FINITE GROUPS OF OTP PROJECTIVE REPRESENTATION TYPE

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Abstract. Let $K$ be a field of characteristic $p > 0$, $K^*$ the multiplicative group of $K$ and $G = G_p \times B$ a finite group, where $G_p$ is a $p$-group and $B$ is a $p'$-group. Denote by $K^\lambda G$ a twisted group algebra of $G$ over $K$ with a 2-cocycle $\lambda \in Z^2(G, K^*)$. We give necessary and sufficient conditions for $G$ to be of OTP projective $K$-representation type, in the sense that there exists a cocycle $\lambda \in Z^2(G, K^*)$ such that every indecomposable $K^\lambda G$-module is isomorphic to the outer tensor product $V \# W$ of an indecomposable $K^\lambda G_p$-module $V$ and a simple $K^\lambda B$-module $W$. We also exhibit finite groups $G = G_p \times B$ such that, for any $\lambda \in Z^2(G, K^*)$, every indecomposable $K^\lambda G$-module satisfies this condition.

0. Introduction. Let $K$ be a field of characteristic $p > 0$ and $G = G_p \times B$ a finite group, where $G_p$ is a Sylow $p$-subgroup and $|G_p| > 1, |B| > 1$. Given $\mu \in Z^2(G_p, K^*)$ and $\nu \in Z^2(B, K^*)$, the map $\mu \times \nu : G \times G \to K^*$ defined by

$$\left(\mu \times \nu\right)_{x_1 b_1, x_2 b_2} = \mu_{x_1, x_2} \cdot \nu_{b_1, b_2},$$

for all $x_1, x_2 \in G_p$, $b_1, b_2 \in B$, belongs to $Z^2(G, K^*)$. Every cocycle $\lambda \in Z^2(G, K^*)$ is cohomologous to $\mu \times \nu$, where $\mu$ is the restriction of $\lambda$ to $G_p \times G_p$ and $\nu$ is the restriction of $\lambda$ to $B \times B$.

From now on, we suppose that each cocycle $\lambda \in Z^2(G, K^*)$ under consideration satisfies the condition $\lambda = \mu \times \nu$, and all $K^\lambda G$-modules are assumed to be left and finite-dimensional (as vector spaces over $K$).

Let $\lambda = \mu \times \nu \in Z^2(G, K^*)$ and $\{u_g : g \in G\}$ be a canonical $K$-basis of $K^\lambda G$. Then $\{u_h : h \in G_p\}$ is a canonical $K$-basis of $K^\mu G_p$ and $\{u_b : b \in B\}$ is a canonical $K$-basis of $K^\nu B$. Moreover, if $g = hb$, where $g \in G$, $h \in G_p$, $b \in B$, then $u_g = u_h u_b = u_h u_h$. It follows that $K^\lambda G \cong K^\mu G_p \otimes_K K^\nu B$.

Given a $K^\mu G_p$-module $V$ and a $K^\nu B$-module $W$, we denote by $V \# W$ the $K^\lambda G$-module whose underlying vector space is $V \otimes_K W$ with the $K^\lambda G$-module structure given by

$$u_{hb}(v \otimes w) = u_h v \otimes u_b w,$$

for all $h \in G_p$, $b \in B$, $v \in V$, $w \in W$, and extended to $K^\lambda G$ and $V \otimes_K W$.

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by $K$-linearity. The module $V \# W$ is called the *outer tensor product* of $V$ and $W$ (see [21, p. 122]).

We recall from [7, p. 10] the following definitions.

(a) The algebra $K^\lambda G$ is defined to be of *OTP representation type* if every indecomposable $K^\lambda G$-module is isomorphic to the outer tensor product $V \# W$, where $V$ is an indecomposable $K^\mu G_p$-module and $W$ is a simple $K^\nu B$-module.

(b) A group $G = G_p \times B$ is defined to be of *OTP projective $K$-representation type* if there exists a cocycle $\lambda \in Z^2(G, K^*)$ such that the algebra $K^\lambda G$ is of OTP representation type.

(c) A group $G = G_p \times B$ is said to be of *purely OTP projective $K$-representation type* if $K^\lambda G$ is of OTP representation type for any $\lambda \in Z^2(G, K^*)$.

In [13] Brauer and Feit proved that if $K$ is algebraically closed, then the group algebra $KG$ is of OTP representation type. Blau [10] and Gudyvok [17, 18] have independently shown that if $K$ is an arbitrary field, then $KG$ is of OTP representation type if and only if $G_p$ is cyclic or $K$ is a splitting field for $B$. Gudyvok [19, 20] also investigated a similar problem for group rings $SG$, where $S$ is a complete discrete valuation ring. In [3, 6], the results of Blau and Gudyvok are generalized to the twisted group rings $S^\lambda G$, where $G = G_p \times B$, $S = K$ or $S$ is a complete discrete valuation ring of characteristic $p > 0$. Let $S = K[[X]]$ be the ring of formal power series in the indeterminate $X$ with coefficients in the field $K$. In [7], necessary and sufficient conditions on $G$ and $K$ are given for $G$ to be of OTP projective $S$-representation type and of purely OTP projective $S$-representation type.

In the present work we determine finite groups $G = G_p \times B$ of OTP projective $K$-representation type and of purely OTP projective $K$-representation type.

Denote by $l_B$ the product of all pairwise distinct prime divisors of $|B|$. Unless stated otherwise, we assume that if $G_p$ is non-abelian, then $[K(\varepsilon): K]$ is not divisible by $p$, where $\varepsilon$ is a primitive $l_B$th root of 1. This condition is satisfied if $K$ contains a primitive $q$th root of 1 for every prime $q$ dividing $|B|$ such that the characteristic $p$ divides $q - 1$. For simplicity of presentation, we set

\[
i(K) = \begin{cases} 
t & \text{if } [K: K^p] = p^t, \\
\infty & \text{if } [K: K^p] = \infty.\end{cases}\]

Let $s$ be the number of invariants of the abelian group $G_p/G'_p$, and $D$ the subgroup of $G_p$ such that $G'_p \subset D$ and $D/G'_p = \text{soc}(G_p/G'_p)$. Suppose that if $p \neq 2$, $s = i(K) + 1$, $G'_p$ is cyclic and $D$ is a non-abelian group of exponent $p$, then $|D: Z(D)| = p^2$, where $Z(D)$ is the center of $D$. We prove in Theorem 3.1 that the group $G = G_p \times B$ is of OTP projective
$K$-representation type if and only if one of the following three conditions is satisfied:

(i) $s \leq i(K)$ and $G_p'$ is cyclic;
(ii) $s = i(K) + 1$, $G_p'$ is cyclic and there exists a cyclic subgroup $T$ of $G_p$ such that $G_p' \subset T$ and $G_p/T$ has $i(K)$ invariants;
(iii) $K$ is a splitting field for $K^\nu B$ for some $\nu \in Z^2(B, K^*)$.

