} \psi^{-1}_{T_z}(\delta|_{n_i}).$$

Notice that for every $z \in Z$ we have $\phi_G(z,\bar{0}) = z$, where $\bar{0}$ is the sequence of zeros. It easily follows from properties (1)–(2) that the function is a bijection. The inverse function $\phi_G^{-1} : U \to Z \times 2^{\mathbb{N}}$ is a pair of functions (S,F) such that S is the restriction of the selector defined in Lemma 4 to U. Note that, by Lemma 4, S is continuous (and by (1), ϕ_G is continuous in the second coordinate). We shall prove that ϕ_G^{-1} is continuous.

Suppose that $\{\beta_n\}$ is a sequence of elements of U convergent to some $\beta \in U$. By Lemma 4, $\lim_{n\to\infty} S(\beta_n) = S(\beta)$. Let l be a natural number. For every i there is a natural number m_i such that for every $n > m_i$, β_n agrees with β at level i. Since $T_\beta(i)$ is the G-orbit of $\beta|_{i+1}$, $T_{\beta_n}(i)$ coincides with $T_\beta(i)$. Then choosing i large enough and $n > m_i$ we have, for $\gamma = \beta_n$,

$$F(\beta)|_l = \psi_{T_\beta}(\beta|_i)|_l = \psi_{T_\gamma}(\gamma|_i)|_l = F(\gamma)|_l.$$

Hence $\lim_{n\to\infty} F(\beta_n) = F(\beta)$. Since a continuous bijection between compact spaces is a homeomorphism, we conclude that ϕ_G is a homeomorphism. ∎

2. Locally compact groups of homeomorphisms of the space $B(T)$. In this section we prove our main results. We shall consider two types of subgroups of the group of all homeomorphisms of the boundary space $B(T)$ of the tree T and their natural actions on $B(T)$.

2.1. *Thompson's type groups.* Let T be a locally finite rooted tree. We will study orbit equivalence relations induced on $B(T)$ by locally compact groups of Thompson's type permutations.

We start with an example of a non-compact closed subgroup of $\mathrm{TH}(2^{\mathbb{N}})$ which is locally compact with respect to the standard tree topology (see Preliminaries). It is worth noting that this group cannot be a subgroup of $\mathrm{Iso}(2^{\mathbb{N}})$, because all closed isometry subgroups are compact.

EXAMPLE. Consider $2^{<\mathbb{N}}$. Let $r : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ be the right shift function $w \mapsto 1^\frown w$, $w \in 2^{<\mathbb{N}}$. For every $n \geq 1$ we define by induction a set $C_n \subset 2^{<\mathbb{N}}$ consisting of 2^{n-1} elements. The definition depends on an appropriate function $q : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$. Let $C_1 = \{q(\emptyset)\}$, $C_2 = \{q(0), q(1)\}$, where $q(\emptyset) = 10$, $q(0) = 1100$, $q(1) = 1101$. At Step $n+1$ let $q(a_1 \ldots a_{n-1} b) = r(q(a_1 \ldots a_{n-1}))b$, where $a_i \in \{0,1\}$ and $b \in \{0,1\}$, and let C_{n+1} consist of all $q(a_1 \ldots a_n)$, $a_i \in \{0,1\}$.

Let c be the 1-letter word 0. We now define by induction permutations g_n on $\{c\} \cup C_1 \cup \cdots \cup C_n \cup \cdots$, which are cyclic on $\{c\} \cup C_1 \cup \cdots \cup C_n$ and preserve each C_m with $m > n$. We demand that $g_1(c) = q(\emptyset)$, $g_1(q(\emptyset)) = c$ and $g_{n-1} = g_n^2$. For each $l \geq 1$ the permutation g_n on C_{n+l} is defined by the following rule. Let $a'_1 \ldots a'_{l-1} a'_l a'_{l+1} \ldots a'_{n+l-1}$ be the 2-adic sum (from left to right) of $0\dots010\dots0$ and $a_1\dots a_{l-1}a_la_{l+1}\dots a_{n+l-1}$ (restricted to sequences of length $n+l-1$). Then let $g_n(q(a_1\dots a_{l-1}a_la_{l+1}\dots a_{n+l-1})^\frown w)$ be $q(a'_1\dots a'_{l-1}a'_la'_{l+1}\dots a'_{n+l-1})^\frown w$, where $w\in 2^{<\mathbb{N}}$.

