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ON μ-COMPATIBLE METRICS AND MEASURABLE SENSITIVITY

BY

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Abstract. We introduce the notion of W-measurable sensitivity, which extends and strictly implies canonical measurable sensitivity, a measure-theoretic version of sensitive dependence on initial conditions. This notion also implies pairwise sensitivity with respect to a large class of metrics. We show that nonsingular ergodic and conservative dynamical systems on standard spaces must be either W-measurably sensitive, or isomorphic mod 0 to a minimal uniformly rigid isometry. In the finite measure-preserving case they are W-measurably sensitive or measurably isomorphic to an ergodic isometry on a compact metric space.

1. Introduction. The notion of sensitive dependence on initial conditions is an extensively studied isomorphism invariant of topological dynamical systems on compact metric spaces (e.g.., [GW93], [AAB96]). In [JKL$^+$08], the authors define two measure-theoretic versions of sensitive dependence: measurable sensitivity and strong measurable sensitivity, and show that, unlike their traditional topologically-dependent counterpart, both of these properties are invariant under measurable-theoretic isomorphism. James et al. introduce these notions for nonsingular transformations and show that measurable sensitivity is implied by double ergodicity (a property equivalent to weak mixing in the finite measure-preserving case) and that strong measurable sensitivity is implied by light mixing in the finite measure-preserving case.

In this paper, we introduce W-measurable sensitivity, a notion that is *a priori* stronger than measurable sensitivity and implies it straightforwardly. We use this new property, together with properties of μ-compatible metrics (see below), to formulate a classification of all nonsingular conservative and ergodic transformations on standard Borel spaces as being either W-measurably sensitive or isomorphic to a minimal uniformly rigid isometry; in the case of finite invariant measure we obtain more, namely the system is

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W-measurably sensitive or isomorphic to a minimal uniformly rigid invertible isometry on a compact metric space. In the course of this proof, we also show that W-measurable sensitivity is in fact equivalent to measurable sensitivity for conservative and ergodic transformations.

In addition, we show (see Appendix A) that W-measurable sensitivity is closely related to *pairwise sensitivity*, a notion introduced in [CJ05] for finite measure-preserving transformations. In their paper, Cadre and Jacob show that weakly mixing finite measure-preserving transformations always exhibit pairwise sensitivity, as also does any ergodic finite measure-preserving transformation satisfying a certain entropy condition. Our results imply that any finite measure-preserving ergodic transformation that is not isomorphic mod 0 to a Kronecker transformation will exhibit pairwise sensitivity with respect to any μ-compatible metric (in addition to W-measurable sensitivity).

The plan of the paper is as follows. Section 2 recalls basic definitions from [JKL$^+$08] and introduces μ-compatible metrics [JKL$^+$08] and some of their properties. In Section 3 we define W-measurable sensitivity. Section 4 starts by constructing 1-Lipschitz metrics from any metric on a dynamical system, and then shows that W-measurable sensitivity can be equivalently expressed in additional ways using properties of μ-compatible metrics. In Section 5, we provide a sufficient condition under which the newly constructed 1-Lipschitz metric is in fact μ-compatible, and discuss consequences of this fact, largely from [AG01]. In Section 6 we discuss the invariance of W-measurable sensitivity under measurable isomorphism, as well as the technical assumptions necessary for it to hold. We also illustrate the main connection between 1-Lipschitz metrics and W-measurable sensitivity, namely that a conservative and ergodic nonsingular dynamical system is W-measurably sensitive if and only if all dynamical systems (X', μ', T') isomorphic mod 0 to it admit no μ'-compatible 1-Lipschitz metrics. Finally, in Section 7 we prove our main result, which classifies all conservative and ergodic, nonsingular transformations on standard Borel spaces as being either W-measurably sensitive, or isomorphic to a minimal uniformly rigid invertible isometry. A corollary of this fact is that for conservative and ergodic transformations, W-measurable sensitivity is equivalent to measurable sensitivity as defined in [JKL$^+$08]. We end the section by obtaining a stronger result in the case of ergodic finite measure-preserving transformations.

In the appendix we elaborate on the relationship between our results and the notion of pairwise sensitivity as introduced in [CJ05], and mention the recent work in [HLY11].

2. Preliminary definitions. A *nonsingular dynamical system* is a quadruple $(X, \mathcal{S}(X), \mu, T)$, where $(X, \mathcal{S}(X), \mu)$ is a standard nonatomic Lebesgue space (i.e., $(X, \mathcal{S}(X))$ is a standardIn Borel space, see e.g. [Sri98], and μ is a σ-finite, nonatomic measure on $\mathcal{S}(X)$). It follows that X must be of cardinality c as the measure is nonatomic. Furthermore, the transformation T is measurable and a nonsingular endomorphism (i.e., for all $A \in \mathcal{S}(X)$, $T^{-1}(A) \in \mathcal{S}(X)$ and $\mu(A) = 0$ if and only if $\mu(T^{-1}(A)) = 0$, see e.g. [Sil08]). In some cases we assume that T is measure-preserving or that the measure space is finite. Recall that T is conservative and ergodic if and only if for all measurable sets A, if $T^{-1}(A) \subset A$, then $\mu(A) = 0$ or $\mu(A^c) = 0$.

We consider metrics or pseudo-metrics on X. We assume throughout this article that all pseudo-metrics $d : X \times X \to \mathbb{R}$ are (Borel) measurable and bounded by 1 (one can replace d by $d/(1 + d)$). It follows that, for each $\varepsilon > 0$, the set $\{(x, y) \in X \times X : d(x, y) < \varepsilon\}$ is measurable. Therefore, by e.g. [Sri98, Exercise 3.1.20], the balls

$$B^d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

are measurable. For a pseudo-metric d define

$$\mathcal{D}^d(x) = \max\{\varepsilon \geq 0 : \mu(B^d(x, \varepsilon)) = 0\},$$
$$\mathrm{Dis}(d) = \{x \in X : \mathcal{D}^d(x) > 0\}.$$

A (measurable) metric d on $(X, \mathcal{S}(X), \mu)$ is said to be *μ-compatible* if μ assigns positive (nonzero) measure to all nonempty, open d-balls in X, equivalently if $\mathrm{Dis}(d) = \emptyset$, or if $\mathcal{D}^d(x) = 0$ for all $x \in X$. If d is a μ-compatible metric on $(X, \mathcal{S}(X), \mu)$, then X is separable under d (see [JKL⁺08, 1.1] and Proposition 2.1 below). Therefore open sets are measurable as they are countable unions of balls. All d-closed sets are also measurable, etc. We say that d is *μ-separable* if $\mu(\mathrm{Dis}(d)) = 0$, or equivalently $\mathcal{D}^d(x) = 0$ a.e. It follows that if d is μ-separable, then the restriction of d to $X \setminus \mathrm{Dis}(d)$ is μ-compatible.

