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GENERALIZED HARDY SPACES ON TUBE DOMAINS OVER CONES

BY

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Abstract. We define a class of spaces H^p_μ, $0 < p < \infty$, of holomorphic functions on the tube, with a norm of Hardy type:

$$\|F\|^p_{H^p_\mu} = \sup_{y\in\Omega} \int_{\overline{\Omega}} \int_{\mathbb{R}^n} |F(x + i(y + t))|^p \, dx \, d\mu(t).$$

We allow μ to be any quasi-invariant measure with respect to a group acting simply transitively on the cone. We show the existence of boundary limits for functions in H^p_μ, and when $p \geq 1$, characterize the boundary values as the functions in L^p_μ satisfying the tangential CR equations. A careful description of the measures μ when their supports lie on the boundary of the cone is also provided.

1. Introduction. Let Ω be an irreducible symmetric cone in $\mathbb{R}^n$, and let

$$T_\Omega = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$$

be the tube domain based on Ω. As in [3], we shall write $r = \operatorname{rank}\Omega$, and $G(\Omega)$ for the group of linear transformations of the cone.

In this paper we study a general family of spaces $H^p_\mu(T_\Omega)$ of holomorphic functions in T_Ω satisfying an integrability condition of Hardy type:

$$\|F\|_{H^p_\mu} := \sup_{y\in\Omega} \Big[\int_{\overline{\Omega}} \int_{\mathbb{R}^n} |F(x + i(y + t))|^p \, dx \, d\mu(t)\Big]^{1/p} < \infty. \tag{1.1}$$

In this definition we let $0 < p < \infty$, and μ be any positive measure in $\mathbb{R}^n$ with the following two geometric assumptions:

1. μ is locally finite in $\mathbb{R}^n$, with $\operatorname{Supp}\mu \subset \overline{\Omega}$;
2. μ is *quasi-invariant* (or *homogeneous*) with respect to a subgroup H

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of $G(\Omega)$, acting simply transitively on the cone; that is,

$$\int f(h^{-1}y)\,d\mu(y)=\chi(h)\int f(y)\,d\mu(y),\qquad \forall f\in L^1(d\mu),\ h\in H,$$

where χ is a character of the group H.

Such measures appear in different contexts related to symmetric cones and Siegel domains, and were completely characterized by Gindikin in [5], [4] (see also §2.3 below).

The particular choice $\mu=\delta_0$ (the delta distribution at the origin) corresponds to the *classical Hardy space* on the tube:

$$H^p(T_\Omega)=\Big\{F\in\mathcal{H}(T_\Omega):\|F\|_{H^p}=\sup_{y\in\Omega}\Big[\int_{\mathbb{R}^n}|F(x+iy)|^p\,dx\Big]^{1/p}<\infty\Big\}.$$

On the other hand, the Lebesgue measure $d\mu(t)=\chi_\Omega(t)\,dt$, quasi-invariant with respect to $G(\Omega)$ (with $\chi(g)=|\det g|$), gives rise to the *Bergman space*

$$A^p(T_\Omega)=\Big\{F\in\mathcal{H}(T_\Omega):\|F\|_{A^p}^p=\int_\Omega\int_{\mathbb{R}^n}|F(x+iy)|^p\,dx\,dy<\infty\Big\}.$$

In this case, the "sup" in (1.1) plays no role by the monotonicity of the integrals (see 3.9 below). The properties of these two spaces have been widely studied, in particular those concerning the existence of boundary values (see, e.g., Chapter III of [13], and [1] for the Bergman case).

Other choices of quasi-invariant measures μ lead to less known holomorphic function spaces in the tube, which for $p=2$ appear in the representation theory of the semisimple Lie group $G(T_\Omega)$ (see [15], [10], [11]). These spaces are "intermediate" between Bergman and Hardy spaces, in the sense that they share many different properties with each of them. Our goal in this paper is to provide a characterization of the boundary values of functions in $H^p_\mu(T_\Omega)$, in the same spirit as for the classical Hardy spaces. The difference is that, in the general situation of $H^p_\mu(T_\Omega)$, the boundary values lie naturally on a "complex manifold"

$$T_\mu=\mathbb{R}^n+i\,\mathrm{Supp}\,\mu\subset\mathbb{C}^n,$$

rather than in the "distinguished boundary" $\mathbb{R}^n+i\{0\}$. To state our first theorem, let us establish the notation:

$$L^p_\mu:=L^p(\mathbb{R}^n+i\,\mathrm{Supp}\,\mu;dxd\mu(t))=L^p(T_\mu;dxd\mu).$$

THEOREM 1.2. *Let $0<p<\infty$ and μ be a measure as above. Then for every $F\in H^p_\mu(T_\Omega)$ there exists $F^{(\mathrm{b})}\in L^p_\mu$ such that*

$$\lim_{\substack{y\to0\\ y\in\Omega_0}}\|F(\cdot+iy)-F^{(\mathrm{b})}\|_{L^p_\mu}=0,$$

$$\lim_{\substack{y\to0\\ y\in\Omega_0}}F(z+iy)=F^{(\mathrm{b})}(z)\qquad\textit{for a.e. } z\in T_\mu,$$

for every proper subcone Ω_0 *of* Ω. *When* $p \geq 1$, *the first limit holds as well with* Ω_0 *replaced by* Ω.

Observe that the function $F^{(\mathrm{b})}$ in the theorem is defined in T_μ, and therefore, it is only a *boundary value* of F when the measure μ is singular (i.e., supported on $\partial\Omega$). In other words, the preceding theorem does not give new information for Bergman type spaces: $F^{(\mathrm{b})} = F$ (see [1] for a different treatment of this case).