For elements $v\in\{c\}\cup C_1\cup\dots\cup C_n$ we define g_n as follows. Let $g_n(c)$ be the q-image of the $(n-1)$-tuple $00\dots0$. The rest of the definition of g_n on $\{c\}\cup C_1\cup\dots\cup C_n$ follows from the assumption that $g_n^2=g_{n-1}$ and the definition of g_{n-1} on C_n and $\{c\}\cup C_1\cup\dots\cup C_{n-1}$. For elements of the form $v^\frown w$ with $v\in\{c\}\cup C_1\cup\dots\cup C_n$ and $w\in 2^{<\mathbb{N}}$ we define $g_n(v^\frown w)=g_n(v)^\frown w$.

If an element $u\in 2^{<\mathbb{N}}$ cannot be represented as a subword of a word of the form $v^\frown w$ with $v\in\{c\}\cup C_1\cup\dots\cup C_k\cup\cdots$ and $w\in 2^{<\mathbb{N}}$ we define $g_n(u)=u$ (this is the case of $111\dots$).

As a result we obtain an action of the Prüfer group C_{2^∞} on $2^{<\mathbb{N}}$ and thus on $2^{\mathbb{N}}$. We consider C_{2^∞} as a topological group under the topology induced from its action, thus under the standard tree topology. The group C_{2^∞} is discrete under this topology. Indeed for any n the element g_n is determined uniquely by its action on the set $\{c\}\cup C_1\cup\dots\cup C_{n-1}$. ∎

LEMMA 7. *Let $G<\mathrm{TH}(B(T))$. For every $n\in\omega$ define $G_n=\{f\in G:$ f is defined by some sequences $(s_i)_{i<l}$ and $(t_i)_{i<l}$ as in Definition* 1 *with* $\max\{\mathrm{lh}(s_i),\mathrm{lh}(t_i):i<l\}\le n\}$. *Then $(G_n)_{n\in\omega}$ is an increasing sequence of closed subgroups of G such that $G=\bigcup_n G_n$ and the equivalence relation induced by G on $B(T)$ is the union of the equivalence relations induced by G_n on $B(T)$. If G is locally compact with respect to the standard tree topology, then all G_n are open in G. In this case G_0 has a subgroup H of countable index which is a closed subgroup of* $\mathrm{Iso}(T)$.

Proof. The first part of the lemma is obvious. Now assume that G is locally compact. Then G is a Baire space. Therefore there is a natural number k such that for every $n\ge k$, G_n is not meager. Then, by Pettis' Theorem, G_n is open for every $n\ge k$.

Let $n<k$ and $f\in G_n$. Since $f\in G_k$, there is $m\ge k$ such that the basic open set $(f|_m)$ is contained in G_k. Thus there are sequences (s_i) and (t_i) of the same length such that $\max_{i,j}(s_i,t_j)\le k$ and any $g\in(f|_m)$ is defined (as in Definition 1) by the map $s_i\mapsto t_i$ and appropriate isometries of the corresponding subtrees. Since $f\in G_n$, there are sequences (s_i^f) and (t_i^f) such that $\max_{i,j}(s_i^f,t_j^f)\le n$ and f is determined by the map $s_i^f\mapsto s_i^f$ and appropriate isometries of the corresponding subtrees. In particular the map $s_i\mapsto t_i$ can be realized by $s_i^f\mapsto t_i^f$ and appropriate isometries. This implies that $(f|_m)\subseteq G_n$. We see that G_n is open.

