A LAW OF THE ITERATED LOGARITHM FOR GENERAL LACUNARY SERIES

BY

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Abstract. We prove a law of the iterated logarithm for sums of the form $\sum_{k=1}^{N} a_k f(n_k x)$ where the n_k satisfy a Hadamard gap condition. Here we assume that f is a Dini continuous function on $\mathbb{R}^n$ which has the property that for every cube Q of sidelength 1 with corners in the lattice $\mathbb{Z}^n$, f vanishes on ∂Q and has mean value zero on Q.

1. Introduction. It has long been appreciated that the partial sums of lacunary series exhibit many of the properties of sums of independent random variables. This is evidenced by many results in analysis which give central limit theorem type behavior or laws of the iterated logarithm (LILs) for lacunary series. In this paper we will show an analysis LIL. We recall the classical LIL of Kolmogorov [K]:

THEOREM 1.1. *Let* $S_m = \sum_{k=1}^{m} X_k$ *where* $\{X_k\}$ *is a sequence of real-valued independent random variables. Let* s_m *be the variance of* S_m. *Suppose* $s_m \to \infty$ *and*

$$|X_m|^2 \leq \frac{K_m s_m^2}{\log \log(e^e + s_m^2)}$$

for some sequence of constants $K_m \to 0$. *Then, almost surely,*

$$\limsup_{m \to \infty} \frac{S_m}{\sqrt{2 s_m^2 \log \log s_m}} = 1.$$

This was first proved for Bernoulli random variables by Khintchine. In [SZ2] Salem and Zygmund considered the case when the X_k are replaced by functions $a_k \cos n_k x$ on $[-\pi, \pi]$ and gave an upper bound (≤ 1) result; this was extended to the full upper and lower bound by Erdős and Gál [EG].

Takahashi [T1] extends the result of Salem and Zygmund: Consider a real measurable function f satisfying $f(x + 1) = f(x)$, $\int_0^1 f(x)\,dx = 0$, and suppose n_k is a lacunary sequence of integers, that is, there is a number q

[95]

such that

$$(1.1) \qquad \frac{n_{k+1}}{n_k} > q > 1$$

for every $k = 1, 2, \ldots$. Suppose that $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha \le 1$. Then

$$(1.2) \qquad \limsup_{N \to \infty} \frac{\sum_{k=1}^{N} f(n_k t)}{\sqrt{N \log \log N}} \le C \qquad \text{a.e.}$$

Several authors: Dhompongsa [D], Takahashi [T2], and Péter [P], have considered versions of this with a gap condition weaker than (1.1). Closely related is the central limit theorem for trigonometric series due to Salem and Zygmund [SZ1] and central limit theorems for more general lacunary sequences of Gaposhkin [G] and Aistleitner and Berkes [AB]. In this paper we will generalize the LIL of Takahashi (1.2). We will retain the gap condition (1.1) but broaden the class of functions f.

We need to introduce some notation and terminology. Throughout, a cube $Q \subseteq \mathbb{R}^n$ will be called *dyadic* if it has the form

$$Q = [k_1 2^l, (k_1 + 1)2^l] \times \cdots \times [k_n 2^l, (k_n + 1)2^l]$$

for some $l, k_1, \ldots, k_n \in \mathbb{Z}$; for such a cube Q we say that Q has *sidelength* 2^l and denote this as $\ell(Q) = 2^l$. We will use the notation $|Q|$ for the Lebesgue measure of Q.

For $m \in \mathbb{Z}$ we let $\mathcal{F}_m$ denote the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength 2^{-m}, and $\mathcal{F}$ will denote the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength ≤ 1. By a slight abuse of notation, we will also use $\mathcal{F}_m$ to denote the σ-field generated by the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength 2^{-m}. (The usage will be clear from the context.) For $x \in \mathbb{R}^n$ we also define $\mathcal{F}^x = \{Q + x : Q \in \mathcal{F}\}$ and $\mathcal{F}_m^x = \{Q + x : Q \in \mathcal{F}_m\}$.