Proposition 2.1. *Let $(X, \mathcal{S}(X), \mu, T)$ be a nonsingular dynamical system and let d be a pseudo-metric on X.*

(1) *The function $\mathcal{D}^d(x)$ is continuous with respect to d and measurable.*
(2) *The pseudo-metric d is separable when restricted to $X \setminus \mathrm{Dis}(d)$. In particular, if d is μ-compatible, then it is separable on X.*
(3) $\mathrm{Dis}(d)$ *is open with respect to d and measurable.*
(4) *A pseudo-metric d is μ-separable if and only if there exists a measure zero subset Z of X such that d restricted to $X \setminus Z$ is separable.*

Proof. (1) Suppose that $\beta < \mathcal{D}^d(x) < \alpha$. Set

$$\delta = \frac{1}{2}\min\{\mathcal{D}^d(x) - \beta, \alpha - \mathcal{D}^d(x)\}.$$

Then for each $y \in B^d(x,\delta)$ we have

$$B^d(y,\beta) \subset B^d(x,\beta+\delta), \qquad B^d(x,\alpha-\delta) \subset B^d(y,\alpha).$$

Since $\beta+\delta < \mathcal{D}^d(x)$, we have $\mu(B^d(x,\beta+\delta)) = 0$, so $\mu(B^d(y,\beta)) = 0$ and $\mathcal{D}^d(y) \geq \beta$. Similarly we obtain $\mathcal{D}^d(y) \leq \alpha$. This implies that $\mathcal{D}^d$ is continuous with respect to d, and therefore measurable.

(2) For $0 < \varepsilon < 1$, let $A_\varepsilon \subset X \setminus \mathrm{Dis}(d)$ be such that if $x, y \in A_\varepsilon$ then $d(x,y) \geq \varepsilon$, and let it be maximal with respect to this property. It follows that

$$\{B^d(x,\varepsilon/2) : x \in A_\varepsilon\}$$

is a collection of disjoint sets of positive measure, and since μ is σ-finite, this collection is countable. This shows that each A_ε is countable. Then the union $\bigcup_{n\in\mathbb{N}} A_{1/n}$ is a countable set that is dense in $X \setminus \mathcal{D}^d$ for the metric d.

(3) Since $\mathcal{D}^d$ is continuous by (1), $\mathrm{Dis}(d)$ is open with respect to d. By part (2), every open set that is contained in $X \setminus \mathrm{Dis}(d)$ is a countable union of balls, hence it is measurable. Similarly, closed sets contained in $X \setminus \mathrm{Dis}(d)$ are measurable. In particular, $X \setminus \mathrm{Dis}(d)$, and so $\mathrm{Dis}(d)$, is measurable.

(4) Suppose that $\mu(\mathrm{Dis}(d)) > 0$ and let $Z \subset X$ be such that $\mu(Z) = 0$. We show that d is not separable on the subset $\mathrm{Dis}(d) \setminus Z$ of $X \setminus Z$. We first note that the collection

$$\{B^d(x,\mathcal{D}(x)) : x \in \mathrm{Dis}(d) \setminus Z\}$$

is an open cover of $\mathrm{Dis}(d) \setminus Z$, and since $\mathrm{Dis}(d) \setminus Z$ has positive measure and each of the balls has measure zero (by definition of $\mathrm{Dis}(d)$), the collection cannot have a countable subcover. Conversely, if $\mu(\mathrm{Dis}(d)) = 0$ we can let $Z = \mathrm{Dis}(d)$ and use part (2). ■

PROPOSITION 2.2. *Let $(X, \mathcal{S}(X), \mu, T)$ be a nonsingular dynamical system and let d be a pseudo-metric on X. Let $\delta > 0$. If $\mathcal{D}^d(x) \geq \delta$ for almost all $x \in X$, then $\mathcal{D}^d(x) \geq \delta/2$ for all $x \in X$.*

Proof. Let

$$Z = \{z \in X : \mathcal{D}^d(z) \geq \delta\} = \{z \in X : \mu(B^d(z,\delta)) = 0\}.$$

We know that $\mu(Z^c) = 0$. Suppose $\mathcal{D}^d(x) < \delta/2$ for some $x \in X$. Then $\mu(B^d(x,\delta/2)) > 0$. So there exists $z \in B^d(x,\delta/2) \cap Z$. By the triangle inequality, $B^d(x,\delta/2) \subset B^d(z,\delta)$. This means that $\mu(B^d(z,\delta)) > 0$, a contradiction. ■

3. W-measurable sensitivity. We start by recalling the definition of measurable sensitivity.

Definition 3.1 ([JKL+08]). A nonsingular dynamical system $(X, \mathcal{S}(X), \mu, T)$ is said to be *measurably sensitive* if for every isomorphic mod 0 dynamical system $(X_1, \mathcal{S}(X_1), \mu_1, T_1)$ and any μ_1-compatible metric d on X_1, there exists $\delta > 0$ such that for $x \in X_1$ and all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\mu_1\{y \in B_\varepsilon(x) : d(T_1^n(x), T_1^n(y)) > \delta\} > 0.$$

We now introduce the definition that we shall be using extensively.

Definition 3.2. For a μ-compatible metric d, a nonsingular dynamical system (X, μ, T) is *W-measurably sensitive with respect to d* if there is a $\delta > 0$ such that for every $x \in X$,

$$\limsup_{n\to\infty} d(T^n x, T^n y) > \delta$$

for almost every $y \in X$. The dynamical system is said to be *W-measurably sensitive* if the above condition holds true for all μ-compatible metrics d.

Remark. (1) As in [JKL+08], it can be shown that a doubly ergodic nonsingular transformation is W-measurably sensitive. (Double ergodicity is a condition for nonsingular transformations that is equivalent to weak mixing in the finite measure-preserving case [Fur81].) There exist both infinite (and finite) measure-preserving and nonsingular type III (i.e., not admitting an equivalent σ-finite invariant measure) invertible transformations that are doubly ergodic (see e.g. [DS09]), and therefore W-measurably sensitive.

(2) If a measure space (X, μ) has atoms, no transformation on it can exhibit W-measurable sensitivity with respect to any metric. Indeed, for any $x \in X$, and any δ, the set of points y such that $\limsup_{n\to\infty} d(T^n x, T^n y) > \delta$ cannot include x. So this set cannot have full measure (i.e., its complement cannot have measure zero) if $\mu(\{x\}) > 0$.

