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TWISTED GROUP RINGS OF STRONGLY UNBOUNDED REPRESENTATION TYPE

BY

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Abstract. Let S be a commutative local ring of characteristic p, which is not a field, S^* the multiplicative group of S, W a subgroup of S^*, G a finite p-group, and $S^\lambda G$ a twisted group ring of the group G and of the ring S with a 2-cocycle $\lambda \in Z^2(G,S^*)$. Denote by $\mathrm{Ind}_m(S^\lambda G)$ the set of isomorphism classes of indecomposable $S^\lambda G$-modules of S-rank m. We exhibit rings $S^\lambda G$ for which there exists a function $f_\lambda : \mathbb{N} \to \mathbb{N}$ such that $f_\lambda(n) \geq n$ and $\mathrm{Ind}_{f_\lambda(n)}(S^\lambda G)$ is an infinite set for every natural $n > 1$. In special cases $f_\lambda(\mathbb{N})$ contains every natural number $m > 1$ such that $\mathrm{Ind}_m(S^\lambda G)$ is an infinite set. We also introduce the concept of projective (S,W)-representation type for the group G and we single out finite groups of every type.

Introduction. Let $p \geq 2$ be a prime. A finite group whose order is a positive power of p is called a *p-group*. Suppose G is a p-group, G' the commutant of G, $\mathrm{rad}\, A$ the Jacobson radical of a ring A, $\overline{A} = A/\mathrm{rad}\, A$ the factor ring of the ring A by $\mathrm{rad}\, A$, S a commutative local ring with an identity element of characteristic p^k, $S^p = \{a^p : a \in S\}$, S^* the multiplicative group of S, and $Z^2(G,S^*)$ the group of all S^*-valued normalized 2-cocycles of the group G that acts trivially on S^*. A *twisted group ring* $S^\lambda G$ of the group G and of the ring S with $\lambda \in Z^2(G,S^*)$ is the S-algebra with S-basis $\{u_g : g \in G\}$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$ ([31, pp. 2–4]). Let e be the identity element of G. We have $u_a u_e = u_e u_a = u_a$ for all $a \in G$. The S-basis $\{u_g : g \in G\}$ of $S^\lambda G$ will be called *natural*. If H is a subgroup of G, then the restriction of a cocycle $\lambda : G \times G \to S^*$ to $H \times H$ will also be denoted by λ. In this case $S^\lambda H$ is a subring of $S^\lambda G$. By an *$S^\lambda G$-module* we mean a finitely generated left $S^\lambda G$-module which is S-free, that is, an $S^\lambda G$-lattice (see [10, p. 140]). The study of S-representations of $S^\lambda G$ is essentially equivalent to the study of $S^\lambda G$-modules (see [9, §10]; [12, p. 74]). The module corresponding to a representation is called the underlying module of that representation ([12, p. 74]).

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Following the terminology of [26], we say that $S^\lambda G$ is of *finite* (resp. *infinite*) *representation type* if the set of all isomorphism classes of indecomposable $S^\lambda G$-modules is finite (resp. infinite). Let $D(S^\lambda G)$ be the set of S-ranks of all indecomposable $S^\lambda G$-modules. If $D(S^\lambda G)$ is finite (resp. infinite), then $S^\lambda G$ is of *bounded* (resp. *unbounded*) *representation type*. Let $\mathrm{Ind}_d(S^\lambda G)$ be the set of isomorphism classes of indecomposable $S^\lambda G$-modules of S-rank d and let $\mathbb{N}$ be the set of positive integers. We say that $S^\lambda G$ is of *SUR-type* (Strongly Unbounded Representation type) if there exists a function $f_\lambda : \mathbb{N} \to \mathbb{N}$ such that $f_\lambda(n) \geq n$ and $\mathrm{Ind}_{f_\lambda(n)}(S^\lambda G)$ is an infinite set for every $n > 1$. A function f_λ will be called an *SUR-dimension-valued function*.

Higman [25] proved that if S is a field of characteristic p, then a group algebra SG is of finite representation type if and only if SG is of bounded representation type. This does not hold in the case when S is not a field [17], [32]. Gudivok [16] and Janusz [27], [28] showed that if S is an infinite field of characteristic p and G is a non-cyclic p-group for which $|G/G'| \neq 4$, then $\mathrm{Ind}_n(SG)$ is an infinite set for every natural $n > 1$. Let G be a finite p-group of order $|G| > 2$, S a commutative local ring of characteristic p^k, and $\mathrm{rad}\, S \neq 0$. Gudivok and Chukhray [19], [20] proved that if $\bar{S}$ is an infinite field or S is an integral domain, then $\mathrm{Ind}_n(SG)$ is infinite for every natural $n > 1$. In paper [24], joint with Sygetij, they obtained a similar result in the case where G is a non-cyclic p-group, $p \neq 2$ and S is an infinite ring of characteristic p or $\bar{S}$ is an infinite field. We note that in [22], [23], Gudivok and Pogorilyak investigate group rings SG of bounded representation type for the case when G is a p-group and S is an arbitrary commutative local ring of characteristic p^k with $\mathrm{rad}\, S \neq 0$. The similar problem was studied in [4] for twisted group rings $S^\lambda G$, where S is a Dedekind domain of characteristic p.

We remark that the investigations mentioned above were considerably stimulated by the well-known Brauer–Thrall conjectures [26] for finite-dimensional algebras over an arbitrary field. For a complete discussion of related problems in the modern representation theory of finite groups, algebras, quivers and vector space categories the reader is referred to the monographs [11], [13] and [33].

In the present paper we describe twisted group rings $S^\lambda G$ of SUR-type. We shall also characterize finite p-groups depending on a projective (S, W)-representation type. Our investigations extend the results of [4], [19] and [20]. We obtain indecomposable $S^\lambda G$-modules of S-rank $f_\lambda(n)$ by applying induction from $S^\lambda H$-modules to $S^\lambda G$-modules, where H is a subgroup of G. If M is an indecomposable $S^\lambda H$-module then the induced module $M^{S^\lambda G}$ is also an indecomposable $S^\lambda G$-module under some assumptions which generalize the hypotheses of the Green Theorems [14], [15]. When $S^\lambda H$ is a group ring and $|H| > 2$, we make use of the indecomposable $S^\lambda H$-

modules which are constructed in [19] (see also [18]) as initial $S^\lambda H$-modules. If $S^\lambda H$ is not a group ring then first we find $\mu \in Z^2(H, S^*)$ such that $S^\lambda H = S^\mu H$ and $S^\mu H$ contains a group ring $S^\mu B$, where B is a subgroup of H and $|B| > 2$. In this case we obtain indecomposable $S^\lambda H$-modules by applying induction from $S^\mu B$-modules to $S^\mu H$-modules.

Let us briefly present the results obtained. In Section 1, we define the kernel of a cocycle and prove its properties. In Section 2, we obtain further information on the infinite series of indecomposable modules of R-rank n over a group ring RH studied in [19], where $n \geq 2$ is an arbitrary natural number, R is a commutative local ring of characteristic p, and H is a cyclic p-group of order $|H| > 2$ or a group of type $(2, 2)$. In particular, we prove that, for every such module V, the ring $\mathrm{End}_{RH}(V)$ is finitely generated as an R-module.

In Section 3, we single out rings $S^\lambda G$ of SUR-type for the case when S is an arbitrary local integral domain of characteristic p, and, in Section 4, for the case when S is a commutative local noetherian ring of characteristic p. We prove that if S is a local integral domain of characteristic p, H the kernel of $\lambda \in Z^2(G, S^*)$, and $|H : G'| > 2$, then for $S^\lambda G$ one can construct the SUR-dimension-valued function $f_\lambda(n) = nd$, where $d = |G : H|$ (Theorem 1). If S is a local noetherian integral domain of characteristic p then in the above statement we can assume that $|H| > 2$ (see Corollary to Theorem 4). Let S be a local integral domain of characteristic p, F a subfield of S, and $\lambda \in Z^2(G, F^*)$ such that $F^\lambda G$ is a non-semisimple algebra. Then for $S^\lambda G$ there exists an SUR-dimension-valued function $f_\lambda(n) = nd$, where $d = \dim_F \overline{F^\lambda G}$. In addition, one should assume that one of the following conditions holds:

1) $p \neq 2$, $d < |G : G'|$ (Theorem 2);
2) $p = 2$, $d < \frac{1}{2}|G : G'|$ (Theorem 3);
3) $p \neq 2$, S is a noetherian ring (Theorem 6).

We remark that if $S^\lambda G = SG$, then $d = 1$ and $f_\lambda(n) = n$, in each of the above cases, and we recover the results of [19], [20]. In Theorem 5, we prove the existence of a ring $S^\lambda G$ with SUR-dimension-valued function $f_\lambda(n) = n \cdot |G : B|$, where B can be an arbitrary subgroup with $G' \subset B \subset G$, and moreover the S-rank of every indecomposable $S^\lambda G$-module is a value of the function f_λ.