DEFINITION 1.2. If f is a function on $\mathbb{R}^n$ we define the *modulus of continuity* ω of f as $\omega(f, \delta) = \sup\{|f(x) - f(y)| : |x - y| < \delta\}$. When f is clear from context, we will write $\omega(f, \delta) = \omega(\delta)$. We say that f is *Dini continuous* if

$$(1.3) \qquad \int_0^1 \frac{\omega(\delta)}{\delta} \, d\delta < \infty.$$

It is easy to see if the integral in (1.3) is finite, then $\int_0^c (\omega(\delta)/\delta) \, d\delta$ is finite for any $c > 0$. In this paper we will show:

THEOREM 1.3. *Suppose f is a Dini continuous function on $\mathbb{R}^n$ with the property that $f(x) = 0$ whenever any coordinate of x is an integer, and $\int_Q f(x) \, dx = 0$ whenever $Q \in \mathcal{F}_0$. Let (n_k) be a sequence of positive numbers satisfying the lacunarity condition $n_{k+1}/n_k \ge q > 1$ and (c_k) be a sequence in $\mathbb{R}^n$. Then there exists a constant C, depending only on n, q,*

and the quantity $\int_0^1 (\omega(\delta)/\delta)\,d\delta$, such that for any sequence (a_k) of numbers with $A_m = \sqrt{\sum_{k=1}^m |a_k|^2} \to \infty$ as $m \to \infty$, we have

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^m a_k f(n_k x + c_k)|}{\sqrt{A_m^2 \log\log A_m^2}} \le C \quad a.e.$$

Notice that we do not assume the n_k are integers, nor do we assume any periodicity of f. We do not know the best possible value of C in this inequality.

COROLLARY 1.4. *Suppose $f(x)$ is a Dini continuous function on $\mathbb{R}$ satisfying $f(x+1) = f(x)$ and $\int_0^1 f(x)\,dx = 0$. Then with n_k, a_k and c_k as in the theorem,*

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^m a_k f(n_k x + c_k)|}{\sqrt{A_m^2 \log\log A_m^2}} \le C \quad a.e.$$

Proof. The conditions on f imply that there exists a $c \in [0,1]$ with $f(c) = 0$. Then $f(c+m) = 0$ for every integer m. Consider $g(x) = f(x+c)$; this satisfies the hypotheses of the theorem. ∎

The proof of the theorem will use a reduction to dyadic martingales. This is not the first time such a theorem has been proved using martingale techniques (e.g. see Péter [P]), but the approach here is very different. In Section 2 we will collect some definitions and lemmas which will be used to prove the theorem in Section 3.

2. Some lemmas

LEMMA 2.1. *Let $n_1 < n_2 < \cdots$ be an infinite sequence of positive numbers satisfying the lacunarity condition $n_{k+1}/n_k \ge q > 1$, $k = 1, 2, \ldots$. If $0 < \alpha < \beta$ then*

$$(2.1) \qquad \sum_{\alpha \le n_k \le \beta} 1 \le \frac{\log(\beta q/\alpha)}{\log q}.$$

Proof. Let k_0 be defined by the inequality $n_{k_0} < \alpha \le n_{k_0+1}$ (put $n_0 = 0$) and $i \ge 0$ be defined by the inequality $n_{k_0+i} \le \beta < n_{k_0+i+1}$. If $i = 0$ then (2.1) is true. If $i \ge 1$ then we have $\beta \ge n_{k_0+i} \ge q^{i-1} n_{k_0+1} \ge q^{i-1}\alpha$. Hence $\beta q/\alpha \ge q^i$ and (2.1) follows immediately. ∎

LEMMA 2.2. *Suppose $k \ge 1$ and $2^{k-1} \le n_k < 2^k$. For any cube $J \subset \mathbb{R}^n$ with $\ell(J) = 1/n_k$, there exists a unique dyadic cube Q of sidelength $1/2^k$ which contains the center of J. Consequently, $J \subseteq \tilde{Q}$ where $\tilde{Q}$ is concentric with Q and $\ell(\tilde{Q}) = 3\ell(Q)$.*

Proof. Because the dyadic cubes of sidelength $1/2^k$ are disjoint and cover $\mathbb{R}^n$, there is a unique cube Q with $\ell(Q) = 1/2^k$ containing the center

of J. Let c_J and c_Q denote the centers of J and Q respectively. Then if $x \in J$,

$$|x - c_Q| \leq |x - c_J| + |c_J - c_Q| \leq \frac{\sqrt{n}}{2 \cdot 2^{k-1}} + \frac{\sqrt{n}}{2 \cdot 2^k} = \frac{3\sqrt{n}}{2 \cdot 2^k},$$

and hence $J \subset \tilde{Q}$. ∎

The following is from Chang, Wilson and Wolff [CWW], where we refer the reader for the proof.