The same is not true about measurable sensitivity. For this reason, throughout this paper we assume that our measure space is nonatomic.

(3) A very important example of an ergodic finite measure-preserving dynamical system which is not W-measurably sensitive is a Kronecker transformation, i.e. an ergodic isometry on an interval of finite length (with the Lebesgue measure and the usual metric). This transformation is not W-measurably sensitive with respect to the usual metric because it is an isometry. There are also examples of conservative and ergodic type III nonsingular invertible transformations that are not W-measurable sensitive. Let $X = \prod_{i=0}^{\infty}\{0, 1\}$, the 2-adic integers, let T be addition of 1, $Tx = x+1$, and d be the 2-adic metric. Then it is well-known that T is a minimal isometry for d. Let $0 < p < 1$ and $\mu_p = \prod_{i=0}^{\infty}\{p, 1-p\}$, a probability measure on the Borel σ-field $\mathcal{B}$. Then μ_p is a nonsingular measure for $(X, \mathcal{B}, T)$ that is conservative and ergodic of type III (when $p \neq 1/2$), see e.g. [DS09]. It is

clear that d is μ_p-compatible, so $(X, \mathcal{B}, T)$ is a conservative ergodic invertible nonsingular transformation that is not finite measure-preserving and is not W-measurably sensitive.

We note that the property of W-measurable sensitivity is preserved under measurable isomorphisms (Proposition 6.2).

W-measurable sensitivity clearly implies measurable sensitivity (see the first part of the proof of Proposition 7.2). In fact, we show that the two notions are equivalent for conservative and ergodic dynamical systems. We first show in Proposition 4.2 that for a transformation to be W-measurably sensitive, it is sufficient for each $y \in Y$ to have *one* value of n that satisfies $d(T^n x, T^n y) > \delta$. The remainder of the equivalence follows from the results in the following sections, culminating with Proposition 7.2.

4. Constructing 1-Lipschitz metrics. We shall use the term *1-Lipschitz metrics* (*with respect to* T) to denote metrics that satisfy the inequality $d(Tx, Ty) \leq d(x, y)$ for all x and y.

First, we provide a way to construct a 1-Lipschitz metric from any other metric.

DEFINITION 4.1. Let (X, μ, T) be a nonsingular dynamical system, and d be a metric on X. Define, for $x, y \in X$,

$$d_T(x, y) = \sup_{n \geq 0} d(T^n x, T^n y).$$

LEMMA 4.1. *d_T is a metric on X (satisfying our standing assumptions: measurable and bounded). Moreover, it is a 1-Lipschitz metric.*

Proof. The first statement is left to the reader. To see that the metric is 1-Lipschitz we compute

$$\begin{aligned} d_T(Tx, Ty) &= \sup_{n \geq 0} d(T^n(Tx), T^n(Ty)) = \sup_{n \geq 1} d(T^n x, T^n y) \\ &\leq \sup_{n \geq 0} d(T^n x, T^n y) = d_T(x, y). \ \blacksquare \end{aligned}$$

REMARK. In general, even if the metric d is μ-compatible, the metric d_T may not be μ-compatible. Consequently, there is no guarantee that the measure space is separable under the topology determined by d_T.

For example, let I be the unit interval, λ be the Lebesgue measure, and d be the usual metric. Let $T : I \to I$ be the doubling map $Tx = 2x \pmod 1$. Note that d is a λ-compatible metric.

The metric d_T is not, however, λ-compatible. Indeed, for any $x \notin \mathbb{Q}$, and any $\varepsilon > 0$, there will be an n such that $d(0, T^n x) > 1 - \varepsilon$. So, since $T(0) = 0$, we have

$$\sup_{n \geq 0} d(T^n(0), T^n y) = \sup_{n \geq 0} d(0, T^n y) = 1.$$

In other words, for any $0 < \delta < 1$, the δ-ball around 0 in the d_T metric may contain only rational points. So, $\lambda(B_{T_\delta}(0)) = 0$, and d_T is not λ-compatible.

In this example, the transformation T turns out to be W-measurably sensitive. In fact, being mixing, it is strongly measurably sensitive (see [JKL$^+$08]). On the other hand, we will see that whenever the 1-Lipschitz metric d_T is μ-compatible, the corresponding transformation T is not W-measurably sensitive.

We now formulate several equivalent definitions of W-measurably sensitive transformations. We start by showing that while the original definition requires the existence of infinitely many times n satisfying the condition, it is sufficient to require the existence of one such n.

PROPOSITION 4.2. *Let (X, μ, T) be a nonsingular dynamical system, and d be a μ-compatible metric. The following are equivalent:*

(1) *The system is W-measurably sensitive with respect to d.*
(2) *There is a $\delta > 0$ such that for each $x \in X$, for almost every $y \in X$,*
$$d_T(x, y) > \delta.$$
(3) *There is a $\delta > 0$ such that for each $x \in X$,*
$$\mu(B^{d_T}(x, \delta)) = 0.$$
(4) *There is a $\delta > 0$ such that for each $x \in X$,*
$$\mathcal{D}^{d_T}(x) \geq \delta.$$
(5) *There is a $\delta > 0$ such that for each $x \in X$,*
$$\mathcal{D}^{d_T}(x) > \delta.$$

Proof. (2)⇒(1). Suppose that there is a $\delta > 0$ such that for each $x \in X$, for almost every $y \in X$, there exists n such that $d(T^n x, T^n y) > \delta$. For every natural number N and $x \in X$ define
$$Y(N, x) = \{y \in X : \exists n > N,\ d(T^n x, T^n y) > \delta\}.$$
We now prove that for all N and x, the set $Y(N, x)$ has full measure. Consider the point $T^N x$. Using our assumption, for almost every $y \in X$, there exists n such that $d(T^n(T^N x), T^n y) > \delta$. In other words, the set
$$Z(N, x) = \{y \in X : \exists n > 0,\ d(T^{N+n} x, T^n y) > \delta\}$$
has full measure. Notice that $Y(N, x) = T^{-N}(Z(N, x))$. Since T is a nonsingular transformation, $Y(N, x)$ must also have full measure.

Finally, let $Y_x = \bigcap_{N=0}^{\infty} Y(N, x)$. Clearly, Y_x has full measure. Furthermore, for every $y \in Y_x$, there are infinitely many values of n such that $d(T^n x, T^n y) > \delta$. So
$$\limsup_{n\to\infty} d(T^n x, T^n y) \geq \delta$$

for almost all $y \in X$. Therefore the system (X, μ, T) is W-measurably sensitive with respect to d.