In Section 5, we introduce the concept of projective (S, W)-representation type for a finite group (finite, infinite, purely infinite, bounded, unbounded, purely unbounded, strongly unbounded, purely strongly unbounded). We prove a number of propositions about p-groups with a given projective (S, W)-representation type over a ring $S = F[[X]]$ (Propositions 5–8).

1. Non-semisimple twisted group algebras

LEMMA 1. *Let G be a p-group, R an integral domain of characteristic p, R^* the multiplicative group of R, W a subgroup of R^*, $\lambda : G \times G \to W$ a 2-cocycle, and A the union of all cyclic subgroups $\langle g \rangle$ of G such that the restriction of λ to $\langle g \rangle \times \langle g \rangle$ is a W-valued coboundary. Then $G' \subset A$, A is a normal subgroup of G, and up to cohomology in $Z^2(G, W)$,*

$$\lambda_{g,a} = \lambda_{a,g} = 1 \tag{1}$$

for all $g \in G$, $a \in A$.

Proof. Evidently if T is a subgroup of G and the restriction of $\lambda : G \times G \to W$ to $T \times T$ is a W-valued coboundary then $T \subset A$. By [29, Corollary 4.10, p. 42], the restriction of λ to $G' \times G'$ is a W-valued coboundary. Hence, $G' \subset A$. Let B be a normal subgroup of G with $G' \subset B$ and suppose the restriction of λ to $B \times B$ is a W-valued coboundary. We may assume $\lambda_{b,b'} = 1$ for all $b, b' \in B$. Let $\{u_g : g \in G\}$ be a natural R-basis of $R^\lambda G$. For any $b \in B$, $g \in G$ we have

$$u_g u_b u_g^{-1} = \gamma u_{b'},$$

where $\gamma \in W$, $b' = gbg^{-1}$. Then

$$u_g u_b^{|b|} u_g^{-1} = \gamma^{|b|} u_{b'}^{|b|},$$

whence $\gamma = 1$. Consequently, $\lambda_{g,b} = \lambda_{b',g}$. Let $\{g_1 = e, g_2, \dots, g_n\}$ be a cross section of B in G ([12, p. 79]). We set $v_{g_i b} = \lambda_{g_i,b} u_{g_i b}$ for every $i \in \{1, \dots, n\}$ and $b \in B$. Then $v_{g_i} = u_{g_i}$, $v_b = u_b$, $v_{g_i} v_b = v_{g_i b}$ and for any $g = g_j c$, $c \in B$, we have

$$v_g v_b = v_{g_j} v_c v_b = v_{g_j} v_{cb} = v_{g_j(cb)} = v_{gb}, \qquad v_b v_g = v_{bg}.$$

Therefore, up to cohomology, $\lambda_{g,b} = \lambda_{b,g} = 1$ for all $g \in G$, $b \in B$.

Let H be a cyclic subgroup of G such that the restriction of λ to $H \times H$ is a W-valued coboundary. Let $D = BH$ and suppose $D \neq B$. Because $G' \subset B$, D is a normal subgroup of G. By hypothesis,

$$\lambda_{h,h'} = \frac{\alpha_h \cdot \alpha_{h'}}{\alpha_{hh'}}$$

for any $h, h' \in H$, where α is a mapping of H into W. If $x, y \in B \cap H$ then

$$\lambda_{x,y} = 1 \quad \text{and} \quad \lambda_{x,y} = \frac{\alpha_x \cdot \alpha_y}{\alpha_{xy}},$$

whence $\alpha_{xy} = \alpha_x \alpha_y$. It follows that $\alpha_x = 1$ for any $x \in B \cap H$.

Let $h_1 = e, h_2, \dots, h_m \in H$ and $\{h_1, \dots, h_m\}$ be a cross section of B in D. If $d \in D$ and $d = bh_i$, $b \in B$, then we set

$$v_d = \alpha_{h_i}^{-1} u_d.$$

Let $d_1 = xh_i$ and $d_2 = yh_j$, where $x, y \in B$, be arbitrary elements of D. Assume that $h_i h_j = bh_r$, $b \in B$, and $z = h_i y h_i^{-1}$. Then $\lambda_{b,h_r} = 1$, and hence

$\alpha_{bh_r} = \alpha_b \alpha_{h_r} = \alpha_{h_r}$, whence $\alpha_{h_i h_j} = \alpha_{h_r}$. Thus, we get

$$v_{d_1} \cdot v_{d_2} = \alpha_{h_i}^{-1} u_x u_{h_i} \cdot \alpha_{h_j}^{-1} u_y u_{h_j} = \alpha_{h_i}^{-1} \alpha_{h_j}^{-1} u_x u_z \lambda_{h_i, h_j} u_{h_i h_j}$$
$$= \alpha_{h_i h_j}^{-1} u_{d_1 d_2} = \alpha_{h_r}^{-1} u_{d_1 d_2} = v_{d_1 d_2}.$$

This proves that the restriction of λ to $D \times D$ is a W-valued coboundary. Let $a_i \in A$, $H_i = \langle a_i \rangle$, $1 \le i \le n$, and $D_n = G' H_1 \cdots H_n$. Applying induction on n, we conclude in view of the above arguments that D_n is a normal subgroup of G, $D_n \subset A$, and up to cohomology in $Z^2(G,W)$ we have $\lambda_{g,d} = \lambda_{d,g} = 1$ for all $g \in G$, $d \in D_n$. This completes the proof, because $A = D_s$ for some s. ∎

Definition. The subgroup A introduced in Lemma 1 is said to be the *kernel* of the cocycle $\lambda \in Z^2(G,W)$. We denote this subgroup by $\mathrm{Ker}(\lambda)$.

In what follows, we assume that every cocycle $\lambda \in Z^2(G,W)$ under consideration satisfies condition (1). We remark that if $\mu_{xA,yA} = \lambda_{x,y}$ for any $x, y \in G$, then $\mu \in Z^2(G/A, W)$ and $\mathrm{Ker}(\mu) = \{A\}$.

Let F be a field of characteristic p, and W a subgroup of F^*. Set $i_F(W) = \sup\{0, m\}$, where m is a natural number such that the algebra

$$F[x]/(x^p - \gamma_1) \otimes_F \cdots \otimes_F F[x]/(x^p - \gamma_m)$$

is a field for some $\gamma_1, \dots, \gamma_m \in W$. By Proposition 1.1 of [6], for any natural number t, there exists a field F such that $i_F(F^*) = t$.

Proposition 1. *Let G be a finite p-group, F a field of characteristic p, W a subgroup of F^*, $\lambda \in Z^2(G,W)$, and $B = \mathrm{Ker}(\lambda)$. Then the set $V = F^\lambda G \cdot \mathrm{rad}\, F^\lambda B$ is a nilpotent ideal of the algebra $F^\lambda G$, and the quotient algebra $F^\lambda G/V$ is isomorphic to $F^\pi H$, where $H = G/B$ and $\pi_{xB,yB} = \lambda_{x,y}$ for any $x, y \in G$. If $d = \dim_F \overline{F^\lambda G}$ then d is a divisor of $|G : B|$. Suppose that $i_F(W) \ge k$, where k is the number of invariants of the group G/G'. Then for every subgroup B of G containing G' there exists a cocycle $\lambda \in Z^2(G,W)$ such that $B = \mathrm{Ker}(\lambda)$ and $\dim_F \overline{F^\lambda G} = |G : B|$.*

Proof. Let $\lambda \in Z^2(G,W)$ and $B = \mathrm{Ker}(\lambda)$. By Lemma 1, B is a normal subgroup of G, $G' \subset B$, and $\lambda_{g,b} = \lambda_{b,g} = 1$ for all $g \in G$, $b \in B$. It follows that $F^\lambda B$ is the group algebra of B over the field F and

$$\mathrm{rad}\, F^\lambda B = \bigoplus_{b \in B,\, b \ne e} F(u_b - u_e).$$

Then $V = F^\lambda G \cdot \mathrm{rad}\, F^\lambda B$ is a nilpotent ideal of $F^\lambda G$. The quotient algebra $F^\lambda G/V$ is the commutative twisted group algebra $F^\pi H$ of the group $H = G/B$ and the field F with the 2-cocycle $\pi \in Z^2(H,W)$, where $\pi_{xB,yB} = \lambda_{x,y}$ for any $x, y \in G$. A natural F-basis of $F^\lambda G/V$ is formed by elements of the form $u_g + V$.

Let $H = \langle h_1\rangle \times \cdots \times \langle h_r\rangle$ be a group of type $(p^{s_1}, \ldots, p^{s_r})$. The algebra $F^\pi H$ has a natural F-basis $\{v_h : h \in H\}$ satisfying the following conditions:

1) if
$$h = h_1^{j_1} \cdots h_r^{j_r}$$
and $0 \le j_i < p^{s_i}$ for every $i = 1, \ldots, r$, then
$$v_h = v_{h_1}^{j_1} \cdots v_{h_r}^{j_r};$$

2) $v_{h_i}^{p^{s_i}} = \alpha_i v_e$, $\alpha_i \in W$ $(i = 1, \ldots, r)$.