Since G is locally compact, there is a compact subgroup $H<G_0$ of countable index. Thus H is closed in $\mathrm{Iso}(T)$. ∎

LEMMA 8. *Let G be a locally compact group of Thompson's type homeomorphisms and E_G be the equivalence relation on $B(T)$ induced by the natural action of G on that space. Let $G_B := H$ be the closed subgroup of* $\mathrm{Iso}(T)$ *defined in Lemma 7.*

(a) *Let $C \subseteq B(T)$ be the closed transversal of the G_B-orbit equivalence relation defined by an application of Lemma 4 to G_B. Then any class of E_G has a non-empty countable intersection with C.*
(b) *Any class of E_G of cardinality $< 2^\omega$ is the union of a countable family of finite E_{G_B}-classes. Any uncountable E_G-class is the union of a countable family of uncountable E_{G_B}-classes. In particular the continuous part of E_G coincides with the continuous part of E_{G_B}, they are G-invariant G_δ-sets and the union of all E_G-classes of countable cardinality is a G-invariant F_σ-set.*

Proof. (a) Observe that C is a section of the equivalence relation induced by G. We claim that C is a countable section. By Lemma 7, G_0 is a clopen subgroup of G, thus it is of countable index in G, so G_B is of countable index in G. Suppose that there is some $\alpha \in C$ such that $[\alpha]_G \cap C$ is uncountable. Then there are two distinct elements $f\alpha, h\alpha \in C$ such that f, h are in the same coset of G_B. Hence $fh^{-1} \in G_B$ and $f\alpha = fh^{-1}(h\alpha)$. Thus C contains two distinct elements $f\alpha, h\alpha$ from the same G_B-orbit, which contradicts the fact that C is a transversal.

(b) Let $\alpha \in B(T)$, $[\alpha]_{G_B}$ be the class of α with respect to the G_B-action and A be a countable set of representatives of all right cosets of G_B in G. We have $[\alpha]_G = \bigcup_{g\in A} g([\alpha]_{G_B})$. Then we are done by Lemma 5. ■

THEOREM 9. *Let $G < \mathrm{TH}(B(T))$ be a locally compact group, E_G be the corresponding orbit equivalence relation on $B(T)$ and $U \subseteq B(T)$ be the continuous part of the relation. Then there is a G_δ set Z which is a countable section of $E_G|_U$ and a homeomorphism $\phi_G : Z \times 2^{\mathbb{N}} \to U$ such that*

$$(\phi_G((z,\delta)), \phi_G((z',\delta'))) \in E_G|_U \iff zE_Gz'.$$

Proof. It follows from Lemma 8 that the continuous part U of E_G is a G_δ-set and coincides with the continuous part of E_{G_B}. Let C be the countable section of E_G defined in the proof of this lemma. Then $Z = C \cap U$ is a G_δ-set which is a countable section of $E_G|_U$.

Now let $\phi_{G_B} : Z \times 2^{\mathbb{N}} \to U$ be the homeomorphism defined in the proof of Theorem 6 applied to G_B. Since $(\phi_{G_B}((z,\delta)), z) \in E_G$ for every z and $\delta \in B(T)$, it satisfies the assertion of the theorem. ■

2.2. *Local isometries*

DEFINITION 10. Let $f : B(T) \to B(T)$ be a homeomorphism, $\alpha \in B(T)$ and $n \in \omega$.

(a) We say that n *stabilizes* f *on* α if for every β and γ from the basic open set $(\alpha|_n)$ we have

$$d(\beta,\gamma) = d(f(\beta), f(\gamma)).$$

(b) We say that n *destabilizes* f *on* α if there are $\beta,\gamma \in (\alpha|_n)$ such that $d(\beta,\gamma) = 2^{-(n+1)}$ whereas $d(f(\alpha), f(\beta)) \neq 2^{-(n+1)}$.

It is obvious that for given α, f and n as above, n stabilizes f on α exactly when no $k \geq n$ destabilizes f on any $\beta \in (\alpha|_n)$.

DEFINITION 11. We say that a homeomorphism $f : B(T) \to B(T)$ is a *local isometry* if for every $\alpha \in B(T)$ there is $n \in \omega$ stabilizing f on α.

It is clear that this definition just says that for every $\delta \in B(T)$ there exists a neighbourhood U such that $d(x_1,x_2) = d(f(x_1), f(x_2))$ for all $x_1, x_2 \in U$. We denote the group of all local isometries of $B(T)$ by $\mathrm{LI}(B(T))$. At the conference "Groups and Group Rings 10" (Ustroń, 2003), Yaroslav Lavrenyuk (Kiev) has announced that the centre of this group is trivial and any automorphism of $\mathrm{LI}(B(T))$ is induced by a conjugation.