We also remark that, if we exclude the classical Hardy space (i.e., $\mu = \delta_0$, for which T_μ is purely real), the function $F^{(\mathrm{b})}$ exhibits a holomorphic behavior in the "complex part" of the manifold $T_\mu = \mathbb{R}^n + i\,\mathrm{Supp}\,\mu \subset \mathbb{C}^n$. This is expressed in terms of the *tangential Cauchy–Riemann equations in* T_μ: $F^{(\mathrm{b})} \in \mathrm{CR}(T_\mu)$. Thus, one can interpret the boundary value $F^{(\mathrm{b})}$ as a function belonging to a "Bergman space" on the manifold T_μ: $A^p_\mu(T_\mu) \equiv L^p(T_\mu; dxd\mu) \cap \mathrm{CR}(T_\mu)$. The next theorem shows under what conditions this property actually characterizes $H^p_\mu(T_\Omega)$. Below, we denote by $\overline{\mathrm{co}}\,(E)$ the *closed convex envelope* of a given $E \subset \mathbb{R}^n$.

THEOREM 1.3. *Let* $1 \leq p < \infty$, *and* μ *be a quasi-invariant measure as above.*

(1) *If* $F \in H^p_\mu(T_\Omega)$, *then its boundary value* $F^{(\mathrm{b})}$ *belongs to* $A^p_\mu(T_\mu)$.

(2) *Suppose, in addition, that* $\overline{\mathrm{co}}\,(\mathrm{Supp}\,\mu) = \overline{\Omega}$. *Then if* $G \in A^p_\mu(T_\mu)$ *there exists a holomorphic function* $F \in H^p_\mu(T_\Omega)$ *such that* $G = F^{(\mathrm{b})}$. *In this case*,

$$F \in H^p_\mu(T_\Omega) \mapsto G = F^{(\mathrm{b})} \in A^p_\mu(T_\mu)$$

is an isometric isomorphism of Banach spaces.

The assumption on μ in the second part of the theorem is made so that every $G \in A^p_\mu(T_\mu)$ has a holomorphic extension to the whole tube T_Ω. This occurs when the support of μ is "large enough" (e.g., when μ is homogeneous under G, see [10]). For general measures there is also a characterization theorem where the extension property is obtained from a condition on the Fourier transform of $G_t := G(\cdot + it) \in L^p(\mathbb{R}^n)$:

THEOREM 1.4. *Let* $1 \leq p < \infty$, *and* μ *a quasi-invariant measure as above. Consider the following closed subspace of* $A^p_\mu(T_\mu)$:

$$A^p_\mu(T_\mu; \Omega) := \{G \in A^p_\mu(T_\mu) : \mathrm{Supp}\,\widehat{G}_t \subset \overline{\Omega} \text{ for a.e. } t \in \mathrm{Supp}\,\mu\}.$$

Then the correspondence $F \mapsto G = F^{(\mathrm{b})}$ *is an isometric isomorphism from* $H^p_\mu(T_\Omega)$ *onto* $A^p_\mu(T_\mu; \Omega)$. *In particular*, $A^p_\mu(T_\mu) = A^p_\mu(T_\mu; \Omega)$ *if and only if* $\overline{\mathrm{co}}\,(\mathrm{Supp}\,\mu) = \overline{\Omega}$.

The previous theorems have been stated assuming only the general homogeneity property in the definition of μ. The proofs we present, however,

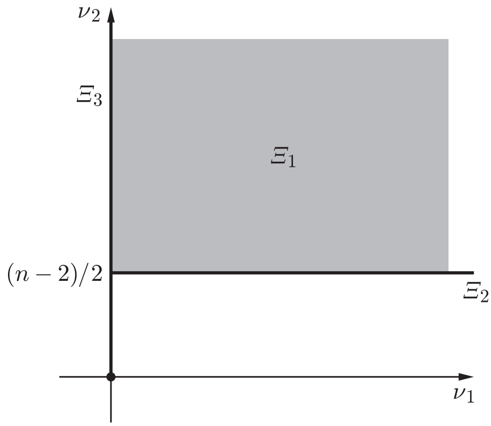

Fig. 1.1. The Wallach set $\boldsymbol{\Xi}$ of a cone of rank 2

will eventually require an explicit expression of the measure, in order to describe the manifold structure of T_μ and obtain the CR equations.

When Ω is an irreducible symmetric cone and H a simply transitive group acting on it, a characterization of quasi-invariant measures was given by Gindikin in [5]. Following the presentation in [3] (see §2 below for details), these measures coincide precisely with the *positive Riesz distributions* in Ω:

$$d\mu_{\boldsymbol{\nu}}(t) = \chi_\Omega(t)\, \frac{\Delta_{\boldsymbol{\nu}}(t)}{\Gamma_\Omega(\boldsymbol{\nu})}\, \frac{dt}{\Delta(t)^{n/r}}, \qquad \boldsymbol{\nu} = (\nu_1, \dots, \nu_r) \in \boldsymbol{\Xi}. \tag{1.5}$$

Here $\boldsymbol{\Xi}$ denotes the *Wallach set* of Ω, consisting of those indices $\boldsymbol{\nu} \in \mathbb{C}^r$ so that $\mu_{\boldsymbol{\nu}}$ is a positive measure. The subset $\Xi_1 = \big\{\boldsymbol{\nu} \mid \nu_j > \frac{j-1}{r-1}\big(\frac{n}{r} - 1\big)\big\}$ corresponds to absolutely continuous measures, while $\boldsymbol{\Xi}\setminus\Xi_1$ comprises those with support in $\partial\Omega$ (see Figure 1).