We denote the algebra $F^\pi H$ also by $[H, F, \alpha_1, \ldots, \alpha_r]$. In view of [5, Theorem 1] we have $\overline{F^\pi H} \cong K$, where K is a finite purely inseparable extension of F and $[K : F]$ divides $|H|$. Since $\overline{F^\lambda G} \cong \overline{F^\pi H}$, d divides $|G : B|$.

Now we prove the final statement. Let B be the subgroup of G with $G' \subset B$ and set $H = G/B$. Assume $H = \langle h_1\rangle \times \cdots \times \langle h_r\rangle$. Then $r \le k$. Since $i_F(W) \ge k$,
$$F[x]/(x^p - \gamma_1) \otimes_F \cdots \otimes_F F[x]/(x^p - \gamma_r)$$
is a field for some $\gamma_1, \ldots, \gamma_r \in W$. The twisted group algebra $F^\mu H = [H, F, \gamma_1, \ldots, \gamma_r]$ is a field. Let $\lambda_{x,y} = \mu_{xB,yB}$ for all $x, y \in G$. Then $\lambda \in Z^2(G, W)$ and $\mathrm{Ker}(\lambda) = B$. Let $V = F^\lambda G \cdot \mathrm{rad}\, F^\lambda B$. Because $F^\lambda G/V \cong F^\mu H$ and $F^\mu H$ is a field, we have $V = \mathrm{rad}\, F^\lambda G$ and $\dim_F \overline{F^\lambda G} = |G : B|$. ■

Proposition 2. *Let G be a finite p-group, F a field of characteristic p, $\lambda \in Z^2(G, F^*)$, and $d = \dim_F \overline{F^\lambda G}$.*

(i) *There exists a homomorphism of $F^\lambda G$ onto a twisted group algebra of the form*

$$A = \bigoplus_{j=0}^{p^m-1} K v_a^j, \qquad v_a^{p^m} = \alpha^{p^l} v_e \ (\alpha \in K^*), \tag{2}$$

where $m > 0$, K is a finite purely inseparable extension of F; $d = [K : F] \cdot p^{m-l}$, $l = 0$ for $d = |G : G'|$ and $1 \le l \le m$ for $d < |G : G'|$; $\alpha \notin K^p$ for $0 \le l < m$ and $\alpha = 1$ for $l = m$.

(ii) *If $d < 1/p|G : G'|$, then there exists a homomorphism of $F^\lambda G$ onto A with $2 \le l \le m$ or onto a twisted group algebra of the form*

$$A' = \bigoplus_{i,j} K v_a^i v_b^j, \qquad v_a v_b = v_b v_a, \qquad v_a^{p^m} = \alpha^p v_e, \qquad v_b^{p^n} = \beta^p v_e, \tag{3}$$

where $m, n > 0$, K is a finite purely inseparable extension of F, $d = [K : F] \cdot p^{m+n-2}$, and $\mathrm{rad}\, A'$ is generated by elements
$$v_a^{p^{m-1}} - \alpha v_e, \qquad v_b^{p^{n-1}} - \beta v_e.$$

Proof. We keep the notations used in the proof of Proposition 1, and we assume that G is non-abelian. Arguing as in that proof, we establish the existence of an algebra homomorphism $F^\lambda G$ onto the algebra $F^\pi H$, where

$H = G/G'$ and $\pi_{xG',yG'} = \lambda_{x,y}$ for all $x, y \in G$. Let $H = \langle h_1\rangle \times \cdots \times \langle h_k\rangle$ be a group of type $(p^{l_1}, \ldots, p^{l_k})$ and $\{u_h : h \in H\}$ a natural F-basis of $F^\pi H$. If $F^\pi H$ is semisimple then $F^\pi H$ is a field and $d = |G : G'|$. We have

$$F^\pi H = \bigoplus_{j=0}^{p^m-1} K v_a^j, \qquad v_a^{p^m} = \alpha v_e \ (\alpha \in F^*),$$

where $m = l_k$, $K = F[u_{h_1}, \ldots, u_{h_{k-1}}]$, and $v_a = u_{h_k}$. In this case $\alpha \notin K^p$. Assume now that the algebra $F^\pi H$ is non-semisimple. Suppose also that $F[u_{h_1}, \ldots, u_{h_{r-1}}]$ is a field and $F[u_{h_1}, \ldots, u_{h_{r-1}}, u_{h_r}]$ is not. Let

$$H_1 = \prod_{i\neq r}\langle h_i\rangle, \qquad H_2 = \langle h_r\rangle, \qquad U = \operatorname{rad} F^\pi H_1, \qquad W = F^\pi H \cdot U,$$

and $F^\pi H_1/U \cong K$, where K is a finite purely inseparable extension of F. Then

$$F^\pi H/W \cong F^\pi H_1/U \otimes_F F^\pi H_2 \cong K \otimes_F F^\pi H_2 \cong K^\pi H_2,$$

and hence, $F^\pi H/W$ is isomorphic to a twisted group algebra A of the form (2), where $m = l_r$. The case when $F[u_{h_i}]$ is not a field for every $i = 1, \ldots, k$ is treated similarly.

Assume that $d < (1/p)|H|$. Then there exists a homomorphism of the algebra $F^\pi H$ onto an algebra of the form (2) with $l \geq 2$ or onto an algebra A' of the form (3), where $\alpha, \beta \in K$, $\alpha \notin K^p$ for $m > 1$, and $\beta \notin K^p$ for $n > 1$. Let $m > 1$ and $L = K(\theta)$, where θ is a root of the polynomial

$$X^{p^{n-1}} - \beta.$$

If $\alpha \in L^p$ then there exists a homomorphism of A' onto

$$\bigoplus_{i=0}^{p^m-1} L v_a^i, \qquad v_a^{p^m} = \gamma^{p^2} v_e \ (\gamma \in L^*),$$

which is of the form (2). ∎

2. Infinite sets of indecomposable underlying modules of representations of a group ring of a p-group. Let $H = \langle a\rangle$ be a cyclic p-group of order $|H| > 2$, and R a commutative local ring of characteristic p. Assume that there is a non-zero element $t \in \operatorname{rad} R$ which is not a zero-divisor. Let E_m be the identity matrix of order m, $J_m(0)$ the upper Jordan block of order m with zeros on the main diagonal, and $\langle 1\rangle$ the $m \times 1$-matrix of the form

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Denote by Γ_i a matrix R-representation of degree n of the group H defined in the following way:

1) if $n = 2$ then
$$\Gamma_i(a) = \begin{pmatrix} 1 & t^i \\ 0 & 1 \end{pmatrix} \qquad (i \in \mathbb{N});$$

2) if $n = 3m$ $(m \geq 1)$ then
$$\Gamma_i(a) = \begin{pmatrix} E_m & t^i E_m & J_m(0) \\ 0 & E_m & t^i E_m \\ 0 & 0 & E_m \end{pmatrix} \qquad (i \in \mathbb{N});$$

3) if $n = 3m + 1$ $(m \geq 1)$ then
$$\Gamma_i(a) = \begin{pmatrix} E_m & t^{2i} E_m & J_m(0) & t\langle 1\rangle \\ 0 & E_m & t^i E_m & 0 \\ 0 & 0 & E_m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (i \in \mathbb{N});$$

4) if $n = 3m + 2$ $(m \geq 1)$ then
$$\Gamma_i(a) = \begin{pmatrix} E_m & t^{i+2} E_m & J_m(0) & t^{2i+4}\langle 1\rangle & t\langle 1\rangle \\ 0 & E_m & t^{2i+4} E_m & 0 & t^2\langle 1\rangle \\ 0 & 0 & E_m & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (i \in \mathbb{N}).$$

Let V_i be the underlying RH-module of this representation.

Note that Γ_i is a slight modification of the representation of H which was constructed in [19, Lemma 4] for the case when R is a local integral domain of characteristic p. One can obtain this representation as a result of the substitution $J_m(0) \mapsto E_m + J_m(0)$.

Lemma 2. *If $i \neq j$, then the RH-modules V_i and V_j are non-isomorphic. The algebra $\operatorname{End}_{RH}(V_i)$ is finitely generated as an R-module and there is an algebra isomorphism*
$$\operatorname{End}_{RH}(V_i)/\operatorname{rad}\operatorname{End}_{RH}(V_i) \cong R/\operatorname{rad} R \quad \text{\textit{for every} } i \in \mathbb{N}.$$

Proof. By direct calculations we find that if $i \neq j$ and $C\Gamma_i(a) = \Gamma_j(a)C$ for some $C \in R^{n\times n}$, then $\det C \notin R^*$. Hence the modules V_i and V_j are non-isomorphic for $i \neq j$. We prove the second and third statement only for the case $n = 3m + 2$, because the proof in the remaining cases is similar.

