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ON THE BROCARD–RAMANUJAN PROBLEM AND GENERALIZATIONS

BY

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Abstract. Let p_i denote the ith prime. We conjecture that there are precisely 28 solutions to the equation $n^2 - 1 = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ in positive integers n and $\alpha_1,\dots,\alpha_k$. This conjecture implies an explicit description of the set of solutions to the Brocard–Ramanujan equation. We also propose another variant of the Brocard–Ramanujan problem: describe the set of solutions in non-negative integers of the equation $n! + A = x_1^2 + x_2^2 + x_3^2$ (A fixed).

1. Introduction. Finding all the positive integer solutions of the equation

$$n! + 1 = y^2 \tag{1}$$

is a famously unsolved problem [6] (the *Brocard–Ramanujan problem*). It is expected that the only solutions are $(y, n) = (5, 4), (11, 5), (71, 7)$. Recent computations by Berndt and Galway [2] showed that there are no other solutions in the range $n < 10^9$. Overholt [12] showed that a weak form of the ABC conjecture implies that (1) has only finitely many solutions.

Dąbrowski [5] showed that if A is a fixed non-zero integer, then the diophantine equation $n! + A = y^2$ has only finitely many solutions if A is not a square, and a weak form of the ABC conjecture implies that it has finitely many solutions if A is a square. Berndt and Galway [2] looked for solutions of

$$n! + B^2 = y^2 \tag{2}$$

for $2 \le B \le 50$, up to $n \le 10^5$, and found either zero or one solution in each case. Ulas [14] found 27 values of B for which (2) has at least two solutions. Luca [8] considered a general diophantine equation $P(y) = n!$, where $P(y)$ is a polynomial of degree ≥ 2 with integer coefficients.

The article by Berend and Harmse [1] contains several related problems, results and bibliography on more general equations of the type $P(x) = H_n$, where $P(x)$ is any polynomial of degree ≥ 2 with integer coefficients, and (H_n) is a 'highly divisible sequence'. Examples of highly divisible sequences include $H_n = n!$ and $H_n = p_1 \cdots p_n$, where $p_1 < p_2 < \cdots$ is the sequence of all primes.

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We propose two generalizations of the Brocard–Ramanujan problem. We conjecture that there are exactly 28 positive integers n such that the set of prime divisors of n^2-1 equals $\{p_1,\dots,p_k\}$, where p_m denotes the mth prime number. This implies, in particular, a conjectural description of all solutions to (2). Another interesting problem is to describe, for fixed positive integer A, the set of solutions of the equation $n!+A=x^2+y^2+z^2$ in non-negative integers n,x,y,z; the same for the equation $n!+A=x^2+y^2$.

2. The conjecture

2.1. Formulation of the conjecture. Let p_n denote the nth prime number. E. Lucas [10] has proved, in particular, that the equation $x^k\pm 1 = p_1\cdots p_n$ $(k\ge 2)$ has no integer solutions. We propose the following

CONJECTURE 1. *The diophantine equation*

$$y^2-1=p_1^{\alpha_1}\cdots p_k^{\alpha_k} \tag{3}$$

has exactly 28 *solutions* $(y;\alpha_1,\dots,\alpha_k)$ *in positive integers*:

(a) $(3;3)$,
(b) $(5;3,1)$, $(7;4,1)$, $(17;5,2)$,
(c) $(11;3,1,1)$, $(19;3,2,1)$, $(31;6,1,1)$, $(49;5,1,2)$, $(161;6,4,1)$,
(d) $(29;3,1,1,1)$, $(41;4,1,1,1)$, $(71;4,2,1,1)$, $(251;3,2,3,1)$, $(449;7,2,2,1)$, $(4801;7,1,2,4)$, $(8749;3,7,4,1)$,
(e) $(769;9,1,1,1,1)$, $(881;5,2,1,2,1)$, $(1079;4,3,1,2,1)$, $(6049;6,3,2,1,2)$, $(19601;5,4,2,2,2)$,
(f) $(3431;4,1,1,3,1,1)$, $(4159;7,3,1,1,1,1)$, $(246401;8,6,2,1,1,2)$,
(g) $(1429;3,1,1,1,1,1,1)$, $(24751;5,2,3,1,1,1,1)$, $(388961;6,4,1,4,1,1,1)$,
(h) $(1267111;4,3,1,1,3,1,1,2)$.