Our last result in this paper gives a complete description of the structure of $\operatorname{Supp}\mu_{\boldsymbol{\nu}}$, in terms of the orbits of H on $\overline{\Omega}$.

Theorem 1.6. *There exists a partition of* $\overline{\Omega} = \Omega_1 \cup \ldots \cup \Omega_s$ *with the following properties*:

(1) *The sets* Ω_j *are orbits of* H. *Further*, *there is a subgroup* H_j *of* H *and a point* $t_j \in \overline{\Omega}$ *such that* $\Omega_j = H_j t_j$.

(2) *The sets* Ω_j *are regular submanifolds of* $\mathbb{R}^n$, *and the measures* $\mu_{\boldsymbol{\nu}}$ *are smooth volume forms.*

(3) *If* $\boldsymbol{\nu} \in \boldsymbol{\Xi}$, *there exists a unique* $j = j(\boldsymbol{\nu})$ *such that*

$$\operatorname{Supp}\mu_{\boldsymbol{\nu}} = \overline{\Omega}_j \quad \textit{and} \quad \mu_{\boldsymbol{\nu}}(\Omega_l) = 0 \quad \forall l \neq j.$$

(4) *Given* $\boldsymbol{\nu} \in \boldsymbol{\Xi}$, *we have* $\overline{\mathrm{co}}\,(\operatorname{Supp}\mu_{\boldsymbol{\nu}}) = \overline{\Omega}$ *if and only if* $\nu_1 \neq 0$.

We point out that the preceding result is not completely new, since parts of it are contained in earlier work of Gindikin [5], [4], and more recent papers of H. Ishi [8], [9]. For the sake of completeness we shall present here a more affordable proof, using the modern notation of [3].

The paper is structured as follows. In §2 we recall the basic notions of symmetric cones and quasi-invariant measures. In §3 we study the spaces $H^p_\mu(T_\Omega)$ and give a proof of Theorem 1.2, and other related results. In §4 we characterize the special case $p = 2$ with a Paley–Wiener type theorem, obtaining as well reproducing formulas for the spaces $H^p_\mu(T_\Omega)$. In §5 we study the boundary of the cone, and give a proof of Theorem 1.6. Finally, §6 is devoted to the tangential CR equations in T_μ, and the proof of Theorems 1.3 and 1.4. Some technical matters on Jordan algebras and symmetric cones are also postponed to the appendix.

2. Symmetric cones and homogeneous measures. We first set some notation and recall well known properties from the theory of symmetric cones. We refer the reader to [3] for proofs and further results.

2.1. *Generalities about symmetric cones.* In this section Ω is an irreducible symmetric cone of rank r in $\mathbb{R}^n$. It is well known that Ω induces in $V \equiv \mathbb{R}^n$ the structure of a *Euclidean Jordan algebra*, in which $\overline{\Omega} = \{x^2 : x \in V\}$. We denote by $\mathbf{e}$ the identity element in V and by $(x|y) = \operatorname{tr}(xy)$ the canonical inner product ([1]).

Let $G(\Omega)$ be the group of transformations of Ω, and G its identity component. Since the cone is homogeneous, the group G acts transitively on Ω. We shall choose a natural subgroup H of G which acts *simply* transitively on Ω. That is, every $y \in \Omega$ can be uniquely written as $y = h\mathbf{e}$ with $h \in H$. This allows us to identify Ω with the quotient G/K, where K is the stabilizer of $\mathbf{e}$:

$$K = \{g \in G : g\mathbf{e} = \mathbf{e}\} = G \cap O(V)$$

(see Chapter I of [3]).

To give a precise description of H we fix a *Jordan frame* $\{c_1, \ldots, c_r\}$ in V. That is, a system of primitive idempotents with the properties

$$c_1 + \ldots + c_r = \mathbf{e} \quad \text{and} \quad c_i c_j = 0, \quad i \neq j.$$

This induces a Peirce decomposition:

$$V = \bigoplus_{1 \le i \le j \le r} V_{i,j}, \tag{2.1}$$

which formally lets us regard V as a space of symmetric matrices (with $V_{i,j}$ as "(i,j)-entry"; see Chapter IV of [3]). More precisely, the subspaces in

([1]) The reader less familiar with Jordan algebras can look at the example $\Omega = \mathrm{Sym}_+(r, \mathbb{R})$. In this case, $V = \mathrm{Sym}(r, \mathbb{R})$, with the Jordan product defined as $x \circ y = (xy + yx)/2$, from the usual matrix multiplication xy.

(2.1) are given by $V_{i,i} = \mathbb{R} \cdot c_i$ and

$$V_{i,j} = V(c_i, 1/2) \cap V(c_j, 1/2) = \{x \in V : c_i x = c_j x = x/2\} \quad \text{for } i < j.$$

For each $i < j$, the dimension of $V_{i,j}$ is a constant integer

$$d = 2\,\frac{n/r - 1}{r - 1}.$$

We define H as the subgroup of matrices $h \in G$ which are *lower triangular* with respect to Peirce decomposition of $\mathbb{R}^n$. That is, given a vector $x = \sum_{i \le j} x_{i,j} \in \bigoplus_{i \le j} V_{i,j} \equiv \mathbb{R}^n$, we have

$$\begin{aligned} (hx_{k,l})_{i,j} &= 0 && \text{if } (i,j) < (k,l), \\ (hx_{i,j})_{i,j} &= \lambda_{i,j} x_{i,j} && \text{for some } \lambda_{i,j} > 0, \end{aligned}$$

where $(i,j) < (k,l)$ denotes the lexicographic order. Then, by Theorem VI.3.6 in [3], H acts simply transitively on Ω. Further, one can write $H = NA = AN$, where N denotes the *strict triangular* subgroup of H (i.e., matrices with $\lambda_{i,j} = 1$), and A the *diagonal* subgroup (i.e., $(hx_{k,l})_{i,j} = 0$ if $(k,l) \ne (i,j)$, and also $\lambda_{i,j} = \lambda_i \lambda_j$).