Suppose that
$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{pmatrix}$$

is a square matrix of order $n = 3m + 2$ with entries from the ring R. In addition, we assume that C_{11}, C_{22}, C_{33} are square matrices of order m and C_{44}, C_{55} square matrices of order 1. If $\Gamma_i(a)C = C\Gamma_i(a)$, then

$$\begin{aligned}
&C_{21} = 0, \quad C_{31} = 0, \quad C_{32} = 0, \quad C_{34} = 0,\\
&C_{41} = 0, \quad C_{51} = 0, \quad C_{52} = 0, \quad C_{54} = 0;\\
&C_{22} = C_{11} - t^{i+2}\langle 1\rangle C_{42}; \quad C_{33} = C_{11} - (t^{i+2} + t^2)\langle 1\rangle C_{42};\\
&C_{53} = t^{2i+4}C_{42}; \quad C_{24} + t^{i+2}\langle 1\rangle C_{44} = t^{i+2}C_{11}\langle 1\rangle;\\
&C_{55} = t^2 C_{42}\langle 1\rangle + C_{44}; \quad C_{24} = t^{2i+4}C_{35} + t^2\langle 1\rangle C_{55} - t^2 C_{22}\langle 1\rangle;\\
&C_{11}J_m(0) - J_m(0)C_{11}\\
&\qquad\qquad = t^{i+2}(C_{23} + t^{i+2}\langle 1\rangle C_{43} + t^{i+3}\langle 1\rangle C_{42} - t^{i+2}C_{12});\\
&C_{14} = t^{i+2}C_{25} + J_m(0)C_{35}\\
&\qquad + t^{2i+4}\langle 1\rangle C_{45} + t\langle 1\rangle C_{55} - tC_{11}\langle 1\rangle - t^2 C_{12}\langle 1\rangle.
\end{aligned} \tag{4}$$

We can find all solutions of this system if we know the solutions of the following system:

$$t^{2i+2}C_{35} + (1 + t^i)\langle 1\rangle C_{55} - (1 + t^i)C_{11}\langle 1\rangle = 0, \tag{5}$$

$$C_{11}J_m(0) - J_m(0)C_{11} = t^{i+2}(C_{23} + t^{i+2}\langle 1\rangle C_{43} + t^{i+3}\langle 1\rangle C_{42} - t^{i+2}C_{12}). \tag{6}$$

Define

$$B = C_{23} + t^{i+2}\langle 1\rangle C_{43} + t^{i+3}\langle 1\rangle C_{42} - t^{i+2}C_{12}; \quad C_{55} = (\alpha);$$

$$B = (b_{kl}), \quad C_{11} = (x_{kl}), \quad 1 \le k, l \le m; \quad C_{35} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_m \end{pmatrix}.$$

Equation (5) yields

$$x_{11} = \alpha + \frac{t^{2i+2}}{1 + t^i}\delta_1; \quad x_{j1} = \frac{t^{2i+2}}{1 + t^i}\delta_j, \quad 2 \le j \le m.$$

We declare α, δ_j for all $j = 1, \dots, m$ to be free unknowns. Equation (6) can be written in the form

$$\begin{pmatrix}
0 & x_{11} & x_{12} & \cdots & x_{1,m-1}\\
0 & x_{21} & x_{22} & \cdots & x_{2,m-1}\\
\cdot & \cdot & \cdot & \cdots & \cdot\\
0 & x_{m-1,1} & x_{m-1,2} & \cdots & x_{m-1,m-1}\\
0 & x_{m1} & x_{m2} & \cdots & x_{m,m-1}
\end{pmatrix} - \begin{pmatrix}
x_{21} & x_{22} & \cdots & x_{2m}\\
x_{31} & x_{32} & \cdots & x_{3m}\\
\cdot & \cdot & \cdots & \cdot\\
x_{m1} & x_{m2} & \cdots & x_{mm}\\
0 & 0 & \cdots & 0
\end{pmatrix}$$
$$= t^{i+2}\begin{pmatrix}
b_{11} & b_{12} & b_{13} & \cdots & b_{1m}\\
b_{21} & b_{22} & b_{23} & \cdots & b_{2m}\\
\cdot & \cdot & \cdot & \cdots & \cdot\\
b_{m-1,1} & b_{m-1,2} & b_{m-1,3} & \cdots & b_{m-1,m}\\
b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mm}
\end{pmatrix}. \tag{7}$$

Equate the first columns on the left side of (7) with those on the right, thereby obtaining

$$b_{k1} = -\frac{t^i}{1+t^i}\delta_{k+1} \quad \text{for } k \in \{1,\dots,m-1\}, \quad b_{m1} = 0.$$

Equating the second columns on both sides of (7), we get

$$\begin{pmatrix} x_{22} \\ \vdots \\ x_{m2} \end{pmatrix} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{m-1,1} \end{pmatrix} - t^{i+2} \begin{pmatrix} b_{12} \\ \vdots \\ b_{m-1,2} \end{pmatrix}, \quad b_{m2} = \frac{t^i}{1+t^i}\delta_m.$$

There is no restriction on $x_{12}, b_{12}, \dots, b_{m-1,2}$. We declare $x_{1l}, b_{1l}, \dots, b_{m-1,l}$ for $l = 2, \dots, m$ to be free unknowns. Taking into consideration the expression of x_{j1} for $2 \le j \le m$, we conclude that t^{i+2} divides x_{j2} for every $j \in \{3, \dots, m\}$. We use induction on q, where $2 \le q \le m$ and q indexes columns in the matrix C_{11}. Let $q \le m-1$, and suppose that x_{kl}, b_{kl} have been determined for all $k \in \{1, \dots, m\}$ and $l \in \{2, \dots, q\}$, where:

1) x_{kl} for $2 \le k \le m$, $2 \le l \le q$ are linear combinations of free unknowns with coefficients in R and t^{i+2} divides the coefficients of x_{jl} for every $j \in \{l+1, \dots, m\}$; moreover $x_{kl} = x_{k-1,l-1} - t^{i+2} b_{k-1,l}$;
2) $t^{i+2} b_{ml} = x_{m,l-1}$.

Equating the $(q+1)$th columns on both sides of (7), we obtain

$$\begin{aligned} &t^{i+2} b_{m,q+1} = x_{mq}, \\ &x_{j,q+1} = x_{j-1,q} - t^{i+2} b_{j-1,q+1} \quad \text{for all } j \in \{2, \dots, m\}. \end{aligned}$$

Since t is not a zero-divisor and t^{i+2} divides the coefficients of x_{mq}, one can solve the first equation for $b_{m,q+1}$. The second equation implies that t^{i+2} divides the coefficients of $x_{j,q+1}$ for every $j \in \{q+2, \dots, m\}$.

Thus the set of pairs (C_{11}, B) is finitely generated as an R-module. For a given matrix B,

$$C_{23} = B - t^{i+2}\langle 1\rangle C_{43} - t^{i+3}\langle 1\rangle C_{42} + t^{i+2} C_{12}.$$

Since the matrices C_{12}, C_{13}, C_{i5} $(i = 1,2,3,4)$, C_{42}, C_{43}, C_{55} are arbitrary, the ring K of matrices C commuting with $\Gamma_i(a)$ is finitely generated as an R-module.

Let $P = \operatorname{rad} R$. We have

$$C_1 \equiv \begin{pmatrix} \alpha & & * \\ & \ddots & \\ 0 & & \alpha \end{pmatrix} \pmod{PR^{m\times m}}.$$

It follows from (4) that

$$C \equiv \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ 0 & C_{11} & C_{23} & 0 & C_{25} \\ 0 & 0 & C_{11} & 0 & C_{35} \\ 0 & C_{42} & C_{43} & C_{55} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{pmatrix} \pmod{PR^{n\times n}},$$

and hence, $\det C \equiv \alpha^n \pmod P$. Since C or $C-E$ is an invertible matrix over R, it follows that C or $C-E$ is invertible in K. Therefore, K is a local ring. We have $C = \alpha E + D$, where $D \in \operatorname{rad} K$. The mapping $f : K/\operatorname{rad} K \to R/P$ defined by $f(C+\operatorname{rad} K) = \alpha + P$ is an isomorphism. This proves that $\overline{\operatorname{End}_{RH}(V_i)} \cong \overline{R}$. ∎

Lemma 3. *Let $H = \langle a\rangle \times \langle b\rangle$ be an abelian group of type $(2,2)$, $t \in \operatorname{rad} R$, $t \neq 0$ and suppose t is not a zero-divisor. Denote by W_i the underlying RH-module of the matrix representation Δ_i of degree n of the group H defined as follows*:

1) *if* $n = 2m$ $(m \geq 1)$, *then*

$$\Delta_i(a) = \begin{pmatrix} E_m & t^i E_m \\ 0 & E_m \end{pmatrix}, \qquad \Delta_i(b) = \begin{pmatrix} E_m & J_m(0) \\ 0 & E_m \end{pmatrix} \qquad (i \in \mathbb{N});$$

2) *if* $n = 2m+1$ $(m \geq 1)$, *then*

$$\Delta_i(a) = \begin{pmatrix} E_m & t^i E_m & 0 \\ 0 & E_m & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Delta_i(b) = \begin{pmatrix} E_m & J_m(0) & t^i\langle 1\rangle \\ 0 & E_m & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (i \in \mathbb{N}).$$

If $i \neq j$, then the modules W_i and W_j are non-isomorphic. Moreover, $\operatorname{End}_{RH}(W_i)$ is finitely generated as an R-module and there is an algebra isomorphism

$$\operatorname{End}_{RH}(W_i)/\operatorname{rad}\operatorname{End}_{RH}(W_i) \cong R/\operatorname{rad} R$$

for all $i \in \mathbb{N}$.