We also prove in Proposition 3.6 that if $G = G_p \times B$ is abelian, then $G$ is of OTP projective $K$-representation type if and only if one of the following conditions is satisfied:

(i) $s \leq i(K) + 1$;
(ii) $B$ has a subgroup $H$ such that $B/H$ is of symmetric type and $K$ contains a primitive $m$th root of 1, where $m = \max\{\exp(B/H), \exp H\}$.

Now suppose that $K$ is an arbitrary field of characteristic $p$. We establish in Proposition 3.11 that if every prime divisor of $|B'|$ is also a divisor of $|B : B'|$, then $G = G_p \times B$ is of purely OTP projective $K$-representation type if and only if either $G_p$ is cyclic, or $K = K^q$ and $K$ contains a primitive $q$th root of 1, for each prime $q$ dividing $|B|$.

In the general case, a finite group $G = G_p \times B$ is of purely OTP projective $K$-representation type if and only if either $G_p$ is cyclic, or there exists a finite central group extension $1 \to A \to \hat{B} \to B \to 1$ such that any projective $K$-representation of $B$ lifts projectively to an ordinary $K$-representation of $\hat{B}$ and $K$ is a splitting field for $\hat{B}$ (Theorem 3.12).

Let $t(K^*)$ denote the torsion subgroup of the multiplicative group $K^*$ of $K$. Assume that either $t(K^*) = t(K^*)^q$ for every prime $q$ dividing $|B'|$, or every prime divisor of $|B'|$ is also a divisor of $|B : B'|$. Then $G$ is of purely OTP projective $K$-representation type if and only if either $G_p$ is cyclic, or there exists a covering group $\hat{B}$ of $B$ over $K$ such that $K$ is a splitting field for $\hat{B}$ (Proposition 3.13).

1. Preliminaries. Throughout the paper, we use the standard group representation theory notation and terminology introduced in the monographs by Alperin [1], Benson [9], Curtis and Reiner [14], and Karpilovsky [21, 22]. The books by Karpilovsky give a systematic account of the projective representation theory. For classical problems and solutions of group representation theory, we refer to [1, 9, 14] and to the old and nice papers [11, 12]. A background of the representation theory of finite-dimensional algebras can be found in the monographs by Assem, Simson and Skowroński [2], Drozd and Kirichenko [16], Simson [23], and Simson and Skowroński [24], where among other things the representation types (finite, tame, wild) of finite groups and algebras are discussed.
In particular, we use the following notation: $p \geq 2$ is a prime; $K$ is a field of characteristic $p$, $K^q = \{ \alpha^q : \alpha \in K \}$; $K^*$ is the multiplicative group of $K$; $t(K^*)$ is the torsion subgroup of $K^*$; $o(\xi)$ is the order of $\xi \in t(K^*)$; $G = G_p \times B$ is a finite group, where $G_p$ is a $p$-group, $B$ is a $p'$-group, $|G_p| > 1$ and $|B| > 1$; $H'$ is the commutant of a group $H$, $Z(H)$ is the center of $H$, $e$ is the identity element of $H$, $|h|$ is the order of $h \in H$ and $\exp H$ is the exponent of $H$; $\soc A$ is the socle of an abelian group $A$. Let $l_B$ be the product of all pairwise distinct prime divisors of $|B|$. Unless stated otherwise, we assume that if $G_p$ is non-abelian, then $[K(\varepsilon) : K]$ is not divisible by $p$, where $\varepsilon$ is a primitive $l_B$th root of $1$. It is not difficult to see that $[K(\varepsilon) : K]$ is not divisible by $p$ if and only if $[K(\xi) : K]$ is not divisible by $p$, where $\xi$ is a primitive $(\exp B)$th root of $1$. Given $\lambda \in Z^2(H, K^*)$, $K^\lambda H$ denotes the twisted group algebra of a group $H$ over $K$ with a 2-cocycle $\lambda$, and $\rad K^\lambda H$ the radical of $K^\lambda H$. A $K$-basis $\{u_h : h \in H\}$ of $K^\lambda H$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in H$ is called canonical (corresponding to $\lambda$). If $D$ is a subgroup of a group $H$, the restriction of $\lambda \in Z^2(H, K^*)$ to $D \times D$ is also denoted by $\lambda$. In this case, $K^\lambda D$ is a subalgebra of $K^\lambda H$.

Throughout this paper we assume that all cocycle groups are defined with respect to the trivial action of the underlying group on $K^*$. By Theorem 4.7 in [21, p. 40], the embedding $t(K^*) \rightarrow K^*$ induces an injective homomorphism

$$H^2(B, t(K^*)) \rightarrow H^2(B, K^*).$$

We shall identify $H^2(B, t(K^*))$ with the subgroup of $H^2(B, K^*)$ which consists of all cohomology classes containing cocycles of finite order.

Given $\mu \in Z^2(G_p, K^*)$, the kernel $\ker(\mu)$ of $\mu$ is the union of all cyclic subgroups $\langle g \rangle$ of $G_p$ such that the restriction of $\mu$ to $\langle g \rangle \times \langle g \rangle$ is a coboundary. We recall from [41, p. 196] that $G'_p \subset \ker(\mu)$, $\ker(\mu)$ is a normal subgroup of $G_p$ and the restriction of $\mu$ to $\ker(\mu) \times \ker(\mu)$ is a coboundary.

Let $M$ be a finite group, $N$ a normal subgroup of $M$ and $T = M/N$. Given $\mu \in Z^2(T, K^*)$, denote by $\inf(\mu)$ (see [21, p. 14]) the element of $Z^2(M, K^*)$ defined by

$$\inf(\mu)_{a,b} = \mu_{a N, b N} \text{ for all } a, b \in M.$$ 

We have $\inf(\mu)_{x,y} = 1$ for all $x, y \in N$. Therefore

$$K^{\inf(\mu)} N = KN.$$ 

Let $\lambda = \inf(\mu)$, $\{v_{a N} : a \in M\}$ be a canonical $K$-basis of $K^\mu T$ corresponding to $\mu$, and $\{u_a : a \in M\}$ a canonical $K$-basis of $K^\lambda M$ corresponding to $\lambda$. The formula

$$f\left(\sum_{a \in M} \alpha_a u_a\right) = \sum_{a \in M} \alpha_a v_{a N}$$

defines a $K$-algebra epimorphism $f : K^\lambda M \rightarrow K^\mu T$ with the kernel $U :=$
$K^\lambda M \cdot I(N)$, where $I(N)$ is the augmentation ideal of the group algebra $KN$ (see [21] p. 88). Hence $K^\lambda M/U \cong K^\mu T$. We recall that

$$I(N) = \bigoplus_{x \in N \setminus \{e\}} K(u_x - u_e).$$

Assume that $N$ and $M$ are groups. An extension of $N$ by $M$ is a short exact sequence of groups

$$E : 1 \xrightarrow{\varphi} N \rightarrow \hat{M} \rightarrow M \rightarrow 1.$$

If $\varphi(N)$ is contained in the center of $\hat{M}$, then $E$ is called a central extension. If $N$ and $M$ are finite groups, then $E$ is a finite extension.