A more explicit expression of all these groups can be given in terms of the endomorphisms of left multiplication in the Jordan algebra V:

$$L(x) : y \mapsto xy \quad \text{for } x \in V.$$

Each endomorphism $L(x)$ is a symmetric operator (with respect to the inner product $(\cdot|\cdot)$) belonging to the Lie algebra $\mathfrak{g}$ of G (see Chapter III of [3]). The main use of L is to define the following two important transformations in a symmetric cone (see Chapters II and VI in [3]):

1. The quadratic *representation*:

$$x \in V \mapsto P(x) = 2L(x)^2 - L(x^2).$$

When $x \in \Omega = \{e^y : y \in V\}$, $P(x)$ can also be written as

$$x = e^y \in \Omega \mapsto P(x) = \exp(2L(y)) \in G.$$

2. The *Frobenius transformation*:

$$z \in V(c_j, 1/2) \mapsto \tau^{(j)}(z) = \exp(L(z) + 2[L(z), L(c_j)]) \in G.$$

With this notation, the statement of Theorem VI.3.6 in [3] can also be read as:

$$A = \Big\{ P(a) : a = \sum_{j=1}^{r} a_j c_j,\ a_j > 0,\ 1 \le j \le r \Big\},$$

$$N = \Big\{ \tau^{(1)}(z_1) \dots \tau^{(r-1)}(z_{r-1}) : z_j \in \bigoplus_{k=j+1}^{r} V_{j,k},\ 1 \le j \le r-1 \Big\},$$

and every $h \in H$ can be uniquely written as

$$h = \tau^{(1)}(z_1) \dots \tau^{(r-1)}(z_{r-1}) P(a) \tag{2.2}$$

for a, z_j as above. In addition, we have the equality $NA = AN$, which follows from the identity

$$P(a)\tau^{(j)}(z) = \tau^{(j)}(\widetilde{z})P(a) \quad \text{where} \quad \widetilde{z} = \sum_{k=j+1}^{r} \frac{a_k}{a_j} z_{j,k} \tag{2.3}$$

and $a = \sum_{j=1}^{r} a_j c_j$, $z = \sum_{k=j+1}^{r} z_{j,k}$ are as above (see Proposition VI.3.7 of [3]).

In the particular case of $\Omega = \mathrm{Sym}_+(r, \mathbb{R})$, (2.2) corresponds to the *Gauss factorization* of a triangular $r \times r$-matrix h. We point out that the theory just described provides two classical decompositions of a semisimple Lie group: $G = NAK$ and $G = KAK$.

2.2. *Determinants and integrals.* As in [3], we let $\Delta(x) = \det(x)$, $x \in V$. Furthermore, we denote by $\Delta_1(x), \dots, \Delta_r(x)$ the *principal minors* of $x \in V$, with respect to the fixed Jordan frame $\{c_1, \dots, c_r\}$. That is, $\Delta_k(x)$ is the determinant of the projection $P_k x$ of x in the Jordan subalgebra $V^{(k)} = \bigoplus_{1 \le i \le j \le k} V_{i,j}$. It is well known (see Chapter VI of [3]) that the action of N leaves invariant each of these forms:

$$\Delta_k(nx) = \Delta_k(x), \qquad n \in \mathbb{N},\ x \in V.$$

Also, for $a = a_1 c_1 + \dots + a_r c_r$ we have $\Delta_k(P(a)x) = a_1^2 \dots a_k^2 \Delta_k(x)$. In particular, $\Delta_k(x) > 0$ for all $k = 1, \dots, r$ and $x \in \Omega$.

The *generalized power function* on Ω is defined as

$$\Delta_{\mathbf{s}}(x) = \Delta_1^{s_1 - s_2}(x) \Delta_2^{s_2 - s_3}(x) \dots \Delta_r^{s_r}(x), \qquad \mathbf{s} = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r,\ x \in \Omega.$$

In the particular case $x = a_1 c_1 + \dots + a_r c_r \in \Omega$, one has $\Delta_{\mathbf{s}}(x) = a_1^{s_1} \dots a_r^{s_r}$.

The next lemma characterizes the characters of H (see also [4]).