The proof of Lemma 3 is similar to that of Lemma 2, and we leave it to the reader.

3. Twisted group rings $S^\lambda G$ of SUR-type if S is an arbitrary local integral domain

Lemma 4. *Let R be a commutative local artinian ring or a complete commutative local noetherian ring of characteristic p, G a finite p-group, $\lambda \in Z^2(G, R^*)$, H a subgroup of G, and V an indecomposable $R^\lambda H$-module. Assume that the quotient algebra*

$$\overline{\operatorname{End}_{R^\lambda H}(V)} = \operatorname{End}_{R^\lambda H}(V)/\operatorname{rad}\operatorname{End}_{R^\lambda H}(V)$$

is isomorphic to a field K containing $\overline{R}$, and one of the following conditions is satisfied:

(i) $G = H \cdot T$, *where* T *is a subgroup of the center of* G;

(ii) *if* K_s *is the separable closure of* $\overline{R} = R/\operatorname{rad} R$ *in* K, *then the order of the group* $\operatorname{Aut}(K_s/\overline{R})$ *is not divisible by* p.

Then $V^{R^\lambda G}$ *is an indecomposable* $R^\lambda G$*-module, and the quotient algebra*

$$\overline{\operatorname{End}_{R^\lambda G}(V^{R^\lambda G})}$$

is isomorphic to a field, which is a finite purely inseparable extension of K.

Lemma 5. *Let* R *be a commutative local ring of characteristic* p^k, G *a finite abelian* p*-group,* H *a subgroup of* G, $\lambda \in Z^2(G, R^*)$, *and* M *an indecomposable* $R^\lambda H$*-module. Assume that* $\operatorname{End}_{R^\lambda H}(M)$ *is finitely generated as an* R*-module and* $\overline{\operatorname{End}_{R^\lambda H}(M)}$ *is isomorphic to a field* K *containing* $\overline{R}$. *Then* $M^{R^\lambda G}$ *is an indecomposable* $R^\lambda G$*-module. Moreover,*

$$\operatorname{End}_{R^\lambda G}(M^{R^\lambda G})$$

is finitely generated as an R*-module and the quotient algebra*

$$\overline{\operatorname{End}_{R^\lambda G}(M^{R^\lambda G})}$$

is isomorphic to a field, which is a finite purely inseparable extension of K.

The proofs of Lemmas 4 and 5 are similar to those of Lemma 2 of [2] and Lemma 2.2 of [3]. These lemmas generalize the results by Green [14], [15], concerning the absolutely indecomposable modules over group rings.

Until the end of this section we assume that S is an arbitrary local integral domain of characteristic p, $P = \operatorname{rad} S$, $P \neq 0$, F is a subfield of S, and G a finite p-group. Denote by $[M]$ the isomorphism class of SG-modules which contains M. Let $\mathfrak{M}_n(SG)$ be the set of all $[M]$ satisfying the following conditions:

(i) the S-rank of M equals n;

(ii) $\operatorname{End}_{SG}(M)$ is finitely generated as an S-module;

(iii) $\overline{\operatorname{End}_{SG}(M)} \cong \overline{S}$.

Lemma 6. *Let* $|G| > 2$. *Then* $\mathfrak{M}_n(SG)$ *is an infinite set for every* $n > 1$.

Lemma 6 follows from Lemmas 2 and 3.

Theorem 1. *Let* $\lambda \in Z^2(G, S^*)$ *and* $H = \operatorname{Ker}(\lambda)$.

(i) *If* $|H| > 2$, *then* $S^\lambda G$ *is of SUR-type with* $f_\lambda(n) = nt_n$, *where* $1 \leq t_n \leq |G : H|$.

(ii) *Assume that* $|H : G'| > 2$. *Then* $f_\lambda(n) = nd$, *where* $d = |G : H|$, *is an SUR-dimension-valued function for* $S^\lambda G$.

Proof. (i) Let $[V] \in \mathfrak{M}_n(SH)$, $\{u_g : g \in G\}$ be a natural S-basis of $S^\lambda G$, and $\{g_1 = e, g_2, \dots, g_m\}$ a cross section of H in G. Then

$$V^{S^\lambda G} = \bigoplus_{i=1}^{m} V_i \quad \text{with} \quad V_i = u_{g_i} \otimes V.$$

Since the SH-module V_i is conjugate to V for every i, there is an algebra isomorphism

$$\mathrm{End}_{SH}(V_i) \cong \mathrm{End}_{SH}(V)$$

for each i. Since the ring of SH-endomorphisms of V_i is local for every $i \in \{1, \dots, m\}$, in view of the Krull–Schmidt Theorem [30, Sect. 7.3] the SH-module $V^{S^\lambda G}$ has a unique decomposition into a finite sum of indecomposable SH-modules, up to isomorphism and the order of summands. Hence, in view of Lemma 6, there are infinitely many non-isomorphic indecomposable $S^\lambda G$-modules M such that M is an $S^\lambda G$-component of a module of the form $V^{S^\lambda G}$. Note that the S-rank of M is divisible by n and does not exceed $n \cdot |G : H|$. Therefore, there exists a natural number t_n such that $1 \leq t_n \leq |G : H|$ and $\mathrm{Ind}_{nt_n}(S^\lambda G)$ is an infinite set.

(ii) Let $A = G/G'$ and

$$U = \bigoplus_{a \in G',\, a \neq e} S(u_a - u_e).$$

The set $V = S^\lambda G \cdot U$ is a two-sided ideal of $S^\lambda G$. The factor ring $S^\lambda G/V$ is isomorphic to $S^\mu A$, where $\mu_{xG', yG'} = \lambda_{x,y}$ for all $x, y \in G$. It contains the group ring SB, where $B = H/G'$. Since $|B| > 2$, by Lemma 6 the set $\mathfrak{M}_n(SB)$ is infinite for every $n > 1$.

Assume that $[M] \in \mathfrak{M}_n(SB)$. By Lemma 5, the induced $S^\mu A$-module $M^{S^\mu A}$ is indecomposable. Its S-rank is equal to $n \cdot |A : B| = n \cdot |G : H|$. Arguing as in case (i), we deduce that $\mathrm{Ind}_{nd}(S^\mu A)$ is infinite for every $n > 1$. It follows that $\mathrm{Ind}_{nd}(S^\lambda G)$ is an infinite set for each $n > 1$. ∎

THEOREM 2. *Let $p \neq 2$ and $\lambda \in Z^2(G, F^*)$. If the algebra $F^\lambda G$ is not semisimple, then the ring $S^\lambda G$ is of SUR-type. Moreover, if $d = \dim_F \overline{F^\lambda G}$ and $d < |G : G'|$, then $f_\lambda(n) = nd$ is an SUR-dimension-valued function for $S^\lambda G$.*

Proof. There exists an algebra homomorphism of $F^\lambda G$ onto $F^\mu \overline{G}$, where $\overline{G} = G/G'$ and $\mu_{xG', yG'} = \lambda_{x,y}$ for all $x, y \in G$. We have $d = \dim_F \overline{F^\mu \overline{G}}$. Taking into account this fact and Theorem 1 we can assume that G is abelian and $F^\lambda G$ is non-semisimple.

In view of Proposition 2, there exists an algebra homomorphism of $F^\lambda G$ onto a twisted group algebra

$$A = \bigoplus_{j=0}^{p^m-1} Kv_a^j, \qquad v_a^{p^m} = \alpha^{p^l} v_e \ (\alpha \in K^*),$$

where K is a finite purely inseparable extension of the field F, $1 \le l \le m$, $\alpha \notin K^p$ for $l < m$ and $d = [K : F] \cdot p^{m-l}$. Since $S^\lambda G \cong S \otimes_F F^\lambda G$, there is an algebra homomorphism of $S^\lambda G$ onto a twisted group ring

$$\Lambda = S \otimes_F A = \bigoplus_{j=0}^{p^m-1} R(1 \otimes v_a)^j,$$

where $R = S \otimes_F Kv_e$. Note that if

$$w = 1 \otimes \alpha^{-1} v_a^{p^{m-l}},$$

then $w^{p^l} = 1 \otimes v_e$. Hence we conclude that the ring

$$\Gamma = \bigoplus_{i=0}^{p^l-1} Rw^i$$

is a twisted group ring of a cyclic group of order p^l and of the ring R.