(1)⇒(2). This is clear from the definitions.

(2)⇔(3). If (2) is satisfied at x for some δ, then $B^{d_T}(x,\delta)$ is contained in the complement of a set of full measure. So $\mu(B^{d_T}(x,\delta)) = 0$.

Conversely, if (3) is satisfied at x for some δ, then $B^{d_T}(x,\delta)$ has measure zero. So in particular, the set $\{y \in X : \forall n \geq 0,\ d(T^n x, T^n y) \leq \delta/2\}$ has measure zero. Therefore, for almost every $y \in X$, there is some n for which $d(T^n x, T^n y) > \delta/2$, and condition (2) is satisfied.

The equivalence of (3) and (4) is clear from the definitions. The equivalence of (4) and (5) is clear since δ does not have to be the same. ■

Remark. From Proposition 2.2 it follows that in the equivalent characterizations of W-measurable sensitivity in Proposition 4.2, one can replace "for each $x \in X$" in parts (2)–(5) with "for a.e. $x \in X$."

5. Conditions for 1-Lipschitz metric d_T to be μ-compatible and consequences. Now, we provide a sufficient condition for the 1-Lipschitz metric d_T to be μ-compatible given that the transformation T is ergodic.

The proof of the following lemma is standard (see for example [ST91, Corollary 2.7]).

Lemma 5.1. *Let (X, μ, T) be a conservative and ergodic nonsingular dynamical system. Let $f : X \to \mathbb{R}$ be a measurable function. If $f \geq f \circ T$ a.e., then $f = f \circ T$ a.e.*

Lemma 5.2. *Let (X, μ, T) be a nonsingular dynamical system, and d be a metric on X. If d is 1-Lipschitz then*

$$\mathcal{D}^d \geq \mathcal{D}^d \circ T \quad \text{on } X.$$

Proof. Let T^*d denote the metric $T^*d(x,y) = d(Tx,Ty)$. First we observe

$$T^{-1}B^d(Tx,\varepsilon) = \{y \in X : d(Tx,Ty) < \varepsilon\} = B^{T^*d}(x,\varepsilon).$$

Since T is nonsingular, $\mu(B^{T^*d}(x,\varepsilon)) = 0$ if and only if $\mu(B^d(Tx,\varepsilon)) = 0$. It follows that

$$\mathcal{D}^{T^*d}(x) = \mathcal{D}^d(Tx) \quad \text{for all } x \in X.$$

Since T is 1-Lipschitz, $d(x,y) \geq d(Tx,Ty)$, which implies

$$\mathcal{D}^d(x) \geq \mathcal{D}^{T^*d}(x) \quad \text{for all } x,$$

completing the proof. ■

Now, we are ready to state the sufficient condition for the 1-Lipschitz metric d_T to be μ-compatible, which is our main tool in proving the main results in Section 7.

LEMMA 5.3. *Let (X, μ, T) be a conservative and ergodic nonsingular dynamical system. Let d be a μ-compatible metric on X. Suppose further that T is not W-measurably sensitive with respect to d. Then there exists a positively invariant measurable set X_1 of full measure (i.e., $X_1 \subset T^{-1}(X_1)$ and $\mu(X \setminus X_1) = 0$) such that d_T is a μ-compatible metric for the system (X_1, μ, T), where μ and T are the restrictions to X_1 of the original measure and transformation.*

Proof. First we observe that

$$T^{-1}(\mathrm{Dis}(d_T)) \subset \mathrm{Dis}(d_T). \tag{5.1}$$

In fact, if $Tx \in \mathrm{Dis}(d_T)$, then $\mathcal{D}^{d_T}(Tx) > 0$. Since d_T is 1-Lipschitz, by Lemma 5.2, $\mathcal{D}^{d_T}(x) > 0$, so $x \in \mathrm{Dis}(d_T)$. Therefore T can be restricted to a transformation on the positively invariant set $X_1 = X \setminus \mathrm{Dis}(d_T)$.

Since T is conservative and ergodic it follows from (5.1) that $\mu(\mathrm{Dis}(d_T)) = 0$ or $\mu(\mathrm{Dis}(d_T)^c) = 0$. If it were the case that $\mu(\mathrm{Dis}(d_T)^c) = 0$ then there would exist $r > 0$ such that $\mathcal{D}^{d_T}(x) > r$ on a set of positive measure, hence by Lemmas 5.2 and 5.1, as T is conservative and ergodic, the condition holds for a.e. x, but this contradicts the hypothesis by the Remark following Proposition 4.2. Therefore $\mu(\mathrm{Dis}(d_T)) = 0$ and X_1 is a set of full measure. (It also follows that $\mathcal{D}^{d_T}(x) = 0$ for a.e. $x \in X$.)

Clearly, d_T is a metric on X_1. To see that it is μ-compatible we calculate, for $x \in X_1$ and $\varepsilon > 0$,

$$\mu(B^{d_T}(x, \varepsilon) \cap X_1) = \mu(B^{d_T}(x, \varepsilon)) > 0. \quad \blacksquare$$

REMARK. In relation to Lemma 5.3, we note that it is possible that a system (X, μ, T) is not W-measurably sensitive but does not itself admit any μ-compatible metric d that is 1-Lipschitz. For example, consider the dynamical system (I, λ, T) where I is the unit interval and λ is the Lebesgue measure. Let α be a fixed irrational number between 0 and 1. For any $x \in I$, we define

$$T(x) = \begin{cases} x & \text{if } x = n \cdot \alpha + m \text{ for some } n, m \in \mathbb{Z}, \\ x + \alpha \pmod 1 & \text{otherwise.} \end{cases}$$

This system is ergodic and not measurably sensitive as it is measurably isomorphic to a rotation. However, there is no λ-compatible 1-Lipschitz metric on I.

Indeed, suppose that there is a λ-compatible metric d such that $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in I$. Let B be a ball of radius $\alpha/2$ around 0. Since d is λ-compatible, B must have positive measure. Furthermore, since $T(0) = 0$, for any point $b \in B$, we must have $d(T(b), 0) \leq d(b, 0) < \alpha/2$, and therefore, $T(b) \in B$. So T maps a set of positive measure into itself. This is impossible for a transformation isomorphic mod 0 to an irrational rotation.

In the rest of this section, we describe some useful consequences of a 1-Lipschitz metric being μ-compatible.