LEMMA 2.3. *There is a positive integer N, $x_1, \ldots, x_N \in \mathbb{R}^n$ and disjoint subsets B^j of $\mathcal{F}$ such that*

$$\left\{ Q \in \mathcal{F} : \ell(Q) \leq \frac{1}{8} \right\} = \bigcup_{j=1}^{N} B^j$$

and

- *if $Q \in B^j$, then $\tilde{Q} \subseteq Q'$ for a unique $Q' \in \mathcal{F}^{x_j}$ with $\ell(Q') = 8\ell(Q)$;*
- *if $Q_1, Q_2 \in B^j$ and $Q_1 \neq Q_2$, then $Q_1' \neq Q_2'$.*

LEMMA 2.4. *Let J be a cube in $\mathbb{R}^n$ and let $\chi_J(x)$ denote the indicator function of J. Suppose f is a function which vanishes on ∂J. Then*

$$\sup_{|x-y| \leq \delta} |f(x)\chi_J(x) - f(y)\chi_J(y)| \leq \sup_{|x-y| \leq \delta} |f(x) - f(y)|.$$

Consequently, $\omega(\chi_J f, \delta) \leq \omega(f, \delta)$ and $\chi_J f$ is Dini continuous if f is.

Proof. Suppose $x, y \in \mathbb{R}^n$ with $|x - y| \leq \delta$. If $x \notin J$ and $y \notin J$, or if both $x, y \in J$, then we easily obtain $|f(x)\chi_J(x) - f(y)\chi_J(y)| \leq \omega(f, \delta)$. If $x \in J$ but $y \notin J$, then choose $z = tx + (1-t)y$, $t \in [0, 1]$, with $z \in \partial J$. Then $f(z) = 0$ whenever $|z - x| \leq \delta$, and so

$$|f(x)\chi_J(x) - f(y)\chi_J(y)| = |f(x) - 0| = |f(x) - f(z)| \leq \omega(f, \delta). ∎$$

LEMMA 2.5. *If f is Dini continuous then for any $c > 0$,*

$$\sum_{l=1}^{\infty} \omega(c2^{-l}) \leq 2 \int_0^c \frac{\omega(\delta)}{\delta} \, d\delta.$$

Proof. Indeed,

$$\int_0^c \frac{\omega(\delta)}{\delta} \, d\delta = \sum_{l=0}^{\infty} \int_{c/2^{l+1}}^{c/2^l} \frac{\omega(\delta)}{\delta} \, d\delta \geq \sum_{l=0}^{\infty} \int_{c/2^{l+1}}^{c/2^l} \frac{\omega(c/2^{l+1})}{c/2^l} \, d\delta = \frac{1}{2} \sum_{l=1}^{\infty} \omega(c2^{-l}). ∎$$

LEMMA 2.6. *Let Q be a dyadic cube in $\mathbb{R}^n$ and let $Q(l)$, $l = 1, \ldots, 2^n$, be the dyadic subcubes of Q obtained by bisecting the edges of Q. Suppose f*

is Dini continuous on Q with modulus of continuity ω. Then for each l,

$$\left| \frac{1}{|Q(l)|} \int_{Q(l)} f(y)\, dy - \frac{1}{|Q|} \int_Q f(y)\, dy \right| \le \omega(\sqrt{n}\, \ell(Q)).$$

Proof. Without loss of generality take $l = 1$. Then

$$\left| \frac{1}{|Q(1)|} \int_{Q(1)} f(y)\, dy - \frac{1}{|Q|} \int_Q f(y)\, dy \right|$$

$$= \left| \frac{1}{|Q(1)|} \int_{Q(1)} f(y)\, dy - \sum_{k=1}^{2^n} \frac{1}{2^n |Q(k)|} \int_{Q(k)} f(y)\, dy \right|$$

$$= \left| \frac{1}{2^n |Q(1)|} \sum_{k=1}^{2^n} \int_{Q(1)} f(y)\, dy - \int_{Q(k)} f(y)\, dy \right| \le \omega(\sqrt{n}\, \ell(Q)). \blacksquare$$