The ring R is a finitely generated S-free S-algebra. By [10, Proposition 5.22, p. 112], we have

$$\overline{R} = R/\operatorname{rad} R \cong (R/PR)/\operatorname{rad}(R/PR) \cong \overline{\overline{S} \otimes_F K},$$

but then ([11, p. 100]) R is a commutative local ring of characteristic p. Let t be a non-zero element of P. The element $t \otimes v_e$ is not a zero-divisor in R and $t \otimes v_e \in \operatorname{rad} R$. In view of Lemma 2, for every $n > 1$, there are infinitely many pairwise non-isomorphic indecomposable Γ-modules $V_1, V_2, \dots$ satisfying the following conditions:

1) the R-rank of V_i is equal to n;
2) $\operatorname{End}_\Gamma(V_i)$ is finitely generated as an R-module;
3) $\overline{\operatorname{End}_\Gamma(V_i)} \cong \overline{R}$.

By Lemma 5, the induced Λ-module V_i^Λ is an indecomposable module of R-rank np^{m-l}. Further, the algebra

$$\overline{\operatorname{End}_\Lambda(V_i^\Lambda)}$$

is isomorphic to a field which is a finite purely inseparable extension of the field $\overline{R}$. Since

$$(V_i^\Lambda)_\Gamma \cong V_i \oplus \cdots \oplus V_i,$$

by the Krull–Schmidt Theorem ([30, Sect. 7.3]) the modules V_i^Λ and V_j^Λ are non-isomorphic for $i \neq j$. The module V_i^Λ is an indecomposable $S^\lambda G$-module of S-rank $[K : F] \cdot np^{m-l} = nd$. ∎

THEOREM 3. *Let* $p=2$, $\lambda\in Z^2(G,F^*)$, *and* $d=\dim_F\overline{F^\lambda G}$.

(i) *If the algebra* $F^\lambda G$ *is not semisimple*, *then the set* $\mathrm{Ind}_l(S^\lambda G)$ *is infinite for some* $l\le|G|$.

(ii) *If* $d<\frac{1}{2}|G:G'|$, *then* $S^\lambda G$ *is of SUR-type. In this case the function* $f_\lambda(n)=nd$ *is an SUR-dimension-valued function.*

Proof. (i) If $|G'|\ne 1$, then by Theorem 1 we may suppose that $|G'|=2$. Let $G'=\langle a\rangle$, $t\in\operatorname{rad}S$, and $t\ne 0$. Denote by M_i the underlying SG'-module of the indecomposable representation

$$\Gamma_i : u_a\mapsto\begin{pmatrix}1 & t^i\\ 0 & 1\end{pmatrix}\qquad (i\in\mathbb{N})$$

of the ring SG'. If $i\ne j$, then the SG'-modules M_i and M_j are non-isomorphic. By the same arguments as in the proof of Theorem 1(i), we can prove that $\mathrm{Ind}_l(S^\lambda G)$ is infinite for some $l\le|G|$.

Suppose that $|G'|=1$, $d=\frac{1}{2}|G|$ and H is the socle of G. Then

$$S^\lambda H=S^\mu H\cong S^\mu H_1\otimes_S SH_2,$$

where $\mu\in Z^2(H,F^*)$, $H=H_1\times H_2$, $H_2\subset\mathrm{Ker}(\mu)$, and $H_2=\langle a\rangle$ is a group of order 2. We assume that Γ_i is a representation of the ring SH_2, and M_i is the underlying module of Γ_i. By Lemma 5,

$$V_i=M_i^{S^\mu H}$$

is an indecomposable $S^\lambda H$-module and $\overline{\mathrm{End}_{S^\lambda H}(V_i)}$ is a finite purely inseparable extension of $\overline{S}$, up to isomorphism. If $i\ne j$, then the $S^\lambda H$-modules V_i and V_j are non-isomorphic. Arguing as in the proof of of Theorem 1(i), we finish the proof in this case.

(ii) If $d<\frac{1}{2}|G:G'|$, then we reason as in the proof of Theorem 2. However, note that if $p=2$, then there are two cases, namely that of an algebra A of the form (2), where $m\ge 2$, and of an algebra A' of the form (3). We apply Lemma 2 in the first case and Lemma 3 in the second. ∎

4. Twisted group rings $S^\lambda G$ of SUR-type if S is a local noetherian ring. In this section we suppose that S is a commutative local noetherian ring of characteristic p, F a subfield of S, $P=\operatorname{rad}S$, and $\widehat{S}$ is the P-adic completion of S. We also assume that S is not a field, and if S is not an integral domain then $\bar{S}=S/P$ is an infinite field. Throughout, we identify S with its canonical image in $\widehat{S}$. It is well known (see [8, p. 205]) that $\widehat{S}$ is a complete commutative local noetherian ring.

Let H be a finite p-group. Denote by $[M]$ the isomorphism class of the $\widehat{S}H$-module M. Let $\mathfrak{M}_n(\widehat{S}H)$ be the set of all classes $[M]$ satisfying the following two conditions:

(i) the $\widehat{S}$-rank of M is equal to n;

(ii) $\overline{\mathrm{End}_{\widehat{S}H}(M)} \cong \widehat{S}/\mathrm{rad}\,\widehat{S}$.

LEMMA 7. *Let H be a finite p-group of order $|H| > 2$, and*

$$\mathfrak{M}_n^0(\widehat{S}H) = \{(V) \in \mathfrak{M}_n(\widehat{S}H) : V \cong \widehat{S} \otimes_S M \textit{ for some } SH\textit{-module } M\}.$$

Then $\mathfrak{M}_n^0(\widehat{S}H)$ is an infinite set for every $n > 1$.

Proof. If S contains a non-zero nilpotent element, then the conclusion follows from Lemma 2 in [19]. Assume that S is not an integral domain and S does not have a non-zero nilpotent element. Then S has two elements u and v such that $uv = 0$, $u \notin \widehat{S}v$, and $v \notin \widehat{S}u$. This allows us to apply the same type of argument as in the proofs of Lemmas 3 and 5 of [19]. Let S be an integral domain, $t \in P$, and $t \neq 0$. Then t is not a zero-divisor in $\widehat{S}$ ([8, p. 204]). In view of Lemmas 2 and 3, the set $\mathfrak{M}_n^0(\widehat{S}H)$ is infinite.

THEOREM 4. *Let G be a p-group and $\lambda \in Z^2(G, S^*)$. Assume that G contains a subgroup H such that $|H| > 2$ and the restriction of λ to $H \times H$ is a coboundary. Then $S^\lambda G$ is of SUR-type with SUR-dimension-valued function $f_\lambda(n) = n \cdot |G : H|$.*

Proof. Without loss of generality, we can suppose that $\lambda_{a,b} = 1$ for all $a, b \in H$. In view of Lemma 7, $\mathfrak{M}_n^0(\widehat{S}H)$ is infinite for each $n > 1$. If $[V] \in \mathfrak{M}_n^0(\widehat{S}H)$ then, by Lemma 4, $V^{\widehat{S}^\lambda G}$ is an indecomposable $\widehat{S}^\lambda G$-module. Since

$$(V^{\widehat{S}^\lambda G})_{\widehat{S}H} \cong V \oplus W,$$

where W is an $\widehat{S}H$-module, the set of all isomorphism classes $[V^{\widehat{S}^\lambda G}]$ is infinite, in view of the Krull–Schmidt Theorem ([10, p. 128]). Then $V \cong \widehat{S} \otimes_S M$, where M is an indecomposable SH-module. It follows that there are infinitely many pairwise non-isomorphic indecomposable $S^\lambda G$-modules of the form $M^{S^\lambda G}$. We also note that the S-rank of $M^{S^\lambda G}$ is $n \cdot |G : H|$. ■

COROLLARY 1. *Let G be a p-group, S a local noetherian integral domain of characteristic p, $\mathrm{rad}\, S \neq 0$, $\lambda \in Z^2(G, S^*)$, and H the kernel of λ. If $|H| > 2$, then $f_\lambda(n) = n \cdot |G : H|$ is an SUR-dimension-valued function.*

Denote by $F[[X_1, \dots, X_m]]$ the F-algebra of formal power series in the indeterminates $X_1, \dots, X_m$ with coefficients in the field F of characteristic p.