LEMMA 2.4. *The characters of the group H are the functions*

$$h \in H \mapsto \Delta_{\mathbf{s}}(h\mathbf{e}) \quad \textit{for every } \mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r. \tag{2.5}$$

Proof. It is easy to see that the functions in (2.5) are characters of H. Indeed, this follows from the properties of principal minors: $\Delta_k(hh'\mathbf{e}) = \Delta_k(h\mathbf{e})\Delta_k(h'\mathbf{e})$ for all $h, h' \in H$ (see Proposition VI.3.10 in [3]). Conversely, if χ is a character of H then we must have $\chi(h) = \chi(khk^{-1})$ for all $k, h \in H$. Since H consists of triangular matrices, this implies that $\chi(h)$ can only depend on the diagonal entries of $h \in H$. Thus, $\chi(nP(a)) = \chi(P(a))$, and the lemma follows immediately, since the characters of the abelian group A are precisely the powers $a_1^{s_1} \dots a_r^{s_r}$ for $s_1, \dots, s_r \in \mathbb{C}$. ∎

Finally, we recall the definition of the *generalized gamma function* on Ω:

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-(\mathbf{e}|\xi)} \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta(\xi)^{n/r}}, \quad \mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r.$$

This integral converges if and only if

$$\operatorname{Re} s_j > (j-1)\frac{n/r-1}{r-1} = (j-1)\frac{d}{2} \quad \text{for all } j = 1, \ldots, r,$$

and in this case it is equal to

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{(n-r)/2} \prod_{j=1}^r \Gamma\left(s_j - (j-1)\frac{n/r-1}{r-1}\right)$$

(see Chapter VII of [3]). As usual, we shall denote $\Gamma_\Omega(\mathbf{s})$ by $\Gamma_\Omega(s)$ when $\mathbf{s} = (s, \ldots, s)$. The main result concerns the *Laplace transform* of the generalized power function, whose formula is not difficult to deduce from the invariance properties of $\Delta_{\mathbf{s}}$ (see Proposition VII.1.2 in [3]).

LEMMA 2.6. *Let* $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ *with* $\operatorname{Re} s_j > (j-1)d/2$, $j = 1, \ldots, r$. *Then for all* $y \in \Omega$ *we have*

$$\int_\Omega e^{-(\xi|y)} \Delta_{\mathbf{s}}(\xi) \frac{d\xi}{\Delta(\xi)^{n/r}} = \Gamma_\Omega(\mathbf{s}) \Delta_{\mathbf{s}}(y^{-1}).$$

REMARK 2.7. The power function $\Delta_{\mathbf{s}}(y^{-1})$ above can also be expressed in terms of the *rotated* Jordan frame $\{c_r, \ldots, c_1\}$. If we denote by Δ_j^*, $j = 1, \ldots, r$, the principal minors with respect to this new frame then

$$\Delta_{\mathbf{s}}(y^{-1}) = [\Delta^*_{\mathbf{s}^*}(y)]^{-1}, \quad \forall \mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r,$$

where we have set $\mathbf{s}^* := (s_r, \ldots, s_1)$ (see Proposition VII.1.5 in [3]).

2.3. *The quasi-invariant measures.* Recall from (1.3) the definition of the measures $\mu_{\mathbf{s}}$. As we pointed out, by analytic continuation one extends this definition to all $\mathbf{s} \in \mathbb{C}^r$, obtaining a family of tempered distributions (see Theorem VII.2.6 in [3]). The following result of Gindikin characterizes the positive measures in the family $\{\mu_{\mathbf{s}}\}_{\mathbf{s} \in \mathbb{C}^r}$ (see [5], or Theorem VII.3.2 in [3]). Below, we denote by $\varepsilon(u)$ the *signum function*: $\varepsilon(u) = 1$ if $u > 0$, and $\varepsilon(0) = 0$.

PROPOSITION 2.8. *Let* $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. *Then* $\mu_{\mathbf{s}}$ *is a positive measure if and only if* $\mathbf{s}$ *belongs to the Wallach set*

$$\boldsymbol{\Xi} = \left\{ \left(u_1, u_2 + \frac{d}{2}\varepsilon(u_1), \ldots, u_r + \frac{d}{2}[\varepsilon(u_1) + \ldots + \varepsilon(u_{r-1})] \right) : u_1, \ldots, u_r \geq 0 \right\}.$$

Let now μ be a positive measure in $\mathbb{R}^n$, locally finite, and with support contained in $\overline{\Omega}$. Suppose also that μ is *quasi-invariant* with respect to the group H in §2.1. That is, for every $h \in H$ there exists $\chi(h) \in \mathbb{C}$ so that

$$\int f(h^{-1}y)\, d\mu(y) = \chi(h) \int f(y)\, d\mu(y), \qquad f \in L^1(d\mu). \tag{2.9}$$

The group composition implies that χ is a character of H: $\chi(hh') = \chi(h)\chi(h')$. Therefore, by Lemma 2.4 there must exist $\mathbf{s} \in \mathbb{C}^r$ such that $\chi(h) = \Delta_{\mathbf{s}}(h\mathbf{e})$. The next proposition tells us that, modulo a constant, μ must be equal to $\mu_{\mathbf{s}}$ (see also [5]).

PROPOSITION 2.10. *Let μ be a positive locally finite measure in $\mathbb{R}^n$ with support in $\overline{\Omega}$. Then μ is quasi-invariant with respect to H if and only if $\mu = c\mu_{\mathbf{s}}$ for some $c > 0$ and $\mathbf{s} \in \mathbf{\Xi}$.*

Proof. It is clear from the results in §2.2 that each measure $\mu_{\mathbf{s}}$ is quasi-invariant. For the converse, let μ be a measure with the above assumptions, and with associated character $\chi(h) = \Delta_{\mathbf{s}}(h\mathbf{e})$. We shall show that necessarily $\mu = c\mu_{\mathbf{s}}$ for some constant $c > 0$. To do this we prove that μ is a tempered distribution, and (modulo a constant) μ and $\mu_{\mathbf{s}}$ have the same Fourier–Laplace transform.