Let $V$ be a finite-dimensional vector space over $K$, $\text{GL}(V)$ the group of all automorphisms of $V$, $1_V$ the identity automorphism of $V$, $M$ a finite group, and let

$$1 \rightarrow N \rightarrow \hat{M} \xrightarrow{\psi} M \rightarrow 1$$

be a finite central group extension. Denote by $\pi : \text{GL}(V) \rightarrow \text{GL}(V)/K^*1_V$ the canonical group epimorphism. Assume that $\Gamma$ is an ordinary $K$-representation of $\hat{M}$ in $V$ with $\Gamma(x) \in K^*1_V$ for any $x \in N$. There exists a projective $K$-representation $\Delta$ of $M$ in $V$ such that the diagram

$$\begin{array}{cccc}
\hat{M} & \xrightarrow{\Gamma} & \text{GL}(V) & \xrightarrow{\pi} & \text{GL}(V)/K^*1_V \\
\psi & & & & \\
M & \xrightarrow{\Delta} & \text{GL}(V) & \xrightarrow{\pi} & \text{GL}(V)/K^*1_V
\end{array}$$

is commutative. We say that $\Delta$ lifts projectively to the ordinary $K$-representation $\Gamma$ of $\hat{M}$. If $|N| = |H^2(M, K^*)|$ and any projective $K$-representation of $M$ lifts projectively to an ordinary $K$-representation of $\hat{M}$, then $\hat{M}$ is called a covering group of $M$ over $K$ [21] p. 138].

We recall that, for any cocycle $\lambda \in Z^2(G_p, K^*)$, the quotient algebra $K^\lambda G_p/\text{rad} K^\lambda G_p$ is $K$-isomorphic to a field that is a finite purely inseparable field extension of $K$ [21] p. 74]. We call $K^\lambda G_p$ uniserial if the left regular and the right regular $K^\lambda G_p$-modules have a unique composition series. It should be noted that some authors use the terminology “uniserial algebra” to mean principal ideal algebras [16] p. 171] and serial algebras (see [15] p. 505] and [16] p. 175]) that are Nakayama algebras [2] p. 168]. By the Morita theorem in [15] p. 507], the algebra $K^\lambda G_p$ is uniserial if and only if $\text{rad} K^\lambda G_p = K^\lambda G_p \cdot v = v \cdot K^\lambda G_p$ for some $v \in K^\lambda G_p$. By [16] p. 170], the algebra $K^\lambda G_p$ is uniserial if and only if $\text{rad} K^\lambda G_p$ is a principal left (equivalently, right) ideal of $K^\lambda G_p$.

We say that an algebra $K^\lambda G_p$ satisfies the $Q$-condition if there exists a $K$-algebra epimorphism $K^\lambda G_p \rightarrow K^\mu T$, where $T$ is a $p$-group and $T$ contains an abelian subgroup $A$ such that $K^\mu A$ is not a uniserial algebra.
The following four facts are proved in [6].

**Lemma 1.1.** Let $K$ be an arbitrary field of characteristic $p$, $G = G_p \times B$, $\mu \in \mathbb{Z}^2(G_p, K*)$, $\nu \in \mathbb{Z}^2(B, K*)$ and $\lambda = \mu \times \nu$. If $K^\mu G_p$ is a uniserial algebra or $K$ is a splitting field for $K^\nu B$, then $K^\lambda G$ is of OTP representation type.

**Lemma 1.2.** Let $G = G_p \times B$, $\mu \in \mathbb{Z}^2(G_p, K*)$, $\nu \in \mathbb{Z}^2(B, K*)$, $\lambda = \mu \times \nu$ and assume that $K^\mu G_p$ satisfies the $Q$-condition. The algebra $K^\lambda G$ is of OTP representation type if and only if either $K^\mu G_p$ is uniserial, or $K$ is a splitting field for $K^\nu B$.

**Theorem 1.3.** Let $G = G_p \times B$, $\mu \in \mathbb{Z}^2(G_p, K*)$, $\nu \in \mathbb{Z}^2(B, K*)$, $\lambda = \mu \times \nu$ and let $d = \dim_K (K^\mu G_p / \text{rad} \, K^\mu G_p)$. Denote by $D$ the subgroup of $G_p$ such that $G_p' \subset D$ and $D/G_p' = \text{soc}(G_p/G_p')$. Assume that if $K^\mu G_p$ is not uniserial, $\text{pd} = |G_p : G_p'|$ and $|G_p'| = p$, then $\text{Ker}(\mu) \neq G_p'$ or $|D : Z(D)| \in \{1, p^2\}$. The $K$-algebra $K^\lambda G$ is of OTP representation type if and only if either $K^\mu G_p$ is uniserial, or $K$ is a splitting field for $K^\nu B$.

**Proposition 1.4.** Let $K$ be an arbitrary field of characteristic $p$, $G = G_p \times B$, $\nu \in \mathbb{Z}^2(B, K*)$ and $K^\lambda G = KG_p \otimes_K K^\nu B$. The $K$-algebra $K^\lambda G$ is of OTP representation type if and only if either $G_p$ is cyclic, or $K$ is a splitting field for $K^\nu B$.

2. On splitting fields for twisted group algebras. We say that an abelian group is of symmetric type if it can be decomposed into a direct product of two isomorphic subgroups.

Let $G$ be an abelian group, $F$ an arbitrary field, $\lambda \in \mathbb{Z}^2(G, F*)$, $\{u_g : g \in G\}$ a canonical $F$-basis of $F^\lambda G$ corresponding to $\lambda$, $Z$ the center of $F^\lambda G$ and $H = \{h \in G : u_h \in Z\}$. Then $H$ is a subgroup of $G$ and $Z = F^\lambda H$. Obviously

$$H = \{h \in G : \lambda_{h,g} = \lambda_{g,h} \text{ for any } g \in G\}.$$  

We call $H$ the $\lambda$-center of $G$.

**Proposition 2.1.** Let $G$ be abelian, $\lambda \in \mathbb{Z}^2(G, F*)$, $H$ the $\lambda$-center of $G$, $\overline{G} = G/H$ and $\overline{x} = xH$ for any $x \in G$. Assume that $G \neq H$.

(i) The algebra $F^\lambda G$ may be viewed as a twisted group ring $Z^\lambda \overline{G}$ of $\overline{G}$ over the ring $Z = F^\lambda H$. Moreover

$$\overline{\lambda}_{\overline{x}, \overline{y}} \cdot \overline{\lambda}_{\overline{y}, \overline{x}}^{-1} \in t(F*) \quad \text{for all } x, y \in G.$$  

(ii) There exists a direct product decomposition $\overline{G} = \overline{C}_1 \times \cdots \times \overline{C}_s$ such that $\overline{C}_i = \langle \overline{a}_i \rangle \times \langle \overline{b}_i \rangle$ is a $q_i$-group of type $(q_i^{n_1}, q_i^{n_2})$,

$$Z^\lambda \overline{G} \cong Z^\lambda \overline{C}_1 \otimes Z \cdots \otimes Z^\lambda \overline{C}_s$$  

and
such that $v_i$-basis $2q_i$, where $\bar{y}_i \subseteq \{ x \in t(F^*) \text{ and } o(x_i) = q_i^{n_i} \text{ for every } i \in \{0, \ldots, s\}$.  