REMARK. (i) The conjecture concerns the diophantine equation $y^2-1 = H_n$, where the sequence (H_n) is not highly divisible.

(ii) We do not know whether the ABC conjecture implies that (3) has only finitely many solutions in positive integers.

(iii) The conjecture seems unattackable at present.

Elementary considerations combined with the results of Bugeaud, Mignotte and Siksek [4] allowed us to check that the 21 solutions listed in (a)–(e) cover all the solutions to (3) satisfying $k\le 5$. The idea was to consider multi-parameter families of diophantine equations $q_1^{\alpha_1}\cdots q_s^{\alpha_s}-q_{s+1}^{\alpha_{s+1}}\cdots q_t^{\alpha_t}=1$, where $q_1,\dots,q_t$ are fixed primes, and use Theorems 1–3 from [4].

For example, $y^2-1=2^\alpha 3^\beta 5^\gamma$ implies $2^{\alpha-2}-3^\beta 5^\gamma=\pm1$ or $3^\beta-2^{\alpha-2}5^\gamma=\pm1$ or $5^\gamma-2^{\alpha-2}3^\beta=\pm1$. Reducing the equation $2^{\alpha-2}-3^\beta 5^\gamma=-1$ modulo 3 and modulo 5 implies that it has no solution. On the other hand, reducing the equation $2^{\alpha-2}-3^\beta 5^\gamma=1$ modulo 3 and modulo 5 implies that $\alpha=2t$, hence $(\alpha, \beta, \gamma) = (6, 1, 1)$ and $y = 31$. If $\gamma = 1$ or 2, then the equation $3^\beta - 2^{\alpha-2}5^\gamma = \pm 1$ has two solutions: $3^4 - 2^4 \cdot 5 = 1$, and $3^2 - 2 \cdot 5 = -1$. For $\gamma \geq 3$ we can consider the equation $3^\beta x^n - 2^{\alpha-2}y^n = \pm 1$ and use [4, Theorem 1]. Similarly, the remaining case leads to the solutions $5^2 - 2^3 \cdot 3 = 1$ and $5 - 2 \cdot 3 = -1$.

The cases $k = 4$ and 5 are basically treated in the same way. The actual calculations however are much longer and (in some cases) complicated. Let us give some details.

The case $k = 4$ leads to considering the following seven equations:

(i) $2^{\alpha-2} - 3^\beta 5^\gamma 7^\delta = \pm 1$,
(ii) $3^\beta - 2^{\alpha-2}5^\gamma 7^\delta = \pm 1$,
(iii) $5^\gamma - 2^{\alpha-2}3^\beta 7^\delta = \pm 1$,
(iv) $7^\delta - 2^{\alpha-2}3^\beta 5^\gamma = \pm 1$,
(v) $2^{\alpha-2}3^\beta - 5^\gamma 7^\delta = \pm 1$,
(vi) $2^{\alpha-2}5^\gamma - 3^\beta 7^\delta = \pm 1$,
(vii) $2^{\alpha-2}7^\delta - 3^\beta 5^\gamma = \pm 1$.

In cases (iii), (iv) and (v) we can use [4] immediately. In the remaining cases we are led to considering additional equations, and then use [4]. For instance, the equation $2^{\alpha-2} - 3^\beta 5^\gamma 7^\delta = -1$ has no solution (reduce modulo 3 and modulo 5). On the other hand, the equation $2^{\alpha-2} - 3^\beta 5^\gamma 7^\delta = 1$ reduces to considering the following six equations (note that α has to be even): $2^m - 3^\beta = \pm 1$, $2^m - 5^\gamma = \pm 1$, … ,$2^m - 5^\gamma 7^\delta = \pm 1$.