Let (X,d) be a metric space and $T : X \to X$ a transformation. Let $\omega(x)$ denote the set of accumulation points of the positive orbit $\{T^n x : n \in \mathbb{N}_0\}$. A point $x \in X$ is *transitive* for T if $\omega(x) = X$. When (X,d) has no isolated points this is equivalent to the (positive) orbit of x being dense in X. As we will only consider μ-compatible metrics where μ is nonatomic, all our metric spaces will have no isolated points. T is *transitive* if it has a transitive point. The transformation T is *minimal* if $\omega(x) = X$ for all $x \in X$. It is *uniformly rigid* if there exists a sequence n_i such that $d(T^{n_i}x, x)$ converges to 0 uniformly on X.

The following lemma is essentially known.

LEMMA 5.4. *Let (X,μ,T) be a conservative and ergodic nonsingular dynamical system. If d is a μ-compatible metric on X, then μ-a.e. point of X is transitive.*

Proof. Since by assumption μ is nonatomic, d has no isolated points. By Proposition 2.1, (X,d) is separable, so there exist $\{x_i : i \in \mathbb{N}\}$ dense in X. For each $r \in \mathbb{Q}$, $r > 0$, and each $i, N \in \mathbb{N}$, set

$$A^*_{i,N,r} = \bigcup_{n \geq N} T^{-n}(B^d(x_i, r)).$$

Since T is conservative and ergodic, each $A^*_{i,N,r}$ is of full measure. Finally let

$$B = \bigcap_{i,N,r} A^*_{i,N,r}.$$

Clearly B is of full measure and each point in B has a dense orbit. ∎

The following proposition is essentially from [AG01].

PROPOSITION 5.5. *Let (X,d) be a metric space and let $T : X \to X$ be a 1-Lipschitz transformation. If T is transitive, then it is a uniformly rigid, minimal isometry.*

Proof. Let x be a point such that $\omega(x) = X$. (This in particular implies that the metric d is separable.) Let $\varepsilon > 0$. There exists an integer $k > 0$ such that $d(x, T^k x) < \varepsilon$. Since T is 1-Lipschitz, for all $n \in \mathbb{N}$, $d(T^n x, T^n(T^k x)) < \varepsilon$. Let $y \in X$. Since T^k is continuous, for n such that $d(y, T^n x)$ is sufficiently small, $d(T^k y, T^k(T^n x)) < \varepsilon$. Then

$$d(y, T^k y) \leq d(y, T^n x) + d(T^n x, T^n(T^k x)) + d(T^k(T^n x), T^k y) < 3\varepsilon.$$

Therefore T is uniformly rigid. Now, in this case there exists a sequence $n_i \to \infty$ such that $d(T^{n_i}x, x) \to 0$ for all $x \in X$. Therefore, for all $x, y \in X$,

$$0 \leq d(T^{n_i}x, T^{n_i}y) - d(x,y) \leq d(T^{n_i}x, x) + d(y, T^{n_i}y) \to 0.$$

If T were not an isometry there would exist $x, y \in X$ such that $d(Tx, Ty) < d(x, y)$, but then $d(T^{n_i}x, T^{n_i}y)$ could not converge to $d(x, y)$.

Finally we show that T is minimal. Again, let $\omega(x) = X$ and $y \in X$. Let $\varepsilon > 0$, $z \in X$. There exists $i \in \mathbb{N}$ such that $d(T^i x, y) < \varepsilon$. Then we can choose $j \in \mathbb{N}$ so that $d(T^{i+j}x, z) < \varepsilon$. So

$$d(T^j y, z) \leq d(T^j y, T^{j+i}x) + d(T^{j+i}x, z) \leq d(y, T^i x) + d(T^{j+i}x, z) < 2\varepsilon.$$

Therefore $\omega(y) = X$. ■

Now, let $\mathcal{C}_d(X, X)$ be the space of continuous maps from X to itself, with the metric $d(S_1, S_2) = \sup_{x \in X}\{d(S_1 x, S_2 x)\}$. We also define a subset

$$\mathcal{J}_T = \{S \in \mathcal{C}_d(X, X) : S \circ T = T \circ S\}.$$

This is clearly a subsemigroup of $\mathcal{C}_d(X, X)$ under composition.

The following proposition is essentially from [AG01]. We are indebted to Ethan Akin for the proof.

Proposition 5.6. *Let (X, d) be a metric space and let T be a transitive and 1-Lipschitz transformation. Then, for each $x \in X$, the evaluation map*

$$\mathrm{ev}_x : \mathcal{J}_T \to X, \qquad S \mapsto Sx,$$

is an isometry. Moreover, the space $\mathcal{J}_T$ is the closure of the sequence $\{\mathrm{id}, T, T^2, \ldots\}$ in $\mathcal{C}_d(X, X)$. If in addition the metric space (X, d) is complete, then the evaluation map ev_x is an invertible isometry. Moreover, the semigroup $\mathcal{J}_T$ is then a group, and therefore $T \in \mathcal{J}_T$ has to be invertible.

Proof. Fix a point $x \in X$ and let $S, S' \in \mathcal{J}_T$. We wish to show that the map ev_x is an isometry. Since S and S' both commute with T, and T is 1-Lipschitz, for all m,

$$d(S(T^m x), S'(T^m x)) \leq d(Sx, S'x).$$

Since S and S' are both continuous and the set of all $\{T^m x\}$ is dense, for all $y \in X$ we have $d(Sy, S'y) \leq d(Sx, S'x)$ and therefore

$$d(S, S')_{\mathcal{C}_d(X,X)} = \sup_{y \in X} d(Sy, S'y)_X = d(Sx, S'x)_X = d(\mathrm{ev}_x S, \mathrm{ev}_x S')_X$$

and so ev_x is an isometry.

Now, the subset $\mathcal{J}_T$ is clearly closed in $\mathcal{C}_d(X, X)$. Fix some $S \in \mathcal{J}_T$ and $x \in X$. Since T is minimal, x is a transitive point, and so there is a sequence $\{n_j\}$ such that $\lim_{j\to\infty} T^{n_j}x = Sx$. In other words, $\lim_{j\to\infty} \mathrm{ev}_x T^{n_j} = \mathrm{ev}_x S$ in X. Since ev_x is an isometry, this implies that $\lim_{j\to\infty} T^{n_j} = S$ in $\mathcal{C}_d(X, X)$, completing the proof of the first part of the proposition.

If we assume that the space (X, d) is complete, so is the space $\mathcal{C}_d(X, X)$. For $x \in X$, we show that ev_x is surjective.

Pick a $y \in X$. There is a sequence of n_j's such that $T^{n_j}x \to y$. In particular, the sequence $\mathrm{ev}_x T^{n_j}$ is Cauchy. Since ev_x is an isometry, the sequence T^{n_j} is Cauchy in $\mathcal{C}_d(X,X)$. By completeness, it has a limit $S \in \mathcal{J}_d$ (since $\mathcal{J}_d$ is closed); clearly $\mathrm{ev}_x S = y$ and ev_x is surjective.