(iii) $\bar{G}$ is a group of symmetric type and $F$ contains a primitive mth root of 1, where $m = \exp \bar{G}$.

Proof. Let $\{g_1, \ldots, g_r\}$ be a cross section of $H$ in $G$ and $g_1 = e$. Then

$$F^G = Zu_{g_1} \oplus \cdots \oplus Zu_{g_r}.$$  

Put $v_{g_i} = u_{g_i}$ for every $i \in \{1, \ldots, r\}$. The algebra $F^G$ may be viewed as a twisted group ring $\mathbb{Z}^\lambda \bar{G}$ of the group $\bar{G}$ over the ring $\mathbb{Z}$ with a canonical $\mathbb{Z}$-basis $v_{g_1}, \ldots, v_{g_r}$. For any $x, y \in \bar{G}$ we have $vu_{g_i}v_{g_j} = \xi_{g_i}v_{g_j}v_{g_j}$, where $\xi \in t(F^*)$.

The ring $\mathbb{Z}$ is the center of $\mathbb{Z}^\lambda \bar{G}$. We also have

$$\mathbb{Z}^\lambda \bar{G} \cong \mathbb{Z}^\lambda \bar{G}_{q_1} \otimes \mathbb{Z} \cdots \otimes \mathbb{Z}^\lambda \bar{G}_{q_k},$$  

where $\bar{G}_{q_i}$ is the Sylow $q_i$-subgroup of $\bar{G}$ for each $i \in \{1, \ldots, k\}$.

Let $q$ be a prime and $\bar{G}_q = \langle \bar{x}_1 \rangle \times \cdots \times \langle \bar{x}_t \rangle$ be a Sylow $q$-subgroup of $\bar{G}$. Assume that $|\bar{x}_j| = q^{m_j}$ and $m_1 \geq \cdots \geq m_t$. The set

$$\{v_{\bar{x}_1}^{k_1} \cdots v_{\bar{x}_t}^{k_t} : k_i = 0, 1, \ldots, q^{m_i} - 1 \text{ for every } i \in \{1, \ldots, t\} \}$$  

is a $\mathbb{Z}$-basis of the algebra $\mathbb{Z}^\lambda \bar{G}_q$. We have

$$v_{\bar{x}_1}v_{\bar{x}_2} = \xi_{v_{\bar{x}_1}v_{\bar{x}_2}}, \quad v_{\bar{x}_1}v_{\bar{x}_2} = \xi_{v_{\bar{x}_1}v_{\bar{x}_2}},$$  

for any $j \in \{2, \ldots, t\}$, where $\xi_j \in F^*$ and $o(\xi_j) \leq q^{m_j}$. If $m_1 > m_2$, then $v_{\bar{x}_1}^{q_2^{m_2}} \neq v_{\bar{x}_1}^{q_2^{m_2}}$ and $\bar{x}_1^{\gamma_{ij}}$ belongs to the center of $\mathbb{Z}^\lambda \bar{G}$. Hence there exists an $\bar{x}_{j_0}$ such that $|\bar{x}_{j_0}| = q^{m_1}$ and $o(\xi_{j_0}) = q^{m_1}$. Let $j_0 = 2$ and $\xi = \xi_2$. We have

$$v_{\bar{x}_1}v_{\bar{x}_2} = \xi_{v_{\bar{x}_1}v_{\bar{x}_2}}, \quad v_{\bar{x}_1}v_{\bar{x}_2} = \xi_{v_{\bar{x}_1}v_{\bar{x}_2}},$$  

for all $i, j$, where $0 \leq \gamma_{ij} < q^{m_1}$ and $o(\xi_{\gamma_{ij}}) \leq \max\{|\bar{x}_i|, |\bar{x}_j|\}$ for all $i, j \in \{1, \ldots, t\}$.

Let $w_{\bar{y}_1} = \bar{x}_1$, $w_{\bar{y}_2} = \bar{x}_2$, $w_{\bar{y}_3} = \bar{x}_1^{\alpha_{31}}\bar{x}_2^{\alpha_{32}}\bar{x}_3$, $\ldots$, $\bar{y}_t = \bar{x}_1^{\alpha_{t1}}\bar{x}_2^{\alpha_{t2}}\bar{x}_t$ and

$$w_{\bar{y}_1} = v_{\bar{x}_1}, \quad w_{\bar{y}_2} = v_{\bar{x}_2}, \quad w_{\bar{y}_3} = v_{\bar{x}_1}^{\alpha_{31}}v_{\bar{x}_2}^{\alpha_{32}}v_{\bar{x}_3}, \ldots, \quad w_{\bar{y}_t} = v_{\bar{x}_1}^{\alpha_{t1}}v_{\bar{x}_2}^{\alpha_{t2}}v_{\bar{x}_t},$$  

where

$$\alpha_{j1} = \gamma_{2j}, \quad \alpha_{j2} = q^{m_1} - \gamma_{1j}$$  

for every $j \in \{3, \ldots, t\}$. Then

$$w_{\bar{y}_1}w_{\bar{y}_j} = w_{\bar{y}_j}w_{\bar{y}_1}, \quad w_{\bar{y}_2}w_{\bar{y}_j} = w_{\bar{y}_j}w_{\bar{y}_2}$$  

for every $j \in \{3, \ldots, t\}$, and $\bar{G}_q = \langle \bar{y}_1 \rangle \times \cdots \times \langle \bar{y}_t \rangle$. Therefore

$$\mathbb{Z}^\lambda \bar{G}_q \cong \mathbb{Z}^\lambda \bar{G}_q^{(1)} \otimes \mathbb{Z}^\lambda \bar{G}_q^{(2)}.$$
where $\mathcal{G}_q^{(1)} = \langle \tilde{y}_1 \rangle \times \langle \tilde{y}_2 \rangle$, $\mathcal{G}_q^{(2)} = \langle \tilde{y}_3 \rangle \times \cdots \times \langle \tilde{y}_t \rangle$ and $Z^\lambda \mathcal{G}_q$ is $Z$-central. By induction on $t$, we conclude that
\[
Z^\lambda \mathcal{G}_q \cong Z^\lambda \mathcal{D}_1 \otimes_Z \cdots \otimes_Z Z^\lambda \mathcal{D}_{s_q},
\]
where $\mathcal{D}_j$ is a $q$-group of type $(q^{k_1}, q^{k_j})$ and $Z^\lambda \mathcal{D}_j$ is a central $Z$-algebra of the form (2.1), for any $j \in \{1, \ldots, s_q\}$.