The following observation shows that a local isometry is a Thompson's type homeomorphism of $B(T)$.

LEMMA 12. *Let $f : B(T) \to B(T)$ be a local isometry. There is a natural number n which stabilizes f on every $\alpha \in B(T)$. Thus f is a Thompson's type homeomorphism where the sequence (s_i) coincides with the sequence (t_i) and consists of all elements of T of length n.*

Proof. We have to show that the set of natural numbers k such that k destabilizes f on some $\alpha \in B(T)$ is finite. Otherwise by König's Lemma, there would be $\alpha \in B(T)$ such that the set of natural numbers k which destabilize f on α is infinite. The latter contradicts the assumption that f is stabilized on α by some natural n. ■

It is easy to verify that $\mathrm{LI}(B(T))$ is a closed subgroup of $\mathrm{TH}(B(T))$. From Lemma 12 we see that Theorem 9 holds for all locally compact subgroups of $\mathrm{LI}(B(T))$.

We finish this section with an example of a locally compact (with respect to the tree topology) group of local isometries which is not compact. In this example the subgroup H arising in Lemma 7 is uncountable.

EXAMPLE. We define the sequence (g_n) of local isometries of the boundary space $2^{\mathbb{N}}$ of the binary tree as follows:

$$g_0 = \mathrm{id},$$
$$g_n(\underbrace{0011\ldots110}_{\text{length } n+1}{}^\frown\alpha) = \underbrace{1100\ldots001}_{\text{length } n+1}{}^\frown\alpha,$$

$$g_n(\underbrace{1100\ldots001}_{\text{length}\,n+1}{}^\frown\alpha) = \underbrace{0011\ldots110}_{\text{length}\,n+1}{}^\frown\alpha,$$
$$g_n(s^\frown\alpha) = s^\frown\alpha \quad \text{for any } s \in 2^{n+1} \setminus \{\underbrace{1100\ldots001}_{\text{length}\,n+1}, \underbrace{0011\ldots110}_{\text{length}\,n+1}\}.$$

Now, let $G < \mathrm{LI}(2^{\mathbb{N}})$ be the group generated by the group $G_L = \langle g_n : n \in \omega\rangle$ and the group G_I of all isometries fixing all $\delta \in 2^{\mathbb{N}}$ of the form $00^\frown\delta'$ and $11^\frown\delta'$. Since no finite union of basic clopen sets of the form $(g|_{n+1})$ covers $\{g_n : n \in \omega\}$, we see that G is not compact with respect to the tree topology. We are going to show that G is locally compact. Observe that G_I is compact, $G = G_L \oplus G_I$ and G_L is an abelian group of exponent 2. Take any $g \in G_L$. Let $n_0 < n_1 < \cdots < n_k$ be an increasing sequence of natural numbers such that $g = g_{n_k} g_{n_{k-1}} \ldots g_{n_0}$. We claim that $(g|_{n_k+1})$ is a compact neighbourhood of g. To prove this suppose that $h \in (g|_{n_k+1}) \cap G$. We have

$$h(\underbrace{0011\ldots11}_{\text{length}\,n_k+1}{}^\frown\alpha) = \underbrace{0011\ldots11}_{\text{length}\,n_k+1}{}^\frown\alpha \quad \text{for any } \alpha \in 2^\omega.$$

Hence if $h \in g_{m_l} g_{m_{k-1}} \ldots g_{m_0} + G_I$ then $m_l \le n_k$. Indeed, otherwise we have the following contradiction with the equality above:

$$\begin{aligned} h(\underbrace{0011\ldots11}_{\text{length}\,n_k+1}{}^\frown \underbrace{11\ldots10}_{\text{length}\,m_l-n_k}{}^\frown\alpha) &= g_{m_l}(\underbrace{0011\ldots11}_{\text{length}\,n_k+1}{}^\frown \underbrace{11\ldots10}_{\text{length}\,m_l-n_k}{}^\frown\alpha) \\ &= \underbrace{1100\ldots00}_{\text{length}\,n_k+1}{}^\frown \underbrace{00\ldots01}_{\text{length}\,m_l-n_k}{}^\frown\alpha \quad \text{for any } \alpha \in 2^\omega. \end{aligned}$$

We now see that $(g|_{n_k+1})$ is contained in the subgroup $\langle g_n : n \le n_k\rangle \oplus G_I$ and thus is compact. The group G_I can be taken as G_B in Lemma 8.