Now, let $S \in \mathcal{J}_d$ be arbitrary. Since the map ev_{Sx} is surjective, we can pick an S' so that

$$S'(Sx) = \mathrm{ev}_{Sx} S' = x.$$

Since $\mathrm{ev}_x(SS') = (S'S)x = x$ and ev_x is injective, $S \circ S'$ is the identity, and $S' = S^{-1}$. So, all maps in $\mathcal{J}_d$ are invertible. ∎

6. W-measurable sensitivity on isomorphic mod 0 dynamical systems. We prove that W-measurable sensitivity is invariant under measurable isomorphism. Here we use the assumption that we are working on standard Borel spaces.

LEMMA 6.1. *Let $(X,\mathcal{S})$ be a standard Borel space, with μ a nonatomic measure on $\mathcal{S}$. Let $U \subset X$ be a Borel subset of full measure and let d be a μ-compatible metric defined on U. Then the metric d can be extended to a μ-compatible metric d_1 on all of X in such a way that d and d_1 agree on a set of full measure.*

Proof. Since the measure is nonatomic and U is Borel, it must have the same cardinality as X. Using e.g. [Sri98, 3.4.23] one can show that there exists a Borel set $Z \subset U$ of measure zero and cardinality c. Therefore there exists a Borel isomorphism $\phi : (X \setminus U) \sqcup Z \to Z$. Then we can define $\phi' : X \to U$ by

$$\phi'(x) = \begin{cases} \phi(x) & \text{if } x \in (X \setminus U) \cup Z, \\ x & \text{if } x \in U \setminus Z. \end{cases}$$

(ϕ' is the identity on the full-measure Borel subset $U \setminus Z$.) For $x, y \in X$ define $d_1(x,y) = d(\phi'(x), \phi'(y))$. Clearly, since d is a measurable metric, so is d_1. Since every d_1-ball corresponds to a d-ball under the map ϕ, which is a Borel isomorphism, d_1 is also a μ-compatible metric and agrees with d on $(U \setminus Z) \times (U \setminus Z)$. ∎

Using Lemma 6.1, we can prove the invariance of W-measurable sensitivity.

PROPOSITION 6.2. *Suppose (X,μ,T) is a W-measurably sensitive nonsingular dynamical system. Let (X',μ',T') be a nonsingular dynamical system isomorphic mod 0 to (X,μ,T). Then (X',μ',T') is also W-measurably sensitive.*

Proof. Suppose (X',μ',T') is not W-measurably sensitive. Then there is a μ'-compatible metric d' on X' such that (X',μ',T') is not W-measurably sensitive with respect to d'.

By the definition of measurable isomorphism, there must be Borel subsets $U \subset X$ and $U' \subset X'$ and a measure-preserving bijection $\phi : U \to U'$ such that $\mu(X \setminus U) = \mu'(X' \setminus U') = 0$, and $\phi \circ T = T' \circ \phi$.

We define a metric d on U by $d(x,y) = d'(\phi(x), \phi(y))$ for $x, y \in U$. It is clearly μ-compatible on U. We apply Lemma 6.1 to extend d to a μ-compatible metric d_1 defined on all of X that agrees with d almost everywhere.

Now, we show that (X, μ, T) is not W-measurably sensitive with respect to d_1. Let $\delta > 0$. Since (X', μ', T') is not W-measurably sensitive with respect to d', by part (3) of Proposition 4.2 there must be an $x' \in X'$ such that the set $Y' = \{y \in X' : \forall n \geq 0,\ d'(T'^n x, T'^n y) < \delta/2\}$ has positive measure. Let Y be the corresponding set in X, that is, $Y = \phi^{-1}(Y' \cap U')$. Note that $\mu(Y) = \mu'(Y') > 0$.

Pick any $x \in Y$. By the triangle inequality, for all $y \in Y$ and all integers n, we have

$$\begin{aligned} d_1(T^n x, T^n y) &= d'(T'^n(\phi(x)), T'^n(\phi(y)) \\ &\leq d'(x', T'^n(\phi(x)) + d'(x', T'^n(\phi(y)) \leq \delta. \end{aligned}$$

Since Y has positive measure, (X, μ, T) cannot be W-measurably sensitive. ∎

PROPOSITION 6.3. *Let (X, μ, T) be a conservative and ergodic nonsingular dynamical system. T is W-measurably sensitive if and only if all measurably isomorphic dynamical systems (X', μ', T') admit no μ'-compatible metrics that are* 1*-Lipschitz.*

Proof. First we note that if a dynamical system (X', μ', T') admits a μ'-compatible 1-Lipschitz metric d', then this system could not be W-measurably sensitive, since for all integers n, $d'(T^n x, T^n y) \leq d'(x, y)$. Now, if a dynamical system (X, μ, T) is W-measurably sensitive, then every measurably isomorphic system (X', μ', T') will also be W-measurably sensitive, and therefore will not admit a μ'-compatible 1-Lipschitz metric d'.

For the converse, suppose (X, μ, T) is *not* W-measurably sensitive. By Lemma 5.3 there is a set $X_1 \subset X$ of full measure such that if T_1 is T restricted to X_1, and μ_1 is μ restricted to X_1, then d_T is a 1-Lipschitz μ_1-compatible metric on (X_1, μ_1, T_1). ∎

REMARK. If d is a μ-compatible metric on (X, μ), then X must be a separable metric space under d (by [JKL+08] and Proposition 2.1(3)), so X has at most the cardinality of the reals. A nonatomic (probability) *Lebesgue space* is defined as a measure space $(X, \mathcal{S}, \mu)$ that is isomorphic mod 0 to the unit interval I with Lebesgue measure λ, i.e., there exist sets of full measure $U \subset X$ and $U' \subset I$ such that there is a (measure-preserving) isomorphism from U to U'. However, there is no restriction on $X \setminus U$ other than being of μ-measure 0, and it could have cardinality greater than the reals. In this case X would admit no μ-compatible metric, and for instance, transformations on this space would be vacuously W-measurably sensitive. We introduce the following definition for Lebesgue spaces.

DEFINITION 6.1. Let (X, μ) be a Lebesgue space (or more generally a σ-finite measure space) and let T be a nonsingular transformation on (X, μ). A dynamical system $(X, \mathcal{S}, \mu, T)$ is *VW-measurably sensitive* if for every positively invariant measurable set of full measure set $U \subset X$, the system $(U, \mathcal{S}(U), \mu, T)$ is W-measurably sensitive.