The group $\mathcal{G}_q$ is of symmetric type. Hence $\mathcal{G}$ is a group of symmetric type. The field $K$ contains a primitive $m_q$th root of 1, where $m_q = \exp \mathcal{G}_q$. It follows that $F$ contains a primitive $m$th root of 1, where $m = \exp \mathcal{G}$. 

We note that Proposition 2.1 is a generalization of Theorem 2.12 in [22, p. 375]. From Proposition 2.1 one can also deduce Corollary 1.12 in [22, p. 368].

**Proposition 2.2.** Let $B$ be an abelian $p'$-group, $\lambda \in Z^2(B, K^*)$, $H$ the $\lambda$-center of $B$ and $\overline{B} = B/H$. Assume that $K$ is a splitting field for $K^\lambda B$.

(i) The field $K$ contains a primitive $(\exp H)$th root of 1, and there exists $\mu \in Z^2(B, K^*)$ such that $\lambda$ is cohomologous to $\inf(\mu)$.

(ii) The algebra $K^\lambda B$ is $K$-algebra isomorphic to $K^{\mu_1} \overline{B} \times \cdots \times K^{\mu_l} \overline{B}$, where $l = |H|$, $\mu_1 = \mu$ and $K^{\mu_i} \overline{B}$ is $K$-algebra isomorphic to $M_n(K)$, $n^2 = |\overline{B}|$, for every $i \in \{1, \ldots, l\}$.

**Proof.** (i) $K$ is a splitting field for $Z = K^\lambda H$. It follows that the restriction of $\lambda$ to $H \times H$ is a coboundary and $K$ contains a primitive $(\exp H)$th root of 1. The algebra $K^\lambda H$ is isomorphic to $KH$. We may assume that $K^\lambda H = KH$. Denote by $I(H)$ the augmentation ideal of $KH$. By Lemma 5.5 in [21, p. 91], $K^\lambda B/K^\lambda B \cdot I(H) \cong K^{\mu_i} \overline{B}$ for some $\mu \in Z^2(B, K^*)$ such that $\lambda$ is cohomologous to $\inf(\mu)$.

(ii) Let $l = |H|$, $e_1, \ldots, e_l$ be a complete system of primitive pairwise orthogonal idempotents of $Z$ and $u_h e_1 = e_1$ for any $h \in H$. Then $Ze_i$ is $K$-algebra isomorphic to $K$ and, by Proposition 2.1, $K^\lambda Be_i \cong (Ze_i)^{\sigma_i} \overline{B} \cong K^{\mu_i} \overline{B}$ for every $i \in \{1, \ldots, l\}$. Moreover, $K^{\mu_i} \overline{B} \cong K^{\mu_i} \overline{B}$ is a central $K$-algebra and $K$ is a splitting field for $K^{\mu_i} \overline{B}$ for each $i$. Hence $K^{\mu_i} \overline{B}$ is $K$-algebra isomorphic to $M_n(K)$, $n^2 = |\overline{B}|$, for every $i \in \{1, \ldots, l\}$. 

**Lemma 2.3.** Let $B$ be an abelian $p'$-group of symmetric type. Assume that the field $K$ contains a primitive $(\exp B)$th root of 1. Then there exists a cocycle $\mu \in Z^2(B, t(K^*))$ such that $K^{\mu} B \cong M_n(K)$, where $n^2 = |B|$.

**Proof.** We may suppose that $B$ is an abelian $q$-group of type $(q^r, q^s)$, where $q \neq p$. Let $\xi$ be a primitive $q^r$th root of 1, $F$ a finite subfield of $K$
which contains $\xi$, $B = \langle x \rangle \times \langle y \rangle$ and

$$F^\mu B = \bigoplus_{i,j=0}^{q^r-1} F u_x^i u_y^j, \quad u_x^q = u_e, \quad u_y^q = u_e, \quad u_x u_y = \xi u_y u_x.$$ 

The $F$-algebra $F^\mu B$ is central. Since a finite division algebra is a field, $F^\mu B$ is $F$-algebra isomorphic to $\mathbb{M}_n(F)$, where $n = q^r$. It follows that the $K$-algebra $K^\mu B := K \otimes_F F^\mu B$ is $K$-isomorphic to $\mathbb{M}_n(K)$. 

**Proposition 2.4.** Assume that $B$ is an abelian $p'$-group and $H$ is a subgroup of $B$ such that $\overline{B} := B/H$ is of symmetric type and $K$ contains a primitive $m$th root of 1, where $m = \max\{\exp \overline{B}, \exp H\}$. Let $\mu \in Z^2(\overline{B}, K^*)$ and $\lambda = \inf(\mu)$.

(i) If $K^\mu \overline{B}$ is a central $K$-algebra then $K^\lambda B$ can be decomposed into a direct product of central twisted group algebras of $\overline{B}$ over $K$.

(ii) If $\mu \in Z^2(\overline{B}, t(K^*))$ and $K^\mu \overline{B}$ is a central $K$-algebra, then $K$ is a splitting field for the algebra $K^\lambda B$.

(iii) Let $K$ contain a primitive $(\exp B)$th root of 1. If $K^\mu \overline{B}$ is $K$-algebra isomorphic to $\mathbb{M}_n(K)$, where $n^2 = |\overline{B}|$, then $K^\lambda B$ is $K$-algebra isomorphic to the direct product of $l$ copies of $\mathbb{M}_n(K)$, where $l = |H|$.

**Proof.** (i) Denote by $\{v_{bH} : b \in B\}$ a canonical $K$-basis of $K^\mu \overline{B}$ corresponding to $\mu$ and by $\{u_b : b \in B\}$ a canonical $K$-basis of $K^\lambda B$ corresponding to $\lambda$.

We have $K^\lambda H = KH$. If $b \in B$ and $h \in H$ then $\lambda_{b,h} = \mu_{bH,H} = 1$ and $\lambda_{h,b} = 1$. It follows that $u_b u_h = u_h u_b$. Therefore $KH \subset Z(K^\lambda B)$. Assume that $u_g \in Z(K^\lambda B)$ for certain $g \in B$. Then $u_g u_b = u_b u_g$ for each $b \in B$. Hence $v_{gH} v_{bH} = v_{bH} v_{gH}$ for any $b \in B$. Since $K^\mu \overline{B}$ is a central $K$-algebra, $gH = H$ and consequently $Z(K^\lambda B) = KH$. This means that $H$ is the $\lambda$-center of $B$. The field $K$ is a splitting field for $KH$. It follows, by Proposition 2.1, that $K^\lambda B$ can be decomposed into a direct product of central twisted group algebras of $\overline{B}$ over $K$.

(ii) Denote by $F$ a finite subfield of $K$ which contains a primitive $m$th root of 1 and all values of the cocycle $\mu$. The algebra $F^\mu \overline{B}$ is a central $F$-algebra. By (i), $F$ is a splitting field for $F^\lambda B$, since each finite division algebra is a field. It follows that $K$ is a splitting field for the algebra $K^\lambda B \cong K \otimes_F F^\lambda B$.