DEFINITION 2.7. Suppose $Q \in \mathcal{F}_0$. A *dyadic martingale* on Q is a sequence $\{g_m\}_{m=0}^\infty$ of integrable functions on Q such that each g_m is $\mathcal{F}_m$ measurable and $g_m = E(g_{m+1}|\mathcal{F}_m)$ for every m. Here $E(g_{m+1}|\mathcal{F}_m)$ denotes the conditional expectation: $E(g_{m+1}|\mathcal{F}_m)(x) = (1/|Q|) \int_Q g_{m+1}\, dy$ if $x \in Q \in \mathcal{F}_m$. For $k \ge 1$, set $d_k = g_k - g_{k-1}$, and we also define the square function $Sf_m = (\sum_{k=1}^m E(d_k^2|\mathcal{F}_{k-1}))^{1/2}$.

LEMMA 2.8 (Upper half LIL for dyadic martingales). *If f_m is a dyadic martingale on Q then*

$$\limsup_{m \to \infty} \frac{|f_m|}{\sqrt{2(Sf_m)^2 \log\log(Sf_m)}} \le 1$$

almost surely on the set where $S(f_m) \to \infty$.

Lemma 2.8 is a special case of a much more general martingale LIL due to Stout [S]. We only need this version, which is much simpler to show. (See [CWW, Corollary 3.2], for a proof.)

3. The proof of the theorem. According to Lemma 2.1, we can assume that for each $k \ge 1$, there exists exactly one n_k with $2^{k-1} \le n_k < 2^k$. We may also assume that $a_1 = a_2 = 0$. For $m \ge 1$, let $f_m(x) := \sum_{k=3}^{m+2} a_k f(n_k x + c_k)$.

For $k = 1, 2, \ldots$, define $\mathcal{G}_k$ as the set of cubes in $\mathbb{R}^n$ of the form

$$\left[\frac{-c_{k1} + l_1}{n_k}, \frac{-c_{k1} + l_1 + 1}{n_k} \right) \times \cdots \times \left[\frac{-c_{kn} + l_n}{n_k}, \frac{-c_{kn} + l_n + 1}{n_k} \right),$$

where $c_k = (c_{k1}, \ldots, c_{kn})$, and $l_1, \ldots, l_n$ are in $\mathbb{Z}$. Then $f(n_k x + c_k)$ vanishes on ∂J for each $J \in \mathcal{G}_k$. Note that $\mathbb{R}^n$ is covered by a disjoint union of cubes in $\mathcal{G}_k$.

For a cube $Q \in \mathcal{F}_k$, of sidelength $\ell(Q) = 1/2^k$, define

(3.1)
$$\lambda_Q(x) = \begin{cases} a_k f(n_k x + c_k) \chi_J(x) & \text{if } Q \text{ contains the center of a cube } J \in \mathcal{G}_k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that each $Q \in \mathcal{F}_k$ contains the center of at most one $J \in \mathcal{G}_k$ and that some cubes $Q \in \mathcal{F}_k$ may not contain the center of any cube in $\mathcal{G}_k$, in which case $\lambda_Q = 0$. By Lemma 2.2, supp $\lambda_Q \subseteq \tilde{Q}$. Apply Lemma 2.3 to decompose $\mathcal{F}$ into the disjoint families B^j.

For $1 \leq j \leq N$, and for each $Q \in \mathcal{F}^{x_j}$, let

$$f_Q^{(j)}(x) = \begin{cases} \lambda_{Q_0}(x) & \text{if } Q = Q_0' \text{ for some } Q_0 \in B^j, \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $Q \in \mathcal{F}^{x_j}$,

(3.2)
$$\text{supp } f_Q^{(j)} \subseteq Q$$

and

(3.3)
$$\int_Q f_Q^{(j)}(x)\, dx = 0.$$

We then define

(3.4)
$$\Lambda_m^{(j)}(x) = \sum_{\substack{Q \in B^j \\ 2^{-m-2} \leq \ell(Q) \leq 2^{-3}}} \lambda_Q(x) = \sum_{\substack{Q \in \mathcal{F}^{x_j} \\ 2^{-m+1} \leq \ell(Q) \leq 1}} f_Q^{(j)}(x),$$

so that with this notation

(3.5)
$$f_m(x) = \sum_{j=1}^N \Lambda_m^{(j)}(x) = \sum_{j=1}^N \sum_{\substack{Q \in B^j \\ 2^{-m-2} \leq \ell(Q) \leq 2^{-3}}} \lambda_Q(x).$$