3. Universal properties of $B(T)$. We close the paper with two remarks concerning the universal character of the space $B(T)$ viewed as a G-space for various $G < \mathrm{Iso}(T)$. Let us recall some terminology.

Let G be a Polish group. Any Borel space U with a Borel measurable action $a : G \times U \to U$ is called a *Borel G-space*. For two Borel G-spaces U_1, U_2, we say that U_1 is *Borel embeddable* into U_2 if there is a Borel measurable, one-to-one map $\pi : U_1 \to U_2$ such that $\pi(g(x)) = g(\pi(x))$ for every $g \in G$ and $x \in U_1$. A Borel G-space $\mathcal{U}$ is *universal* if any Borel G-space U can be Borel embedded into $\mathcal{U}$.

The following example of a universal Borel G-space is given by H. Becker and A. Kechris in [2]. By $\mathcal{F}(G)$ we denote the standard Borel space of closed subsets of G with the Effros Borel structure. It is proved in [2] that

$(\mathcal{F}(G))^{\mathbb{N}}$ with the left actions of G by $g(F_n)_{n\in\omega} = (gF_n)_{n\in\omega}$ is a universal Borel G-space.

Our observation concerns actions of profinite groups.

PROPOSITION 13. *For any countably based profinite group G, there is a locally finite tree T and an isometric action of G on T such that the G-space $B(T)$ is a universal Borel G-space.*

Proof. Let G be a countably based profinite group. We want to show that there is a locally finite tree T and an isometric action of G on T such that the universal Borel G-space $\mathcal{U}_G = (\mathcal{F}(G))^{\mathbb{N}}$ with the left action of G can be Borel embedded into $B(T)$ with this action.

By Proposition 4.1.3 of [10], there is a chain of open normal subgroups $G = M_0 \geq M_1 \geq \cdots$ such that the set of all their cosets forms a base of G. For every $i \in \mathbb{N}$ let $n_i = |G : M_i|$ and $\{A_{ij} : j < 2^{n_i}\}$ be any enumeration of the set of all unions of subfamilies of the family of cosets of M_i. Let T be the spherically homogeneous tree such that for every $i > 1$, any point at level $i-1$ has valency $2^{n_i}+1$ (the root has valency 2^{n_1}). Define an isometric action of G on T as follows. Let $g \in G$. For $s, s' \in T(n)$ put $g(s) = s'$ iff $(\forall i \leq n)(gA_{is(i)} = A_{is'(i)})$.

We now want to define a G-embedding of $(\mathcal{F}(G))^{\mathbb{N}}$ into $B(T)$. First, to every $F \in \mathcal{F}(G)$ and $i \in \mathbb{N}$, we assign a natural number $j_{iF} < 2^{n_i}$ such that $F \subseteq A_{ij_{iF}}$ and $(\forall j < 2^{n_i})(F \subseteq A_{ij} \Rightarrow A_{ij_{iF}} \subseteq A_{ij})$. Also fix some $f : \mathbb{N} \to \mathbb{N}$ such that for every natural i, we have $i + 1 > f(i+1)$ and the preimage $f^{-1}[i]$ is infinite.

We define an embedding $\pi : (\mathcal{F}(G))^{\mathbb{N}} \to B(T)$ as follows. For every $(F_i)_{i\in\omega} \in (\mathcal{F}(G))^{\mathbb{N}}$, we put

$$\pi((F_0, F_1, \ldots, F_i, \ldots)) = \alpha \ \text{ iff } \ \alpha \in B(T) \text{ and } (\forall i \in \mathbb{N})(\alpha(i) = j_{iF_{f(i)}}).$$