(iii) By Theorem 6.1 in [25, p. 179], $K^\lambda B$ can be decomposed into a direct product of mutually isomorphic simple algebras over $K$. Since $K^\mu \overline{B}$ is a simple component of $K^\lambda B$, the algebra $K^\lambda B$ is $K$-algebra isomorphic to $K^\mu \overline{B} \times \cdots \times K^\mu \overline{B}$. 

**Proposition 2.5 ([7, p. 20]).** Let $B$ be an abelian $p'$-group. The field $K$ is a splitting field for some $K$-algebra $K^\lambda B$ if and only if $B$ has a subgroup
such that \( B/H \) is of symmetric type and \( K \) contains a primitive \( m \)th root of 1, where \( m = \max\{\exp(B/H), \exp H\} \).

Proof. Apply Propositions 2.1, 2.2, 2.4 and Lemma 2.3.

**Proposition 2.6.** Let \( K \) be a finite field of characteristic \( p \), \( B \) an abelian \( p' \)-group, \( \lambda \in Z^2(B, K^*) \) and \( H \) the \( \lambda \)-center of \( B \). The field \( K \) is a splitting field for \( K^\lambda B \) if and only if the restriction of \( \lambda \) to \( H \times H \) is a coboundary and \( K \) contains a primitive \((\exp H)\)th root of 1.

Proof. Apply Propositions 2.1 and 2.2.

**Proposition 2.7.** Let \( B \) be a nilpotent \( p' \)-group. If \( K \) is a splitting field for some twisted group algebra of \( B \) over \( K \), then \( K \) contains a primitive \( q \)th root of 1 for each prime \( q \) that divides \( |B| \).

Proof. Assume that \( K \) is a splitting field for an algebra \( K^\lambda B \) and \( K \) does not contain a primitive \( q \)th root of 1 for a certain prime \( q \) dividing \( |B| \). Denote by \( B_q \) the Sylow \( q \)-subgroup of \( B \). The center of \( B_q \) contains an element \( b \) of order \( q \). Let \( \{u_q : g \in B\} \) be a canonical \( K \)-basis of \( K^\lambda B \) corresponding to \( \lambda \). Then \( u_b \) lies in the center \( Z \) of \( K^\lambda B \). Let \( \{f_1, \ldots, f_s\} \) be a complete system of pairwise orthogonal primitive idempotents of \( Z \). We have \( u_b = \beta_1 f_1 + \ldots + \beta_s f_s \), where \( \beta_j \in K \) for every \( j \in \{1, \ldots, s\} \). If \( u^q_b = \gamma u_b \), \( \gamma \in K^* \), then \( \gamma = \beta^q_j \) for each \( j \). It follows that \( \beta_1 = \cdots = \beta_s \) and \( u_b = \beta_1 u_e \). This contradiction proves that \( K \) contains a primitive \( q \)th root of 1 for each prime \( q \) that divides \( |B| \).

**Proposition 2.8.** Let \( B \) be a \( p' \)-group.

(i) If the field \( K \) is a splitting field for all twisted group algebras of \( B \) over \( K \), then \( K = K^q \) and \( K \) contains a primitive \( q \)th root of 1 for each prime \( q \) that divides \(|B : B'| \).

(ii) If \( K = K^q \) and \( K \) contains a primitive \( q \)th root of 1 for any prime \( q \) that divides \(|B| \), then \( K \) is a splitting field for every twisted group algebra of \( B \) over \( K \).

(iii) Assume that every prime divisor of \(|B'| \) is also a divisor of \(|B : B'| \). Then \( K \) is a splitting field for any twisted group algebra of \( B \) over \( K \) if and only if \( K = K^q \) and \( K \) contains a primitive \( q \)th root of 1 for each prime \( q \) that divides \(|B| \).

Proof. (i) Let \( B \neq B' \) and \( q \) be a prime divisor of \(|B : B'| \). Denote by \( D \) a normal subgroup of \( B \) such that \(|B/D| = q \). Let \( \tilde{B} := B/D = \langle xD \rangle \), \( \alpha \in K^* \) and

\[
K^\mu \tilde{B} = \bigoplus_{i=0}^{q-1} K v_{xD}^i, \quad v_{xD}^q = \alpha v_{xD}.
\]
Denote $\lambda = \inf(\mu)$. There exists a $K$-algebra homomorphism of $K^\lambda B$ onto $K^\mu \hat{B}$. It follows that $K$ is a splitting field for $K^\mu \hat{B}$. Hence $\alpha = \beta^q$ for some $\beta \in K^*$ and $K$ contains a primitive $q$th root of 1.

(ii) Denote by $n$ the order of a cohomology class $[\lambda] \in H^2(B, K^*)$. It is well known that $n$ divides $|B|$. Arguing as in the proof of Theorem 53.3 in [14, p. 359], we show that $[\lambda]$ contains a cocycle $\alpha$ whose order is equal to $n$. By Theorem 1.3 in [21, p. 137], there exists a central group extension $1 \to A \to \hat{B} \to B \to 1$ such that $A$ is a cyclic group of order $n$ and

$$K\hat{B} \cong \prod_{i=0}^{n-1} K^{\alpha_i}B.$$ 

Since any prime divisor of $|\hat{B}|$ is also a divisor of $B$, the field $K$ contains a primitive $m$th root of 1, where $m = \exp \hat{B}$. By the Brauer theorem, $K$ is a splitting field for $K\hat{B}$. Hence $K$ is a splitting field for $K^\alpha B$.

(iii) Apply (i) and (ii). □

PROPOSITION 2.9. Let $B$ be a $p'$-group. The field $K$ is a splitting field for all twisted group algebras of $B$ over $K$ if and only if there exists a finite central group extension $1 \to A \to \hat{B} \to B \to 1$ such that any projective $K$-representation of $B$ lifts projectively to an ordinary $K$-representation of $\hat{B}$ and $K$ is a splitting field for $K\hat{B}$.

Proof. Assume that $K$ is a splitting field for all twisted group algebras of $B$ over $K$. By Proposition 2.8, $K^* = (K^*)^m$, where $m$ is the exponent of $B/B'$. In view of Corollary 2.5 in [21, p. 142], $H^2(B, K^*) = H^2(B, t(K^*))$. Arguing as in the proof of Theorem 2.3 in [21, p. 141], we conclude that there exists a finite central group extension $1 \to A \to \hat{B} \to B \to 1$ such that the following conditions hold:

(i) If $r$ is the exponent of $A$, then $K^*$ contains a primitive $r$th root of 1.

(ii) Every projective $K$-representation of $B$ lifts projectively to an ordinary $K$-representation of $\hat{B}$.

By Theorem 4.2 in [21, p. 80], $K\hat{B} \cong \prod_i K^{\lambda_i}B$. It follows that $K$ is a splitting field for $K\hat{B}$. This completes the proof of the necessity.