THEOREM 5. *Let $S = F[[X]]$, W be a subgroup of F^*, G a finite p-group, t the number of invariants of the group G/G', $i_F(W) \geq t$, and B a subgroup of G such that $G' \subset B$. If $|B| \geq 2$, then there is a cocycle $\lambda \in Z^2(G, W)$ such that $\mathrm{Ker}(\lambda) = B$, $\dim_F \overline{F^\lambda G} = |G : B|$, $S^\lambda G$ is of SUR-type and satisfies the following conditions:*

(i) *the function* $f_\lambda(n) = n\cdot|G : B|$ *is an SUR-dimension-valued function for* $S^\lambda G$;

(ii) *the* S*-rank of every* $S^\lambda G$*-module is a value of* f_λ;

(iii) *there is only one* $S^\lambda G$*-module of* S*-rank* $f_\lambda(1)$, *up to isomorphism.*

Proof. In view of Proposition 1, there is a cocycle $\lambda \in Z^2(G, W)$ such that $B = \mathrm{Ker}(\lambda)$ and $\dim_F \overline{F^\lambda G} = |G : B|$. By Theorem 4, the function $f_\lambda(n) = n \cdot |G : B|$ is an SUR-dimension-valued function for $S^\lambda G$. Let M be an $S^\lambda G$-module. Then M/XM is an $F^\lambda G$-module and $\dim_F(M/XM)$ is divisible by $|G : B|$, because $F^\lambda G$ is a local algebra. Since the S-rank of M equals $\dim_F(M/XM)$, it is a value of f_λ.

Let K be the quotient field of S. Obviously, the ring $S^\lambda G$ is an S-order in the algebra $K^\lambda G$. Let M be an $S^\lambda G$-module of S-rank $f_\lambda(1)$. We embed M in the irreducible $K^\lambda G$-module $M^* = K \otimes_S M$. Since the set

$$U = \bigoplus_{b\in B} K^\lambda G(u_b - u_e)$$

is a nilpotent ideal of $K^\lambda G$, we have $U \subset \mathrm{rad}\, K^\lambda G$. Note also that

$$V = \bigoplus_{b\in B} S^\lambda G(u_b - u_e)$$

is an ideal of $S^\lambda G$. Since $\mathrm{rad}\, K^\lambda G \cdot M^* = 0$ and $V \subset \mathrm{rad}\, K^\lambda G$, we have $VM = 0$ and M may be viewed as a module over $S^\lambda G/V$. But $S^\lambda G/V \cong S^\mu H$, where $H = G/B$ and $\mu_{xB,yB} = \lambda_{x,y}$ for all $x, y \in G$. If $L = F^\mu H$ and $T = L[[X]]$, then $L \cong \overline{F^\lambda G}$, $T \cong S^\mu H$, and L is a finite purely inseparable extension of F. Therefore M is T-torsion free. Since T is a principal ideal ring, we get $M \cong S^\mu H$. ■

THEOREM 6. *Let* $p \neq 2$, S *be a local noetherian integral domain of characteristic* p, $\mathrm{rad}\, S \neq 0$, F *a subfield of* S, G *a finite* p*-group,* $\lambda \in Z^2(G, F^*)$, *and* $d = \dim_F \overline{F^\lambda G}$. *If the algebra* $F^\lambda G$ *is not semisimple, then* $S^\lambda G$ *is of SUR-type with SUR-dimension-valued function* $f_\lambda(n) = nd$.

Proof. If $d = |G : G'|$, then $G' \neq \{e\}$. In this case, $|\mathrm{Ker}(\lambda)| > 2$ and Theorem 4 applies. If $d < |G : G'|$, then Theorem 2 applies. ■

PROPOSITION 3. *Let* $p \neq 2$, F *be a perfect field of characteristic* p, $S = F[[X]]$, G *an abelian* p*-group,* $\overline{G}$ *the socle of* G, *and* $\lambda \in Z^2(G, S^*)$. *Suppose that* $S^\lambda\overline{G}/X^2S^\lambda\overline{G}$ *is not the group ring of* $\overline{G}$ *over the ring* S/X^2S. *If* $|\overline{G}| > p$, *then* $S^\lambda G$ *is of SUR-type. If* $|\overline{G}| = p$, *then* $S^\lambda G$ *is of finite representation type.*

Proof. Arguing as in the proof of Proposition 4.4 of [4], we show that if $|\overline{G}| > p$, then $S^\lambda\overline{G} = S^\mu\overline{G}$, where $\mu \in Z^2(\overline{G}, S^*)$ and $\mathrm{Ker}(\mu) \neq \{e\}$. Applying induction from $S^\mu\,\mathrm{Ker}(\mu)$-modules to $S^\mu\overline{G}$-modules and next from

$S^\lambda\overline{G}$-modules to $S^\lambda G$-modules, we deduce, in view of Lemmas 5 and 7, that $S^\lambda G$ is of SUR-type. If $|\overline{G}| = p$ then, by Proposition 4.4 of [4], $S^\lambda G$ is of finite representation type. ∎

PROPOSITION 4. *Let F be a perfect field of characteristic* 2, $S = F[[X]]$, *G an abelian* 2*-group, and $\lambda \in Z^2(G, S^*)$. Assume that G contains a cyclic subgroup H of order* 4 *such that $S^\lambda H/X^2 S^\lambda H$ is not the group ring of H over the ring S/X^2S. Then*:

(i) *the ring $S^\lambda G$ is of bounded representation type if and only if G is a cyclic group or a group of type* $(2^n, 2)$;
(ii) *the ring $S^\lambda G$ is of SUR-type if and only if it is of unbounded representation type.*

Proof. Let $D = \{g \in G : g^4 = e\}$. By the same type of argument as in the proof of Proposition 4.5 of [4], one can establish that if G is neither a cyclic group nor a group of type $(2^n, 2)$, then $S^\lambda D = S^\mu D$, where $|\mathrm{Ker}(\mu)| \geq 4$. Arguing as in the proof of Proposition 3, we conclude that $S^\lambda G$ is of SUR-type. If G is a cyclic group or a group of type $(2^n, 2)$, then, by Proposition 4.5 of [4], $S^\lambda G$ is of finite representation type. ∎

5. The projective representation type of finite groups over local rings. Let S be a commutative ring with identity, S^* the multiplicative group of S, W a subgroup of S^*, $\mathrm{GL}(n, S)$ the group of all unimodular matrices of order n over S, G a finite group, and $Z^2(G, W)$ the group of all W-valued normalized 2-cocycles of the group G that acts trivially on W. A *projective* (S, W)*-representation* of the group G of degree n is defined [1] as a mapping $\Gamma : G \to \mathrm{GL}(n, S)$ such that $\Gamma(e) = E$ and $\Gamma(a)\Gamma(b) = \lambda_{a,b}\Gamma(ab)$, where $\lambda_{a,b} \in W$ for all $a, b \in G$. It is easy to see that $\lambda : (a, b) \mapsto \lambda_{a,b}$ belongs to $Z^2(G, W)$. We also say that Γ is a projective (S, W)-representation of G *with cocycle* λ. Two projective (S, W)-representations Γ_1 and Γ_2 of G are called *equivalent* if there exists a unimodular matrix C over S and elements $\alpha_g \in W$ $(g \in G)$ such that

$$C^{-1}\Gamma_1(g)C = \alpha_g\Gamma_2(g)$$

for all $g \in G$. If $W = S^*$ then Γ is called a *projective S-representation* of G. If $W = \{1\}$ then Γ is said to be a *linear* or *ordinary S-representation* of G. By analogy with indecomposable projective S-representations of the group G, we can introduce the concept of an indecomposable projective (S, W)-representation of G ([9, §51]).

We say that a group G is of *finite projective* (S, W)*-representation type* if the number of (inequivalent) indecomposable projective (S, W)-representations of G with cocycle λ is finite for any $\lambda \in Z^2(G, W)$. Otherwise, G is said to be of *infinite projective* (S, W)*-representation type*. If the number of indecomposable projective (S, W)-representations of G with cocycle λ is infinite for every $\lambda \in Z^2(G, W)$, we say that G is of *purely infinite projective* (S, W)*-representation type*. A group G is defined to be of *bounded projective* (S, W)*-representation type* if the set of degrees of all indecomposable projective (S, W)-representations of G with cocycle λ is finite for each $\lambda \in Z^2(G, W)$. Otherwise, G is said to be of *unbounded projective* (S, W)*-representation type*. If the set of degrees of all indecomposable projective (S, W)-representations of G with cocycle λ is infinite for each $\lambda \in Z^2(G, W)$, G is defined to be of *purely unbounded projective* (S, W)*-representation type*. A group G is of *strongly unbounded projective* (S, W)*-representation type* if for some cocycle $\lambda \in Z^2(G, W)$ there is a function $f_\lambda : \mathbb{N} \to \mathbb{N}$ such that $f_\lambda(n) \geq n$ and the number of indecomposable projective (S, W)-representations of G with cocycle λ and of degree $f_\lambda(n)$ is infinite for all $n > 1$. If there is such a function f_λ for every $\lambda \in Z^2(G, W)$, then G is of *purely strongly unbounded projective* (S, W)*-representation type*.