It is clear that π is injective. By a straightforward argument we see that for every $(F_i)_{i\in\omega} \in (\mathcal{F}(G))^{\mathbb{N}}$ and $g \in G$,

$$\pi(g(F_0, \ldots, F_i, \ldots)) = g(\pi(F_0, \ldots, F_i, \ldots)).$$

To prove π is a Borel map consider preimages of basic open sets of the form $(j_1 j_2 \ldots j_i)$, where $j_1 j_2 \ldots j_i \in \prod_{l\leq i}\{0, 1, \ldots, 2^{n_l}-1\}$ for some natural number i. We have

$$\pi^{-1}[(j_1 j_2 \ldots j_i)] = \{(F_i)_{i\in\omega} \in (\mathcal{F}(G))^{\mathbb{N}} : (\forall l \leq i)((F_{f(l)} \subseteq A_{lj_l}) \wedge (\forall k < 2^{n_l})(\emptyset \neq A_{lk} \subseteq A_{l_{j_l}} \Rightarrow F_{f(l)} \cap A_{lk} \neq \emptyset))\},$$

which is a Borel subset of $(\mathcal{F}(G))^{\mathbb{N}}$. ■

Our final observation does not concern *closed* subgroups of $\mathrm{Iso}(B)$. It reveals a variety of different actions of countable subgroups of $\mathrm{Iso}(T)$ on the space $B(T)$. We transfer the example of S. Thomas of two incomparable actions of the same countable group to our context. We need some more terminology.

Given two Borel equivalence relations E_1, E_2 on X_1 and X_2 respectively, we say that E_1 is *Borel reducible* to E_2 if there is a Borel measurable function $f : X_1 \to X_2$ such that $x\, E_1\, y \Leftrightarrow f(x)\, E_2\, f(y)$, for all $x, y \in X_1$. We say that E_1 and E_2 are *incomparable* if neither E_1 is reducible to E_2, nor E_2 is reducible to E_1.

Let $n \geq 3$ be some fixed odd integer, $J \subseteq \mathbb{P}$ be a non-empty subset of primes and let $\{p_1, p_2, \ldots, p_i, \ldots\}$ be the increasing enumeration of J. Put

$$K(J) = \prod_{i \in \mathbb{N}} \mathrm{SL}_n(\mathbb{Z}_{p_i}),$$

where $\mathbb{Z}_p$ is the ring of p-adic integers. The group $\mathrm{SL}_n(\mathbb{Z})$ can be regarded as a subgroup of $K(J)$ via the diagonal embedding. Then it naturally acts on $K(J)$ via left translations. Let E_J denote the orbit equivalence relation arising from that action. In [9] S. Thomas has proved the following theorem.

Let $J_1 \neq J_2$ be two distinct non-empty subsets of primes. Then E_{J_1} and E_{J_2} are incomparable Borel equivalence relations.

Observe that $\mathrm{SL}_n(\mathbb{Z}_p)$ is a profinite group with respect to the canonical maps $\pi_r : \mathrm{SL}_n(\mathbb{Z}_p) \to \mathrm{SL}_n(\mathbb{Z}_p/p^r\mathbb{Z}_p)$, $r > 0$, determined by applying the quotient maps $\mathbb{Z}_p \to \mathbb{Z}_p/p^r\mathbb{Z}_p$ to each matrix entry (see [10] for details). The profinite topology on $\mathrm{SL}_n(\mathbb{Z}_p)$ is given by the family of cosets of open normal subgroups

$$K^p_r = \mathrm{Ker}(\pi_r) = \{g \in \mathrm{SL}_n(\mathbb{Z}_p) : g - 1 \in p^r\mathrm{SL}_n(\mathbb{Z}_p)\}, \qquad r > 0.$$

Then also $K(J)$ is a profinite group endowed with a sequence $K(J) = M_0 > M_1 > \cdots > M_i > \cdots$ of open normal subgroups of the form

$$M_i = K^{p_1}_i \times \cdots \times K^{p_i}_i \times \mathrm{SL}_n(\mathbb{Z}_{p_{i+1}}) \times \mathrm{SL}_n(\mathbb{Z}_{p_{i+2}}) \times \cdots, \qquad i \in \mathbb{N},$$

whose cosets form a base of the topology on $K(J)$.