Let us prove the sufficiency. The group algebra $KA$ lies in the center of $K\hat{B}$, hence $K$ contains a primitive $m$th root of 1, where $m$ is the exponent of $A$. It follows, by Theorem 4.2 in [21, p. 80] and Lemma 2.1 in [21, p. 139], that $K\hat{B}$ is $K$-algebra isomorphic to $K^{\sigma_1}B \times \cdots \times K^{\sigma_r}B$ and every algebra $K^{\lambda}B$ is isomorphic to some $K^{\sigma_i}B$. Hence $K$ is a splitting field for every twisted group algebra of $B$ over $K$. □

PROPOSITION 2.10. Let $B$ be a $p'$-group. Assume that either $t(K^*) = t(K^*)^q$ for every prime $q$ that divides $|B'|$, or every prime divisor of $|B'|$ is
also a divisor of $|B : B'|$. Then $K$ is a splitting field for any twisted group algebra of $B$ over $K$ if and only if there exists a covering group $\hat{B}$ of $B$ over $K$ such that $K$ is a splitting field for $\hat{B}$.

Proof. Assume that $K$ is a splitting field for any twisted group algebra of $B$ over $K$. In view of Proposition 2.8, $K = K^q$ for each prime $q$ dividing $|B : B'|$. It follows that $t(K) = t(K^q)$ for every prime $q$ that divides $B$. Arguing as in the proof of Theorem 53.3 in [14, p. 359], we show that each cohomology class $[\lambda] \in H^2(B, t(K))$ contains a cocycle whose order is equal to the order of $[\lambda]$. In view of Theorem 2.3 in [21, p. 140], there exists a finite central group extension $1 \to A \to \hat{B} \to B \to 1$ such that $A \cong H^2(B, t(K))$ and any projective $K$-representation of $B$ lifts projectively to an ordinary $K$-representation of $\hat{B}$. By Corollary 2.5 in [21, p. 142], we have

$$H^2(B, K^*) \cong H^2(B, t(K^*))$$

Hence $\hat{B}$ is a covering group of $B$ over $K$. Theorem 4.2 in [21, p. 80] yields

$$K\hat{B} \cong \prod_i K^{\sigma_i}B,$$

since $K^*$ contains a primitive $(\exp A)$th root of $1$. It follows that $K$ is a splitting field for $\hat{B}$. This proves the necessity.

The sufficiency follows from Proposition 2.9.

We note that in [25] Yamazaki proved Theorem 4.2 from [21, p. 80] while Theorem 2.3 from [21, p. 140] and Corollary 2.5 from [21, p. 142] are proved in [26].

3. Groups of OTP projective representation type. We recall that $K$ is a field of characteristic $p$ and $G = G_p \times B$ is a finite group, where $G_p$ is a $p$-group, $B$ is a $p'$-group and $|G_p| \neq 1, |B| \neq 1$. We assume that if $G_p$ is non-abelian then $[K(\xi) : K]$ is not divisible by $p$, where $\xi$ is a primitive $(\exp B)$th root of $1$.

Theorem 3.1. Let $G = G_p \times B$, $s$ be the number of invariants of the group $G_p/G'_p$ and $D$ the subgroup of $G_p$ such that $G'_p \subset D$ and $D/G'_p = \soc(G_p/G'_p)$. Assume that if $p \neq 2$, $s = i(K) + 1$, $|G'_p| = p$ and $D$ is a non-abelian group of exponent $p$, then $|D : Z(D)| = p^2$. The group $G$ is of OTP projective $K$-representation type if and only if one of the following conditions is satisfied:

(i) $s \leq i(K)$ and $G'_p$ is cyclic;
(ii) $s = i(K) + 1$, $G'_p$ is cyclic and there exists a cyclic subgroup $T$ of $G_p$ such that $G'_p \subset T$ and $G_p/T$ has $i(K)$ invariants;
(iii) $K$ is a splitting field for some $K^\nu B$. 

Proof. Suppose (ii). Let $\tilde{G}_p = G_p/T$. There is a cocycle $\sigma \in Z^2(\tilde{G}_p, K^*)$ such that $K^\sigma \tilde{G}_p$ is a field. Let $\mu = \inf(\sigma)$. If $V := K^\mu G_p \cdot I(T)$ then $V$ is the radical of $K^\mu G_p$ and $K^\mu G_p/V$ is $K$-algebra isomorphic to $K^\sigma \tilde{G}_p$. Therefore $K^\mu G_p$ is a uniserial algebra. Hence, if $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. In view of Theorem 1.3, $K^\lambda G$ is of OTP representation type.

Arguing as in the case (ii) we prove that if (i) holds, then there exists a cocycle $\lambda \in Z^2(G, K^*)$ such that $K^\lambda G$ is of OTP representation type.

Assume that $K$ is a splitting field for some $K^\nu B$. Let $K^\lambda G = KG_p \otimes_K K^\nu B$. By Theorem 1.3, $K^\lambda G$ is of OTP representation type.

If $s \geq i(K) + 2$ or $G_p'$ is non-cyclic then $K^\mu G_p$ is not a uniserial algebra for any $\mu \in Z^2(G_p, K^*)$. Moreover, in the case $s \geq i(K) + 2$, we have $|G_p : G_p'| \geq p^2$, where

$$d = \dim_K(K^\mu G_p/\text{rad} K^\mu G_p).$$

Let $\nu \in Z^2(B, K^*)$ and $\lambda = \mu \times \nu$. By Theorem 1.3, an algebra $K^\lambda G$ is of OTP representation type if and only if $K$ is a splitting field for $K^\nu B$.

Assume now that $s = i(K) + 1$, $G_p' = \langle c \rangle$ and $G_p$ does not contain a cyclic subgroup $T$ such that $G_p' \subset T$ and $G_p/T$ has $i(K)$ invariants. Let $H = \langle c^p \rangle$ and $G_p/G_p' = \langle a_1 G_p' \rangle \times \cdots \times \langle a_s G_p' \rangle$, where $|a_j G_p'| = p^{n_j}$ for every $j \in \{1, \ldots, s\}$. We have

$$a_j^{p^{n_j}} \in H \quad \text{for each } j \in \{1, \ldots, s\}.$$

First, we examine the case $p = 2$. Let $N_{r,t}$ be the subgroup of $G_2$ generated by the elements $a_r$, $a_t$, and $c$, where $r, t \in \{1, \ldots, s\}$ and $r \neq t$. If $|N_{r,t} : G_2'| = 4$ and $N_{r,t}' = G_2'$, then $N_{r,t}$ is metacyclic. There exists a cyclic subgroup $T$ of $N_{r,t}$ such that $G_2' \subset T$ and $G_2/T$ has $i(K)$ invariants, a contradiction. Hence, if $|N_{r,t} : G_2'| = 4$, we have $[a_r, a_t] \in H$ and

$$D/H = \langle cH \rangle \times \langle b_1 H \rangle \times \cdots \times \langle b_s H \rangle,$$

where $b_j = a_j^{2^{n_j}-1}$ for each $j \in \{1, \ldots, s\}$. Each twisted group algebra of the group $D/H$ over the field $K$ is non-uniserial. Consequently, every $K^\mu G_2$ satisfies the $Q$-condition. By Lemma 1.2, the group $G = G_2 \times B$ is of OTP projective $K$-representation type if and only if condition (iii) holds.