The case $k = 5$ leads to considering the following 15 equations: $2^{\alpha-2} - 3^\beta 5^\gamma 7^\delta 11^\epsilon = \pm 1, \ldots, 7^\delta 11^\epsilon - 2^{\alpha-2}3^\beta 5^\gamma \pm 1$. Let us illustrate the method for the first equation (one of the easiest). The equation $2^{\alpha-2} - 3^\beta 5^\gamma 7^\delta 11^\epsilon = -1$ has no solution. The equation $2^{\alpha-2} - 3^\beta 5^\gamma 7^\delta 11^\epsilon = 1$ leads to considering the following 14 equations: $2^t - 3^\beta = \pm 1, \ldots, 2^t - 7^\delta 11^\epsilon = \pm 1$, $2^t - 3^\beta 5^\gamma 7^\delta = \pm 1$, $2^t - 3^\beta 5^\gamma 11^\epsilon = \pm 1$, $2^t - 3^\beta 7^\delta 11^\epsilon = \pm 1$, $2^t - 5^\gamma 7^\delta 11^\epsilon = \pm 1$. For the first ten equations we can use [4]. Each of the remaining four equations leads to considering six equations, for which we can apply [4].

We can also use the MWRANK program and the MAGMA system. Let me illustrate the method for the equation $5^c - 2^a 3^b 7^d 11^e = -1$. Note that necessarily $a = 1$. Reducing modulo 3 we obtain c odd, reducing modulo 7 we obtain $c = 6m + 3$, and reducing modulo 5 we obtain $b + d$ odd. We may therefore consider four diophantine equations:

(i) $x^3 - 2 \cdot 7y^2 = -1$. Using MAGMA we conclude that the model $Y^2 = X^3 + 14^3$ has the following integral points: $(-7, \pm 49)$, $(70, \pm 588)$, $(-14, 0)$.
(ii) $x^3 - 2 \cdot 7 \cdot 11y^2 = -1$. Using MAGMA we conclude that the model $Y^2 = X^3 + 11^3 \cdot 14^3$ has the following integral points: $(330, \pm 6292)$, $(-7, \pm 1911)$, $(-154, 0)$.

(iii) $x^3 - 2 \cdot 3y^2 = -1$. Using MWRANK we conclude that the rank of $Y^2 = X^3 + 6^3$ is zero, hence it has only one integral point $(-6, 0)$.

(iv) $x^3 - 2 \cdot 3 \cdot 11y^2 = -1$. Using MWRANK we conclude that the rank of $Y^2 = X^3 + 66^3$ is zero, hence it has only one integral point $(-66, 0)$.

Collecting all this information we conclude that $5^c - 2^a 3^b 7^d 11^e = -1$ has no solution in positive integers a, b, c, d, e.

M. Wieczorek searched numerically for solutions of (3) up to $y \leq 10^8$ and found additional seven solutions, listed in (f)–(h) above.

F. Luca and F. Najman [9] have computed all the solutions to the inequality $P(x^2 - 1) < 100$, where $P(n)$ is the largest prime factor of n. Their calculations confirm our conjecture for $k \leq 25$.

2.2. Some applications. We give two applications of the conjecture.

(i) *Exact description of the set of solutions to* (2). Conjecture 1 gives the exact description of the set of solutions to (2). For example:

$B = 1$: $(y, n) = (5, 4), (11, 5), (71, 7)$;
$B = 2$: no solution;
$B = 3$: $(y, n) = (27, 6)$;
$B = 4$: no solution.

As a consequence, for $2 \leq B \leq 50$, we should obtain at most one solution. This is in agreement with the calculations by Berndt and Galway mentioned in the Introduction.

Remark. Such a description also follows from (a weak) effective version of the ABC conjecture formulated by Browkin [3].