The first claim follows easily from the quasi-invariance. Indeed, one just notices that the measure of a ball $B(0, R)$ grows at most polynomially with the radius R:

$$\int \chi_{B(0,R)}(y)\, d\mu(y) = \int \chi_{B(0,1)}(y/R)\, d\mu(y) = R^{s_1+\ldots+s_r}\, \mu(B(0,1)).$$

For the second assertion, we first show that the integral defining the Laplace transform $\mathcal{L}\mu(\xi)$, $\xi \in \Omega$, converges absolutely. Indeed, by Lemma I.1.5 in [3], there is a constant $C_\xi > 0$ so that $(\xi|t) \geq C_\xi |t|$ for all $t \in \overline{\Omega}$. Thus, using the condition $\operatorname{Supp} \mu \subset \overline{\Omega}$, we see that the integral

$$\mathcal{L}\mu(\xi) = \int e^{-(\xi|y)}\, d\mu(y), \qquad \xi \in \Omega,$$

converges absolutely, and moreover, the Fourier–Laplace transform

$$\mathcal{F}\mu(z) = \int e^{i(z|y)}\, d\mu(y), \qquad z \in T_\Omega,$$

defines a holomorphic function on the tube T_Ω. Since the measures $\mu_{\mathbf{s}}$ are also supported in $\overline{\Omega}$, and a holomorphic function in T_Ω is determined by its values in $i\Omega$, it suffices to show the equality $\mathcal{L}\mu(\xi) = c\mathcal{L}\mu_{\mathbf{s}}(\xi)$ for all $\xi \in \Omega$.

To do this, let $\xi = h^*\mathbf{e}$, for $h \in H$, be an arbitrary point in Ω. Then, choosing $c(\mu) = \mathcal{L}\mu(\mathbf{e})$, we have

$$\begin{aligned}\mathcal{L}\mu(\xi) &= \int e^{-(\xi|y)}\, d\mu(y) = \int e^{-(\mathbf{e}|hy)}\, d\mu(y) \\ &= \Delta_{\mathbf{s}}(h^{-1}\mathbf{e})\mathcal{L}\mu(\mathbf{e}) = c(\mu)\Delta_{\mathbf{s}}(\xi^{-1}) = c(\mu)\mathcal{L}\mu_{\mathbf{s}}(\xi),\end{aligned}$$

where we have used the identity $h^{-1}\mathbf{e} = (h^*\mathbf{e})^{-1}$ (see p. 124 of [3]), and Lemma 2.6. ∎

We conclude this section with a simple lemma, valid for all measures, which will be useful later.

LEMMA 2.11. *Let μ be a locally finite positive measure in $\mathbb{R}^n$, not null in a neighborhood of the origin. Then there exist $c, c' > 0$ so that*

$$c \int_{B_{1/8}(\mathbf{e})} f(y)\,dy \le \int_{B_{1/4}(0)} \int_{B_{1/4}(\mathbf{e})} f(y+t)\,dy\,d\mu(t) \le c' \int_{B_{1/2}(\mathbf{e})} f(y)\,dy$$

for all non-negative f.

Proof. The second inequality is obvious with $c' = \mu(B_{1/4}(0))$. For the first inequality, note that given $t \in B_{1/8}(0)$ we have

$$\int_{B_{1/8}(\mathbf{e})} f(y)\,dy = \int_{B_{1/8}(\mathbf{e})-t} f(y+t)\,dy \le \int_{B_{1/4}(\mathbf{e})} f(y+t)\,dy.$$

Integrating with respect to $d\mu(t)$ we obtain

$$\mu(B_{1/8}(0)) \int_{B_{1/8}(\mathbf{e})} f(y)\,dy \le \int_{B_{1/8}(0)} \int_{B_{1/4}(\mathbf{e})} f(y+t)\,dy\,d\mu(t). \ \blacksquare$$

3. The spaces $H^p_\mu(T_\Omega)$. Throughout this section $0 < p < \infty$ is fixed and μ is a quasi-invariant measure with respect to H. We assume that μ has associated character $\chi(h) = \Delta_{\mathbf{s}}(h\mathbf{e})$ for some $\mathbf{s} \in \boldsymbol{\Xi}$. This implies that, modulo a constant, $\mu = \mu_{\mathbf{s}}$, although on the formal level we shall not use this fact.

3.1. *Basic properties of the norm.* Our first result tells us that the spaces $H^p_\mu(T_\Omega)$ are invariant under transformations in H.

PROPOSITION 3.1. *Let $h \in H$. Then $F \in H^p_\mu(T_\Omega)$ if and only if $F \circ h \in H^p_\mu(T_\Omega)$. In this case,*

$$\|F\|_{H^p_\mu} = \Delta_{\mathbf{s}+n/r}(h\mathbf{e})^{1/p} \|F \circ h\|_{H^p_\mu}.$$

Proof. The proof is an immediate consequence of the quasi-invariance of μ:

$$\begin{aligned}
\|F \circ h^{-1}\|^p_{H^p_\mu} &= \sup_{y\in\Omega} \int \int_{\mathbb{R}^n} |F \circ h^{-1}(x + i(y+t))|^p\,dx\,d\mu(t) \\
&= (\operatorname{Det} h) \sup_{y\in\Omega} \int \int_{\mathbb{R}^n} |F(x + i(y + h^{-1}t))|^p\,dx\,d\mu(t) \\
&= \Delta^{n/r}(h\mathbf{e}) \Delta_{\mathbf{s}}(h\mathbf{e}) \|F\|^p_{H^p_\mu},
\end{aligned}$$

where we used the identity $\operatorname{Det} h = \Delta(h\mathbf{e})^{n/r}$ (see III.4.3 of [3]). $\blacksquare$