Now we consider the case $p \neq 2$. By [5, p. 288], $|D'| \leq p$. If $|G_p'| \geq p^2$ then $D/H = \langle cH \rangle \times \langle b_1 H \rangle \times \cdots \times \langle b_s H \rangle$, where

$$b_j = a_j^{p^{n_j}-1} \quad \text{for each } j \in \{1, \ldots, s\}.$$
and, for every $\mu \in \mathbb{Z}^2(G_p, K^*)$, the algebra $K^{\mu}G_p/K^{\mu}G_p \cdot \text{rad} KG_p'$ is not a field. In view of Lemma 1.7 in [3] p. 177], $K^{\mu}G_p$ is not a uniserial algebra. By Theorem [3.3], $G$ is of OTP projective $K$-representation type if and only if one of the following conditions is satisfied:

**Corollary 3.2.** Let $G = G_p \times B$ and $K$ be an arbitrary perfect field of characteristic $p$. The group $G$ is of OTP projective $K$-representation type if and only if $G_p$ is cyclic or $K$ is a splitting field for some $K^vB$.

**Corollary 3.3.** Let $G = G_p \times B$ and $[K : K^p] = p$. Then $G$ is of OTP projective $K$-representation type if and only if either $G_p$ is metacyclic or $K$ is a splitting field for some $K^vB$.

**Corollary 3.4.** Let $G = G_p \times B$, $s$ be the number of invariants of $G_p/G_p'$ and $[K : K^p] = p^2$. The group $G$ is of OTP projective $K$-representation type if and only if one of the following conditions is satisfied:

(i) $s \leq 2$ and $G_p'$ is cyclic;
(ii) $s = 3$ and there exists a cyclic subgroup $T$ of $G_p$ such that $G_p' \subset T$ and $G_p/T$ has two invariants;
(iii) $K$ is a splitting field for some $K^vB$.

**Proof.** Keep the notation of Theorem [3.1]. Assume that $p \neq 2$, $s = 3$, $[G_p'] = p$ and $D$ is a non-abelian group of exponent $p$. Moreover, let $D/G_p' = \langle b_1G_p' \rangle \times \langle b_2G_p' \rangle \times \langle b_3G_p' \rangle$, $G_p' = \langle c \rangle$ and $[b_1, b_2] = c$, $[b_1, b_3] = c^r$, $[b_2, b_3] = c^t$, where $0 \leq r, t < p$. Set $h = b_1^r b_2^{-t} b_3$. Then $b_1 h = h b_1$, $b_2 h = h b_2$. It follows that $Z(D)$ is generated by $h, c$. Hence $|D : Z(D)| = p^2$. Applying Theorem [3.1] we conclude that $G$ is of OTP projective $K$-representation type if and only if one of the present conditions (i)–(iii) is satisfied.

**Corollary 3.5.** Let $G = G_p \times B$ and $[K : K^p] = \infty$. The group $G$ is of OTP projective $K$-representation type if and only if either $G_p'$ is cyclic, or $K$ is a splitting field for some $K^vB$.

**Proposition 3.6.** Let $G = G_p \times B$ be an abelian group and $s$ the number of invariants of $G_p$. The group $G$ is of OTP projective $K$-representation type if and only if one of the following conditions is satisfied:

(i) $s \leq i(K) + 1$;
(ii) $B$ has a subgroup $H$ such that $B/H$ is of symmetric type and $K$ contains a primitive $m$th root of 1, where $m = \max\{\exp(B/H), \exp H\}$.

**Proof.** Apply Proposition [2.5] and Theorem [3.1].

**Proposition 3.7.** Let $G_p$ be an abelian $p$-group, $s$ the number of invariants of $G_p$, $B$ a nilpotent $p'$-group and $G = G_p \times B$. Assume that $K$ does not contain a primitive $q$th root of 1 for some prime $q$ dividing $|B|$. The group $G$ is of OTP projective $K$-representation type if and only if $s \leq i(K) + 1$. 


Proof. Apply Proposition 2.7 and Theorem 3.1.

From now on, $K$ denotes an arbitrary field of characteristic $p$.

**Proposition 3.8.** A group $G = G_p \times B$ is of purely OTP projective $K$-representation type if and only if either $G_p$ is cyclic, or $K$ is a splitting field for every twisted group algebra of $B$ over $K$.

Proof. Let $\nu \in Z^2(B, K^*)$ be an arbitrary cocycle and $K^\lambda G = KG_p \otimes_K K^\nu B$. By Proposition 1.4, $K^\lambda G$ is of OTP representation type if and only if either $G_p$ is cyclic, or $K$ is a splitting field for $K^\nu B$. Assume now that $G_p$ is cyclic, $\mu \in Z^2(G_p, K^*)$ is an arbitrary cocycle and $\lambda = \mu \times \nu$. Since the algebra $K^\nu G_p$ is uniserial, by Lemma 1.1, $K^\lambda G$ is of OTP representation type.

**Proposition 3.9.** Let $G = G_p \times B$. Assume that $K = K^q$ and $K$ contains a primitive $q$th root of 1 for each prime $q$ that divides $|B|$. Then $G$ is of purely OTP projective $K$-representation type.

Proof. Apply Propositions 2.8 and 3.8.

**Corollary 3.10.** If $K$ is a separably closed field then every group $G = G_p \times B$ is of purely OTP projective $K$-representation type.

**Proposition 3.11.** Let $G = G_p \times B$. Assume that every prime divisor of $|B'|$ is also a divisor of $|B : B'|$. The group $G$ is of purely OTP projective $K$-representation type if and only if either $G_p$ is cyclic, or $K = K^q$ and $K$ contains a primitive $q$th root of 1 for each prime $q$ that divides $|B|$.

Proof. Again apply Propositions 2.8 and 3.8.

**Theorem 3.12.** A group $G = G_p \times B$ is of purely OTP projective $K$-representation type if and only if either $G_p$ is cyclic, or there exists a finite central group extension $1 \to A \to \hat{B} \to B \to 1$ such that any projective $K$-representation of $B$ lifts projectively to an ordinary $K$-representation of $\hat{B}$ and $K$ is a splitting field for $\hat{B}$.

Proof. Apply Propositions 2.9 and 3.8.

**Proposition 3.13.** Let $G = G_p \times B$. Assume that either $t(K^*) = t(K^*)^q$ for any prime $q$ dividing $|B'|$, or every prime divisor of $|B'|$ is also divisor of $|B : B'|$. Then $G$ is of purely OTP projective $K$-representation type if and only if either $G_p$ is cyclic, or there exists a covering group $\hat{B}$ of $B$ over $K$ such that $K$ is a splitting field for $\hat{B}$.

Proof. Apply Propositions 2.10 and 3.8.

**References**


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