(ii) *Application to ternary diophantine equations.* Consider the equation $ax^p + by^p = cz^p$, where $p \geq 5$ is a prime number, and a, b, c are pairwise coprime integers. Assume that the integers $a+b$, $a-b$ and $b-a$ do not belong to $c\mathbb{Z}^2$. Ivorra and Kraus [7] conjecture that in this case there exists a constant $f(a, b, c)$ such that for $p > f(a, b, c)$ the above equation has no primitive solutions (x, y, z) ((x, y, z) is primitive means $xyz \neq 0$ and $\gcd(x, y, z) = 1$). Let p_r denote the rth prime number. Assume that $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ (with positive integers $\alpha_1, \ldots, \alpha_r$) do not belong to the set of 28 numbers described by Conjecture 1. Combining the conjecture formulated by Ivorra and Kraus and our Conjecture 1, we obtain the following

Conjecture 2. *There exists a positive constant* $f = f(p_1, \ldots, p_r; \alpha_1, \ldots, \alpha_r)$ *such that for any prime* $p > f$ *the diophantine equation*

$$x^p + p_1^{\alpha_1} \cdots p_r^{\alpha_r} y^p = z^2$$

has no primitive solutions.

2.3. Generalizations. We give two variants of Conjecture 1.

(i) *Variant for primitive polynomials* $P(y) \in \mathbb{Z}[y]$. One can formulate a variant of Conjecture 1 for a general primitive polynomial $P(y)$ of degree ≥ 2 with integer coefficients, different from $(x+a)^n$. Of course, there are polynomials for which such a conjecture can easily be verified: take, for instance, $P(y) = y^{2m} + 1$. As an application, one can give an explicit description of the set of solutions to the diophantine equation $P(y) = n!$ studied in [8].

(ii) *Variant for the Fibonacci sequence.* Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence. Marques [11] proved the Fibonacci version of the Brocard–Ramanujan problem: the diophantine equation $F_1F_2\cdots F_n + 1 = F_m^2$ has no solution in positive integers n, m.

Question. Describe the set of solutions $(m; \alpha_1, \dots, \alpha_k)$ in positive integers of the diophantine equation $F_m^2 - 1 = F_{p_1^{\alpha_1}} \cdots F_{p_k^{\alpha_k}}$.

3. Sums of squares in the sequence $n! + A$. It is well known (Lagrange's theorem) that every non-negative integer is representable as a sum of four squares of integers. We propose the following variants of the Brocard–Ramanujan problem.

3.1. Sums of three squares. Let us recall the following result from elementary theory of numbers (see, for instance, [13, Chapter 11,4]): A natural number n is the sum of three squares of integers if and only if it is not of the form $4^l(8k+7)$, where k, l are non-negative integers. Using this criterion, we immediately see that $n! + 1$ is a sum of three squares for any $n \neq 3$. In particular, the set of non-negative integer solutions of the equation $n! + 1 = x^2 + y^2 + z^2$ is infinite. More generally, we obtain

Theorem.

(a) *Assume* $4 \nmid A$ *and* $n \geq 4$. *Then* $n! + A$ *is a sum of three squares if and only if* $A \equiv 1, 2 \bmod 4$ *or* $A \equiv 3 \bmod 8$.
(b) *Assume* $A = 8k + 4$ *and* $n \geq 8$. *Then* $n! + A$ *is a sum of three squares if and only if* $k \neq 4m + 3$.
(c) *Assume* $A = 8k$ *and* $v_2(n!) \geq 3 + v_2(A)$. *Then* $n! + A$ *is a sum of three squares if and only if* $k \neq 2^{2l-3}(8m+7)$.

Question. *Describe the set of non-negative integer solutions of the equation* $n! + A = x^2 + y^2 + z^2$ *(A fixed).*

3.2. Sums of two squares. Here we have the following criterion [13]: A natural number n is a sum of two squares of integers if and only if all prime factors of n of the form $4m+3$ have even exponent in the prime factorization of n. Using this criterion, we find that $n! + A$ is not a sum of two squares if $n \geq 4$ and $A \equiv 3, 6, 7 \bmod 8$. The remaining cases are more difficult to deal with. Let us consider the sequence $n!+1$, and write down all $n \leq 50$ such that $n!+1$ is a sum of two squares: $1, 4, 5, 7, 8, 11, 12, 17, 25, 26, 27, 28, 29, 37, 38,$ 41, 48.

QUESTION. *Is the set of integer solutions of the equation* $n!+1 = x^2+y^2$ *infinite?*

QUESTION. *Describe the set of non-negative integer solutions of the equation* $n!+A = x^2+y^2$ *(A fixed).*

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