PROPOSITION 3.2. *There exists a constant $c > 0$ so that, for all $F \in H^p_\mu(T_\Omega)$,*

$$|F(x+iy)| \le c\Delta_{\mathbf{s}+n/r}(y)^{-1/p} \|F\|_{H^p_\mu}, \qquad \forall x + iy \in T_\Omega. \tag{3.3}$$

Proof. Since $H^p_\mu(T_\Omega)$ is invariant under translation by $x \in \mathbb{R}^n$ we may assume $x = 0$. Next, we show (3.3) for $y = \mathbf{e}$. The mean value property applied to the subharmonic function $|F|^p$, together with Lemma 2.11, gives us

$$\begin{aligned}|F(i\mathbf{e})|^p &\le c \int\limits_{B_{1/8}(\mathbf{e})} \int\limits_{B_1(0)} |F(x+iy)|^p\,dx\,dy\\ &\le c' \int\limits_{B_{1/4}(\mathbf{e})} \int\limits_{B_{1/4}(0)} \int\limits_{B_1(0)} |F(x+i(y+t))|^p\,dx\,d\mu(t)\,dy \le c''\|F\|^p_{H^p_\mu}.\end{aligned}$$

In general, if $y = h\mathbf{e}$ for some $h \in H$, we apply the previous inequality to $F \circ h$ and use Proposition 3.1:

$$|F(iy)|^p = |F\circ h(i\mathbf{e})|^p \le c''\|F\circ h\|^p_{H^p_\mu} = c''\Delta_{\mathbf{s}+n/r}(y)^{-1}\|F\|^p_{H^p_\mu}. \ \blacksquare$$

The previous proposition tells us that the pointwise evaluation is a continuous linear functional in $H^p_\mu(T_\Omega)$. A standard argument, using convergence on compact sets, gives the following corollary.

Corollary 3.4. *$H^p_\mu(T_\Omega)$ is a complete metric space.*

A slight refinement in the proof of Proposition 3.2 provides a result which will be of crucial importance to us.

Proposition 3.5. *There exists a constant $c > 0$ so that, for all $F \in H^p_\mu(T_\Omega)$ and $y \in \Omega$,*

$$\|F(\cdot + iy)\|_{L^p(\mathbb{R}^n)} \le c\Delta_{\mathbf{s}}(y)^{-1/p}\|F\|_{H^p_\mu}. \tag{3.6}$$

Proof. We first show (3.6) for $y = \mathbf{e}$. Now, the first part of the proof of Proposition 3.2 applied to $F(\cdot + x)$ gives

$$|F(x+i\mathbf{e})|^p \le c \int\limits_{B_{1/4}(\mathbf{e})} \int\limits_{B_{1/4}(0)} \int\limits_{B_1(0)} |F(x+x'+i(y+t))|^p\,dx'\,d\mu(t)\,dy.$$

Integrating on $x \in \mathbb{R}^n$ we obtain

$$\begin{aligned}&\|F(\cdot + i\mathbf{e})\|^p_{L^p(\mathbb{R}^n)}\\ &\qquad\le c' \int\limits_{B_{1/4}(\mathbf{e})} \int\limits_{B_{1/4}(0)} \|F(\cdot + i(y+t))\|^p_{L^p(\mathbb{R}^n)}\,d\mu(t)\,dy \le c''\|F\|^p_{H^p_\mu}.\end{aligned}$$

For a general $y = h\mathbf{e} \in \Omega$, we apply the previous inequality to $F \circ h$ and obtain

$$\begin{aligned}\|F(\cdot + iy)\|_{L^p(\mathbb{R}^n)} &= (\operatorname{Det} h)^{1/p}\|F\circ h(\cdot + i\mathbf{e})\|_{L^p(\mathbb{R}^n)}\\ &\le c(\operatorname{Det} h)^{1/p}\|F\circ h\|^p_{H^p_\mu} = c\Delta_{\mathbf{s}}(y)^{-1/p}\|F\|_{H^p_\mu}. \ \blacksquare\end{aligned}$$

Corollary 3.7. *If $F \in H^p_\mu(T_\Omega)$ and $y \in \Omega$, then $F_y := F(\cdot + iy) \in H^p(T_\Omega)$ and there is a constant $c > 0$ so that*

$$\|F_y\|_{H^p} \le c\Delta_{\mathbf{s}}(y)^{-1/p}\|F\|_{H^p_\mu}.$$

The proof is an immediate consequence of Proposition 3.5 and the next general lemma on symmetric cones, whose proof is postponed to the appendix.

LEMMA 3.8. *Let* $s_1, \ldots, s_r \geq 0$ *and* $\mathbf{s} = (s_1, \ldots, s_r)$. *Then*

$$\Delta_{\mathbf{s}}(y + y') \geq \Delta_{\mathbf{s}}(y) \quad \forall y, y' \in \Omega.$$

A second corollary of Proposition 3.5 gives us the monotonicity of the integrals defining the norm $\|F\|_{H^p_\mu}$. Recall that $L^p_\mu = L^p(\mathbb{R}^n + i\overline{\Omega}; dx d\mu(t))$.