REMARK. (1) By Lemma 6.1, on standard Borel spaces, the notions of W-measurable sensitivity and VW-measurable sensitivity are equivalent. Also, it follows from the definition that VW-measurable sensitivity is invariant under isomorphism.

(2) Here we note that nonsingular dynamical systems (on standard Borel spaces) (X, μ, T) do admit μ-compatible measures. In fact we know that if $(X, \mathcal{S})$ is a standard Borel space and μ is a continuous measure on $\mathcal{S}$, which we may assume a probability measure, then there exists a Borel isomorphism ϕ from $(X, \mathcal{S}, \mu)$ to the unit interval with Lebesgue measure $(I, \mathcal{B}, \lambda)$ (see e.g. [Sri98, 3.4.23]). Clearly the Euclidean distance d on I is a λ-compatible measure on $(I, \mathcal{L}, \lambda)$. Then d' defined by $d'(x, y) = d(\phi(x), \phi(y))$ is a μ-compatible metric on X.

7. Characterization of W-measurable sensitivity. We shall now prove our main result, that such a transformation is either W-measurably sensitive or measurably isomorphic to a minimal uniformly rigid isometry. This can be seen as a measurable version of the dichotomy theorem of Auslander and Yorke [AY80] for topological dynamical systems (continuous surjective maps on compact metric spaces), which states that a transitive map on a topological system is either sensitive or almost equicontinuous. Related topological dynamical results are in [GW93], [AG01] and the references therein.

THEOREM 1. *Let (X, μ, T) be a conservative and ergodic nonsingular dynamical system. Then T is either W-measurably sensitive, or isomorphic mod 0 to an invertible minimal uniformly rigid isometry on a Polish space.*

Proof. Suppose T is not W-measurably sensitive. Then, by Lemma 5.3, there exists a positively invariant set X_1 of full measure such that d_T is μ-compatible for the system (X_1, μ_1, T_1), where μ_1 is the restriction of μ to X_1, and T_1 the restriction of T to X_1. By Lemma 5.4, T_1 is transitive with respect to d_T. Since T_1 is 1-Lipschitz with respect to d_T, by Proposition 5.5, T_1 is a uniformly rigid minimal isometry on (X_1, d_T).

Now, let (X_2, d_2) be the topological completion of the metric space (X_1, d_T). Since d_T is separable, d_2 is also separable so (X_2, d_2) is Polish. We extend the measure μ_1 to X_2 by defining a set $S \subset X_2$ to be measurable if $S \cap X_1$ is measurable, with $\mu_2(S) = \mu_1(S \cap X)$. Since T_1 is an isometry, it is continuous on (X_1, d_T), so there is a unique way to extend it to a continuous transformation T_2 on (X_2, d_2). It is easy to verify that T_2 must also be an isometry with respect to d_2. It is invertible by Proposition 5.6.

Clearly, the dynamical system (X_2, μ_2, T_2) is measurably isomorphic to (X, μ, T). ■

Invertible examples of W-measurably sensitive transformations are mentioned in Section 3, but we have the following direct consequence of the theorem.

Corollary 7.1. *If a conservative and ergodic nonsingular transformation is not invertible a.e. then it cannot be isomorphic mod* 0 *to an invertible isometry, so it must be W-measurably sensitive.*

As a first application of Theorem 1, we show the following proposition.

Proposition 7.2. *If a dynamical system is W-measurably sensitive, then it is measurably sensitive. If a dynamical system is conservative ergodic and measurably sensitive, then it is W-measurably sensitive.*

Proof. First, suppose (X, μ, T) is a W-measurably sensitive nonsingular dynamical system. By Proposition 6.2, every isomorphic mod 0 dynamical system (X_1, μ_1, T_1) is also W-measurably sensitive. So, for any μ_1-compatible metric d_1 on X_1, there is a $\delta > 0$ such that for all $x \in X_1$, we have $\limsup_{n\to\infty} d(T^n x, T^n y) > \delta$ for almost all $y \in X_1$.

In particular,

$$\mu_1\{y \in B^{d_1}(x,\varepsilon) : \exists n > 0,\, d_1(T_1^n(x), T_1^n(y)) > \delta\} = \mu_1(B^{d_1}(x,\varepsilon)) > 0.$$

This implies that there is an $n > 0$ for which the set

$$\{y \in B^{d_1}(x,\varepsilon) : d_1(T_1^n(x), T_1^n(y)) > \delta\}$$

has positive measure. Thus (X, μ, T) is measurably sensitive.

To show the converse, suppose (X, μ, T) is a conservative and ergodic dynamical system that is not W-measurably sensitive. Then, by Theorem 1, there is, an isomorphic mod 0 dynamical system (X_1, μ_1, T_1) and a μ_1-compatible metric d_1 on X_1 that is an isometry. For all $\delta > 0$, choose any $\varepsilon < \delta$, and then for any $x \in X_1$ with $d_1(x, y) < \varepsilon$, for all integers n,

$d_1(T_1^n(x), T_1^n(y)) = d_1(x, y) < \delta$. So neither (X_1, μ_1, T_1) nor (X, μ, T) can be measurably sensitive. ∎

REMARK. Note that the assumption that the dynamical system is ergodic is crucial to the above statement. For example, as we mentioned in Section 3, no transformation can be W-measurably sensitive on a space with points of positive measure. Nonetheless, there are (nonergodic) transformations on such spaces which are measurably sensitive according to the definition in [JKL+08].

In the case when the measure space is finite (and a conservative transformation is measure-preserving), we can prove more.

THEOREM 2. *Let (X, μ, T) be a finite measure-preserving ergodic dynamical system. Then T is either W-measurably sensitive, or isomorphic to a minimal, uniformly rigid compact group rotation (i.e., a Kronecker transformation).*

Proof. We first show that X is a totally bounded space with respect to any μ-compatible metric d that is an isometry for T.

Let $\varepsilon > 0$. Let $C = \mu(B(x, \varepsilon/2))$ for some $x_0 \in X$. Since the metric is μ-compatible, $C > 0$. We claim that this is a constant independent of x. In fact, if we let $f(x) = \mu B(x, \varepsilon/2)$, then

$$f(Tx) = \mu(B(Tx, \varepsilon/2)) = \mu(T^{-1}B(x, \varepsilon/2)) = \mu(B(x, \varepsilon/2)) = f(x).$$

As f is continuous and T is transitive (Lemma 5.4), f is constant.