For every $i > 0$, let $n_i = |M_{i-1} : M_i|$ and $\{g_{ij} : j < n_i\}$ be an enumeration of some transversal of the family of all cosets of M_i in M_{i-1}. Then we have $M_{i-1} = \bigcup_{j<n_i} g_{ij} M_i$ and $K(J) = \bigcup \{g_{1j_1} g_{2j_2} \ldots g_{ij_i} M_i : j_1 < n_1, \ldots, j_i < n_i\}$.

Let T be a locally finite spherically homogeneous rooted tree such that for every $i > 1$ any vertex of level i–1 has valency n_i+1 (the root is of valency n_1). For every $x \in K(J)$, every $i \in \mathbb{N}$ and $l < i$, there is exactly one $j_l(x) < n_l$ such that $x \in g_{1j_1(x)} \ldots g_{ij_i(x)} M_i$. Hence $x = \lim_{i\to\infty} g_{1j_1(x)} \ldots g_{ij_i(x)}$. We define $\pi : K(J) \to B(T)$ by $\pi_J(x) = \lim_{i\to\infty} j_1(x) \ldots j_i(x)$.

It is easily seen that π is a homeomorphism. Moreover, for every $g \in \mathrm{SL}_n(\mathbb{Z}) < K(J)$ there is exactly one $\widehat{g} \in \mathrm{Iso}(T)$ such that $\pi_J(g(x)) =$

$\widehat{g}(\pi_J(x))$ for every $x \in K(J)$. So, the function $\sigma_J : g \to \widehat{g}$ is an isomorphic embedding of $\mathrm{SL}_n(\mathbb{Z})$ into $\mathrm{Iso}(T)$. Denote by G_J the image of σ_J. Then the equivalence relation arising from the action of $\mathrm{SL}_n(\mathbb{Z})$ on $K(J)$ is isomorphic to the equivalence relation arising from the action of G_J on $B(T)$.

Let $J_1 \neq J_2$ be any non-empty subsets of primes. Then G_{J_1} and G_{J_2} are isomorphic subgroups of $\mathrm{Iso}(T_1)$ and $\mathrm{Iso}(T_2)$ respectively. According to S. Thomas, the corresponding equivalence relations on $B(T_1)$ and $B(T_2)$ are incomparable.

Thus, we have obtained the following variant of Thomas' theorem.

PROPOSITION 14. *There are locally finite rooted trees T_1 and T_2 and two isomorphic finitely generated subgroups $G_1 < \mathrm{Iso}(T_1)$, $G_2 < \mathrm{Iso}(T_2)$ such that the orbit equivalence relations E_1 and E_2 arising from the isometry actions of these groups on $B(T_i)$ are incomparable with respect to Borel reducibility.*

REFERENCES

[1] H. Bass, M. Otero-Espinar, M. Rockmore and C. P. L. Tresser, *Cyclic Renormalization and the Automorphism Groups of Rooted Trees*, Springer, Berlin, 1995.

[2] H. Becker and A. S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, Cambridge Univ. Press, 1996.

[3] J. W. Cannon, W. J. Floyd and W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) 42 (1996), 215–256.

[4] R. Grigorchuk, *Just infinite branch groups*, in: New Horizons in Pro-p Groups, M. du Sautoy *et al.* (eds.), Birkhäuser, Boston, 2000, 121–180.

[5] R. Grigorchuk, V. Nekrashevich and V. Sushchanskiĭ, *Automata, dynamical systems, and groups*, Proc. Steklov Inst. Math. 231 (2000), 128–203.

[6] A. Kechris, *Countable sections for locally compact group actions II*, Proc. Amer. Math. Soc. 120 (1994), 241–247.

[7] S. Solecki, *Actions of non-compact and non-locally compact Polish groups*, J. Symbolic Logic 65 (2000), 1881–1894.

[8] S. M. Srivastava, *A Course on Borel Sets*, Springer, New York, 1998.

[9] S. Thomas, *Superrigidity and countable Borel equivalence relations*, Ann. Pure Appl. Logic 120 (2003), 237–262.

[10] J. S. Wilson, *Profinite Groups*, Oxford Univ. Press, Oxford, 1998.

[11] —, *On just infinite abstract and profinite groups*, in: New Horizons in Pro-p Groups, M. du Sautoy *et al.* (eds.), Birkhaüser, Boston, 2000, 181–204.

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