Define dyadic martingales $g^{(j)} = \{g_m^{(j)}\}_{m=0}^\infty$ by $g_m^{(j)} = E(\Lambda_m^{(j)} | \mathcal{F}_m^{x_j})$, $m \geq 1$, and $g_0^{(j)} = 0$. To see that $g^{(j)}$ is a martingale, note that

$$E(g_{m+1}^{(j)} | \mathcal{F}_m^{x_j}) = E(\Lambda_{m+1}^{(j)} | \mathcal{F}_m^{x_j}) = E(\Lambda_m^{(j)} | \mathcal{F}_m^{x_j}) + \sum_{Q \in \mathcal{F}^{x_j} : \ell(Q) = 2^{-m}} E(f_Q^{(j)} | \mathcal{F}_m^{x_j})$$

and the terms in the sum vanish due to (3.2) and (3.3). This is a small abuse of terminology, because the $g^{(j)}$ are defined on all of $\mathbb{R}^n$, which is not a probability space. However, the restriction of $g^{(j)}$ to each cube $Q \in \mathcal{F}^{x_j}$ of sidelength 1 is a martingale on the probability space Q, and $\mathbb{R}^n$ can be exhausted by a countable number of such cubes.

For $x \in \mathbb{R}^n$, let us denote by $Q_m^{x_j}(x)$ the unique dyadic cube of sidelength 2^{-m} in $\mathcal{F}^{x_j}$ containing x. Then, using (3.5), the definition of the $g^{(j)}$,

and (3.4), we have

$$\Big| f_m(x) - \sum_{j=1}^{N} g_m^{(j)}(x) \Big| \leq \sum_{j=1}^{N} \sum_{\substack{Q \in \mathcal{F}^{x_j} \\ 2^{-m+1} \leq \ell(Q) \leq 1}} |f_Q^{(j)}(x) - E(f_Q^{(j)}|\mathcal{F}_m^{x_j})(x)|$$

$$\leq \sum_{j=1}^{N} \sum_{\substack{Q \in \mathcal{F}^{x_j} \\ 2^{-m+1} \leq \ell(Q) \leq 1}} \frac{1}{|Q_m^{x_j}(x)|} \int_{Q_m^{x_j}(x)} |f_Q^{(j)}(x) - f_Q^{(j)}(y)| \, dy.$$

If $\ell(Q) = 2^{-k}$, $k \leq m-1$, and $y \in Q_m^{x_j}(x)$, then by the definition of $f_Q^{(j)}$, and that of λ_Q in (3.1), and by Lemma 2.4,

$$|f_Q^{(j)}(x) - f_Q^{(j)}(y)| \leq |a_{k+3}|\omega(n_{k+3}\sqrt{n}\,\ell(Q_m^{x_j}(x))).$$

Thus,

$$\Big| f_m(x) - \sum_{j=1}^{N} g_m^{(j)}(x) \Big| \leq \sum_{j=1}^{N} \sum_{k=0}^{m-1} |a_{k+3}|\omega(n_{k+3}\sqrt{n}\,\ell(Q_m^{x_j}(x)))$$

$$\leq \sum_{j=1}^{N} \sum_{k=3}^{m+2} |a_k|\omega\Big(\sqrt{n}\,\frac{2^k}{2^m}\Big) = N \sum_{k=3}^{m+2} |a_k|\omega\Big(8\sqrt{n}\,\frac{2^{k-3}}{2^m}\Big)$$

$$\leq N \Big(\sum_{k=3}^{m+2} |a_k|^2\Big)^{1/2} \Big(\sum_{k=3}^{m+2} \omega\Big(8\sqrt{n}\,\frac{2^{k-3}}{2^m}\Big)^2\Big)^{1/2}$$

$$= C A_{m+2},$$

where for the last inequality we have used Lemma 2.5.