Now choose a largest possible collection of points $\{x_1, \dots, x_n\}$ such that the balls $B(x_i, \varepsilon/2)$ are all disjoint. Note that the size of any such collection will be no greater than $\mu(X)/C$, as they all have the same measure. By the triangle inequality, for any point $x \in X$, there must be an i such that $d(x, x_i) < \varepsilon$, as otherwise the ball $B(x, \varepsilon/2)$ would be disjoint from all the balls $B(x_i, \varepsilon/2)$. So $X = \bigcup_{i=1}^n B(x_i, \varepsilon)$. Since ε was arbitrary, X is totally bounded.

Now, as we have seen before, if T is not W-measurably sensitive, there exists a positively invariant set X_1 of full measure such that d_T is μ-compatible for the system (X_1, μ, T), and T is a minimal uniformly rigid isometry. Let (X_2, d_2) be the topological completion of the metric space (X_1, d). It is complete and totally bounded, and therefore compact. As before, we extend the measure μ to X_2, and T extends to a continuous transformation T_2 on (X_2, d_2) that is an isometry with respect to d_2.

Clearly, the dynamical system (X_2, μ_2, T_2) is measurably isomorphic to (X, μ, T), and T_2 is an ergodic isometry on the compact metric space X_2, as desired. Finally, every d_2-ball is measurable and contains a d-ball, so the metric d_2 is μ_2-compatible. ∎

REMARK. Theorems 1 and 2 also hold for VW-measurable sensitivity.

Appendix. Connections to pairwise sensitivity and other literature. In their paper [CJ05], Cadre and Jacob introduce the notion of *pairwise sensitivity*, which they define as follows. They only consider finite measure-preserving transformations, so we will also restrict to them in this appendix.

DEFINITION A.1. Let (X, μ) be a Lebesgue probability space and fix a metric d on X. An endomorphism T is said to be *pairwise sensitive* (with respect to initial conditions) if there exists $\delta > 0$ a (*sensitivity constant*) such that for $\mu^{\otimes 2}$-a.e. $(x, y) \in X \times X$, one can find $n \geq 0$ with $d(T^n x, T^n y) \geq \delta$.

Since this concept depends on the choice of the metric d, we will often refer to T as being *pairwise sensitive with respect to* d.

Cadre and Jacob prove that weakly mixing finite measure-preserving transformations are pairwise sensitive, and that a certain entropy condition implies pairwise sensitivity for ergodic transformations.

This notion is very closely related to the notion of W-measurable sensitivity, as the following proposition shows.

PROPOSITION A.1. *Let (X, μ, T) be a dynamical system and d be a μ-compatible metric on X. Then T is pairwise sensitive with respect to d if and only if it is W-measurably sensitive with respect to d.*

Proof. First suppose that the system is W-measurably sensitive with respect to d. Then there is a $\delta > 0$ such that for every $x \in X$ the set

$$Y_x = \{y \in X : \exists n,\, d(T^n x, T^n y) > \delta\}$$

has full measure. By Fubini's theorem (for the version we use, see [EG92]), the set $Y = \{(x, y) \in X \times X : \exists n,\, d(T^n x, T^n y) > \delta\}$ must have full $\mu^{\otimes 2}$-measure in $X \times X$. So T is pairwise sensitive with respect to d.

Now, suppose that the system is pairwise sensitive with respect to a μ-compatible metric d. That is, there is a $\delta > 0$ such that the set Y, defined as before, has full $\mu^{\otimes 2}$-measure in $X \times X$.

Take any $x \in X$. We claim that for almost every $y \in X$, there is an n such that $d(T^n x, T^n y) > \delta/2$. Once we have this claim, Proposition 4.2 implies that T is W-measurably sensitive with respect to d.

To prove the above claim, we need to show that the set $S_x = \{y \in X : \forall n,\, d(T^n x, T^n y) \leq \delta/2\}$ has measure zero. Take any $y_1, y_2 \in S_x$. By the triangle inequality, for all n we have $d(T^n y_1, T^n y_2) \leq \delta$. So the pair (y_1, y_2) does not belong to the set $Y \subset X \times X$. In other words, the Cartesian product $S_x \times S_x$ lies wholly inside the $\mu^{\otimes 2}$-measure-zero set $(X \times X) \setminus Y$. Again by

Fubini, this is only possible if the set S_x is measurable and has μ-measure zero. ∎

With this in mind, we see that our Theorem 2 implies the following result concerning pairwise sensitivity.

THEOREM 3. *Let (X, μ, T) be a nonatomic ergodic finite measure-preserving dynamical system. Suppose further that this dynamical system is not isomorphic mod 0 to a Kronecker transformation. Then, for any μ-compatible metric d, T is pairwise sensitive with respect to d.*

We do need to assume that the metric d is μ-compatible. However, while Cadre and Jacob never specify any restrictions on their metric, they also tacitly use several very similar properties. For example, they extensively use the notion of the *support* of a measure (i.e., the complement of the largest open set of zero measure), which is not well-defined without the assumptions that open and closed sets are measurable, and that the space is separable (if the space were not separable, the union of all open sets of measure zero may have positive measure even if measurable). Together, these two properties are almost sufficient to force the metric to be μ-compatible, as the following proposition shows.

PROPOSITION A.2. *A metric d on a measure space (X, μ) is μ-compatible if and only if the following three conditions are satisfied:*

(1) *Every d-ball is μ-measurable.*
(2) *The space X is separable under d.*
(3) *The support of μ is the whole of X.*

Proof. The fact that if d is μ-compatible then X is separable under d is shown in [JKL+08]. The other two properties are obvious.

Now, suppose that d satisfies the first two properties. Then the notion of support is well-defined. Clearly, the support of the measure is the whole space if and only if every nonempty open set has positive measure, i.e., if d is μ-compatible. ∎

According to Proposition A.2, to go from μ-compatible metrics to the metrics Cadre and Jacob use, we only need to require that the support of the measure is the whole space. This can always be achieved by removing a set of measure zero from the space.

With this assumption, Theorem 3 sharpens the results of [CJ05].

We also mention a recent work that we learned of from Ethan Akin after the research for this paper was completed. In [HLY11], Huang, Lu, and Ye introduce the notion of μ-sensitivity for topological dynamical systems and study its properties, and in particular show that it is equivalent to pairwise sensitivity [HLY11, 2.4]. We note that in [HLY11], the authors consider topological dynamical systems (continuous maps on compact metric spaces) and put invariant probability measures on them, while we consider measurable dynamical systems (nonsingular maps on standard Borel spaces) and put compatible metrics on them. We also note that one of the theorems of Huang, Lu, and Ye, [HLY11, Theorem 5.4], is related to our Theorem 2.

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