PROPOSITION 5. *Let S be a local integral domain of characteristic p, $\operatorname{rad} S \neq 0$, F a subfield of S, W a subgroup of S^*, and G a finite p-group.*

(i) *If $|G| > 2$, then G is of strongly unbounded projective (S, W)-representation type.*
(ii) *If $|G'| > 2$, then G is of purely strongly unbounded projective (S, S^*)-representation type.*
(iii) *Let $W \subset F^*$ and G/G' be a direct product of r cyclic subgroups, where $r \geq i_F(W) + 1$ for $p > 2$ and $r \geq i_F(W) + 2$ for $p = 2$. Then G is of purely strongly unbounded projective (S, W)-representation type.*

Proof. Statement (i) follows immediately from the results of [19], [20] (see also Lemmas 2 and 3). Statement (ii) follows from Theorem 1. Now we prove (iii). Let $H = G/G'$, and $\overline{H}$ be the socle of H. For any cocycle $\mu \in Z^2(H, W)$ we have $S^\mu \overline{H} = S^\sigma \overline{H}$, where $\sigma \in Z^2(\overline{H}, W)$ and $B := \operatorname{Ker}(\sigma)$ satisfies the following conditions: if $p > 2$, then $|B| \geq p$; if $p = 2$, then $|B| \geq 4$. Applying induction from $S^\sigma B$-modules to $S^\sigma \overline{H}$-modules, and then from $S^\mu \overline{H}$-modules to $S^\mu H$-modules, we conclude, in view of Lemmas 5 and 7, that $S^\mu H$ is of SUR-type. Since for every $\lambda \in Z^2(G, W)$ there exists a homomorphism of $S^\lambda G$ onto $S^\mu H$, where $\mu_{xG', yG'} = \lambda_{x,y}$ for all $x, y \in G$, it follows that G is of purely strongly unbounded projective (S, W)-representation type. ∎

PROPOSITION 6. *Let G be a finite p-group, F a field of characteristic p, $S = F[[X]]$, and W a subgroup of S^*.*

(i) *G is of bounded projective (S, W)-representation type if and only if $|G| = 2$. Moreover, G is of unbounded projective (S, W)-representa-*

tion type if and only if G is of strongly unbounded projective (S,W)-representation type.

(ii) *Let $W \subset F^*$ and $p \neq 2$. Then G is of purely strongly unbounded projective (S,W)-representation type if and only if $|G'| \neq 1$ or G is a direct product of l cyclic subgroups and $l \geq i_F(W)+1$. In addition, G is of purely strongly unbounded projective (S,W)-representation type if and only if G is of purely unbounded projective (S,W)-representation type.*

(iii) *Let $p = 2$ and $|G'| \neq 2$. Then G is of purely strongly unbounded projective (S,F^*)-representation type if and only if one of the following conditions is satisfied: 1) $|G'| > 2$; 2) G is a direct product of l cyclic subgroups and $l \geq i_F(F^*)+2$; 3) G is a direct product of $i_F(F^*)+1$ cyclic subgroups whose orders are not equal to 2. Furthermore, G is of purely strongly unbounded projective (S,F^*)-representation type if and only if G is of purely unbounded projective (S,F^*)-representation type.*

Proof. (i) It follows from Lemma 6 (or Lemma 7) that if G is of bounded projective (S,W)-representation type, then $|G| = 2$. Let us prove the sufficiency. Let $|G| = 2$ and $\lambda \in Z^2(G,W)$. If $S^\lambda G = SG$ then the S-rank of every indecomposable $S^\lambda G$-module is 1 or 2 (see [17]). Assume that $S^\lambda G \neq SG$. Then $S^\lambda G \cong S[\theta]$, where θ is a root of the polynomial $Y^2 - \alpha$, $\alpha \in S^*$, which is irreducible over S. Let $\alpha = a_0 + a_1X + a_2X^2 + \cdots$, $a_i \in F$. Denote by K the quotient field of S and by T the integral closure of S in $K(\theta)$. If $a_0 \notin F^2$, then $T = S[\theta]$. Let $a_0 \in F^2$. Obviously, we can assume $a_0 = 1$. Then $T = S + S\omega$, where $\omega = X^{-n}(1 + b_1X + \cdots + b_{n-1}X^{n-1} + \theta)$ and

$$\alpha = 1 + b_1^2X^2 + \cdots + b_{n-1}^2X^{2n-2} + \sum_{j\geq 2n} a_jX^j, \qquad a_{2n} \notin F^2 \text{ or } a_{2n+1} \neq 0.$$

It is clear that the ring $S[\theta]$ is noetherian and T is finitely generated as an $S[\theta]$-module. Since S is a principal ideal domain, every ideal in $S[\theta]$ can be generated by two elements. Moreover, any ring L with $S[\theta] \subset L \subset T$ is local. Applying Theorem 1.7 of [7], we show that each indecomposable torsion free $S[\theta]$-module is isomorphic to a ring L with $S[\theta] \subset L \subset T$. Hence the S-rank of each indecomposable $S^\lambda G$-module equals 2. The second statement follows from Theorem 1 and the first statement.

(ii) Apply Proposition 5.

(iii) Let $p = 2$, $m = i_F(F^*)$, and G be a direct product of $m+1$ cyclic subgroups of order 4 each. We show that $\dim_F \overline{F^\lambda G} \leq \frac{1}{4}|G|$ for all $\lambda \in Z^2(G,F^*)$. Obviously, it is sufficient to prove this for

$$F^\lambda G = \bigoplus_{i_1,\ldots,i_{m+1}} Fu_{a_1}^{i_1}\ldots u_{a_{m+1}}^{i_{m+1}}, \quad \text{with} \quad u_{a_j}^4 = \alpha_j u_e \ (j = 1,\ldots,m+1),$$

where $K = F[u_{a_1}, \dots, u_{a_m}]$ is a field. Let $L = F[u_{a_1}^2, \dots, u_{a_m}^2]$. For each $\alpha \in F$ there exists $\beta \in L$ such that $\alpha = \beta^2$. The element β is uniquely expressible as

$$\beta = \sum_{i_1,\dots,i_m} \gamma_{i_1,\dots,i_m} u_{a_1}^{2i_1} \cdots u_{a_m}^{2i_m},$$

where $i_j = 0, 1$ and $\gamma_{i_1,\dots,i_m} \in F$. However, $\gamma_{i_1,\dots,i_m} = \delta_{i_1,\dots,i_m}^2$ for some $\delta_{i_1,\dots,i_m} \in L$. It follows that $\beta = \varrho^2$ for $\varrho \in K$, and hence $\alpha = \varrho^4$. This allows us to assume that $\alpha_{m+1} = 1$. But then $\dim_F \overline{F^\lambda G} = 4^m$, $4^m = \frac{1}{4}|G|$.

If condition 1) holds, we apply Proposition 5. If 2) or 3) holds, we apply Theorem 3. ∎

Proposition 7. *Let G be a finite p-group, F a field of characteristic p, $S = F[[X]]$, and W a subgroup of S^*.*

(a) *G is of infinite projective (S, W)-representation type.*
(b) *If $W \subset F^*$, then G is of purely infinite projective (S, W)-representation type if and only if one of the following two conditions is satisfied*: 1) $|G'| \neq 1$; 2) *G is a direct product of l cyclic subgroups*, *where $l \geq i_F(W) + 1$.*

Proof. Statement (a) follows from Theorems 1 and 3.

(b) Let $W \subset F^*$. If 1) or 2) is satisfied, then in view of Theorems 2 and 3, G is of purely infinite projective (S, W)-representation type. Let G be a direct product of r cyclic subgroups, where $r \leq i_F(W)$. Then there is a cocycle $\lambda \in Z^2(G, W)$ such that $F^\lambda G$ is a field. Let $K = F^\lambda G$. We have $S^\lambda G \cong K[[X]]$, and so every indecomposable $S^\lambda G$-module is isomorphic to $S^\lambda G$. Hence G is not of purely infinite projective (S, W)-representation type. ∎

Proposition 8. *Let G be a finite 2-group, $|G'| = 2$, F a field of characteristic 2, and $S = F[[X_1, \dots, X_m]]$. If $m > 1$ then G is of purely strongly unbounded projective (S, S^*)-representation type.*

Proof. By our assumption, $S^\lambda G' = SG'$ for every cocycle $\lambda \in Z^2(G, S^*)$, and the set $\mathrm{Ind}_n(SG')$ is infinite for each $n > 1$ (see [21]). Since S is a complete commutative noetherian local ring, the Krull–Schmidt Theorem holds for SG'-modules ([10, p. 128]). Then, arguing as in the proof of Theorem 1, we prove that for every $n > 1$ there exists a natural number t_n such that $1 \leq t_n \leq \frac{1}{2}|G|$ and $\mathrm{Ind}_{nt_n}(S^\lambda G)$ is infinite. ∎

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