We now estimate the square functions of the martingales $g_k^{(j)}$. For $1 \leq j \leq N$, let $d_k^{(j)} = |g_k^{(j)} - g_{k-1}^{(j)}|$, $k = 1, 2, \ldots$. Then, using Lemma 2.6,

$$|d_k^{(j)}(x)| = |E(\Lambda_k^{(j)}|\mathcal{F}_k^{x_j})(x) - E(\Lambda_{k-1}^{(j)}|\mathcal{F}_{k-1}^{x_j})(x)|$$

$$= |E(\Lambda_k^{(j)}|\mathcal{F}_k^{x_j})(x) - E(\Lambda_k^{(j)}|\mathcal{F}_{k-1}^{x_j})(x)|$$

$$= \Big| \frac{1}{|Q_k^{x_j}(x)|} \int_{Q_k^{x_j}(x)} \Lambda_k^{(j)}(y) \, dy - \frac{1}{|Q_{k-1}^{x_j}(x)|} \int_{Q_{k-1}^{x_j}(x)} \Lambda_k^{(j)}(y) \, dy \Big|$$

$$\leq \sum_{\substack{Q \in \mathcal{F}^{x_j} \\ 2^{-k+1} \leq \ell(Q) \leq 1}} \Big| \frac{1}{|Q_k^{x_j}(x)|} \int_{Q_k^{x_j}(x)} f_Q^{(j)}(y) \, dy - \frac{1}{|Q_{k-1}^{x_j}(x)|} \int_{Q_{k-1}^{x_j}(x)} f_Q^{(j)}(y) \, dy \Big|$$

$$\leq \sum_{l=0}^{k-1} |a_{l+3}|\omega(n_{l+3}\sqrt{n}\,\ell(Q_k^{x_j}(x))) \leq \sum_{l=3}^{k+2} |a_l|\omega\Big(\sqrt{n}\,\frac{2^l}{2^k}\Big)$$

$$\leq \left(\sum_{l=3}^{k+2}|a_l|^2\omega\left(8\sqrt{n}\,\frac{2^{l-3}}{2^k}\right)\right)^{1/2}\left(\sum_{l=3}^{k+2}\omega\left(8\sqrt{n}\,\frac{2^{l-3}}{2^k}\right)\right)^{1/2}$$

$$\leq M\left(\sum_{l=3}^{k+2}|a_l|^2\omega\left(8\sqrt{n}\,\frac{2^{l-3}}{2^k}\right)\right)^{1/2}.$$

Then

$$(Sg_m^{(j)}(x))^2 = \sum_{k=1}^{m} E((d_k^{(j)})^2|\mathcal{F}_{k-1}) \leq M^2 \sum_{k=1}^{m}\sum_{l=3}^{k+2}|a_l|^2\omega\left(8\sqrt{n}\,\frac{2^{l-3}}{2^k}\right)$$

$$\leq M^2 \sum_{l=3}^{m+2}|a_l|^2 \sum_{k=l-2}^{m} \omega\left(8\sqrt{n}\,\frac{2^{l-3}}{2^k}\right) \leq M^2 M \sum_{l=3}^{m+2}|a_l|^2$$

$$= M^3 A_{m+2}^2.$$

Therefore,

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^{m+2}a_k f(n_k x + c_k)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}}$$

$$\leq \limsup_{m\to\infty} \frac{|f_m(x) - \sum_{j=1}^{N} g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} + \limsup_{m\to\infty} \frac{\sum_{j=1}^{N}|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}}$$

$$\leq \limsup_{m\to\infty} \frac{C}{\sqrt{\log\log A_{m+2}^2}} + \sum_{j=1}^{N}\limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}}$$

$$= \sum_{j=1}^{N}\limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}}.$$

For j fixed,

$$\limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{(Sg_m^{(j)}(x))^2 \log\log(Sg_m^{(j)}(x))^2}} \leq \sqrt{2}$$

almost surely on the set $\{Sg_m^{(j)}(x) \to \infty\}$ by Lemma 2.8. But then, for such x, $(Sg_m^{(j)}(x))^2 \leq M^3 A_{m+2}^2$ and hence

$$\limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} \leq C$$

almost surely on this set. Because $\{Sg_m^{(j)}(x)$ is bounded$\} = \{|g_m^{(j)}(x)|$ is

bounded} almost surely (see [BG]),

$$\limsup_{m\to\infty} \frac{|g_m^{(j)}(x)|}{\sqrt{A_{m+2}^2 \log\log A_{m+2}^2}} = 0$$

almost surely on the set $\{Sg_m^{(j)}(x)$ is bounded$\}$ and we obtain the conclusion of the theorem. ∎

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