COROLLARY 3.9. *Let* $F \in H^p_\mu(T_\Omega)$ *and* $y, y' \in \Omega$. *Then*

$$\|F(\cdot + i(y + y'))\|_{L^p_\mu} \leq \|F(\cdot + iy)\|_{L^p_\mu}.$$

Furthermore,

$$\|F\|_{H^p_\mu} = \lim_{\substack{y \to 0 \\ y \in \Omega}} \Big[\int \int_{\mathbb{R}^n} |F(x + i(y + t))|^p \, dx \, d\mu(t) \Big]^{1/p}.$$

Proof. The first inequality follows directly from the previous corollary and the properties of Hardy spaces, since these imply

$$\|F(\cdot + i(y + y' + t))\|_{L^p(\mathbb{R}^n)} \leq \|F(\cdot + i(y + t))\|_{L^p(\mathbb{R}^n)} \quad \forall y, y' \in \Omega, \ t \in \overline{\Omega}.$$

As a consequence we have

$$\lim_{y_n \to 0} \|F(\cdot + iy_n)\|_{L^p_\mu} = \sup_{y \in \Omega} \|F(\cdot + iy)\|_{L^p_\mu} = \|F\|_{H^p_\mu}$$

for any *decreasing* sequence $y_n \searrow 0$ in Ω (i.e., decreasing with respect to the partial order of the cone: $y < y'$ iff $y' - y \in \Omega$). To establish the convergence of the limit within all the cone, it suffices to see that every sequence $y_n \to 0$ in Ω has a decreasing subsequence. But this is easy to construct by induction, since $y > \lambda \mathbf{e}$ if λ is smaller than all the eigenvalues of y. ∎

Finally, as a scholium of the previous corollary we obtain the following:

COROLLARY 3.10. *If* $F \in H^p_\mu(T_\Omega)$ *and* $y \in \Omega$, *then* $F_y = F(\cdot + iy) \in H^p_\mu(T_\Omega)$ *and* $\|F_y\|_{H^p_\mu} = \|F_y\|_{L^p_\mu}$. *Further*,

$$\|F\|_{H^p_\mu} = \lim_{\substack{y \to 0 \\ y \in \Omega}} \|F_y\|_{H^p_\mu}.$$

3.2. *Boundary values in* $H^p_\mu(T_\Omega)$. With the background in the previous subsection we are now in a position to prove the following:

THEOREM 3.11. *Let* $0 < p < \infty$ *and* μ *be a measure as above. Let* Ω_0 *be a proper subcone of* Ω *and define*, *for* $F \in H^p_\mu(T_\Omega)$,

$$F^*(z) = \sup_{y \in \Omega_0} |F(z + iy)|, \quad z \in \mathbb{R}^n + i \operatorname{Supp} \mu. \tag{3.12}$$

Then $F^* \in L^p_\mu$, and there is a constant $c = c(\Omega_0) > 0$ such that

$$\|F\|_{H^p_\mu} \le \|F^*\|_{L^p_\mu} \le c\|F\|_{H^p_\mu} \quad \text{for all } F \in H^p_\mu(T_\Omega).$$

Proof. Let $\varepsilon > 0$, $t \in \operatorname{Supp}\mu$ and write $\eta = t + \varepsilon\mathbf{e} \in \Omega$. Then, by Corollary 3.7, we have $F_\eta = F(\cdot + i\eta) \in H^p_\mu \cap H^p$. Now, properties of Hardy spaces (see 5.13 in Chapter 3 of [12]) imply that

$$(F_\eta)^*(x) := \sup_{y\in\Omega_0} |F_\eta(x+iy)| \in L^p(\mathbb{R}^n), \tag{3.13}$$

and there exists a constant $c = c(\Omega_0) > 0$ so that

$$\int_{\mathbb{R}^n} |(F_\eta)^*(x)|^p\,dx \le c \int_{\mathbb{R}^n} |(F_\eta)(x)|^p\,dx.$$

We can write the last inequality as

$$\int_{\mathbb{R}^n} \sup_{y\in\Omega_0} |F(x+i(y+t+\varepsilon\mathbf{e}))|^p\,dx \le c \int_{\mathbb{R}^n} |F(x+i(t+\varepsilon\mathbf{e}))|^p\,dx,$$

and therefore, after integrating with respect to $d\mu(t)$ we obtain

$$\int\int_{\mathbb{R}^n} \sup_{y\in\Omega_0} |F_{\varepsilon\mathbf{e}}(x+i(y+t))|^p\,dx\,d\mu(t) \le c\int\int_{\mathbb{R}^n} |F_{\varepsilon\mathbf{e}}(x+it)|^p\,dx\,d\mu(t).$$

We may now let $\varepsilon \to 0$. Using Corollary 3.10 on the right hand side and the Monotone Convergence Theorem on the left hand side, we obtain

$$\|F^*\|^p_{L^p_\mu} = \int\int_{\mathbb{R}^n} \lim_{\varepsilon\to 0} \sup_{y\in\Omega_0+\varepsilon\mathbf{e}} |F(x+i(y+t))|^p\,dx\,d\mu(t) \le c\|F\|^p_{H^p_\mu}.$$

The reverse inequality is clear since for any $y_0 \in \Omega_0$ we have

$$\|F\|_{H^p_\mu} = \lim_{\varepsilon\to 0} \|F_{\varepsilon y_0}\|_{L^p_\mu} \le \|F^*\|_{L^p_\mu}. \ \blacksquare$$

For simplicity, we have stated the previous theorem using the *vertical* (*restricted*) maximal function

$$F^*(x) := \sup_{y\in\Omega_0} |F(x+iy)|, \qquad x \in \mathbb{R}^n.$$

But we could have as well taken a *non-tangential* (*restricted*) *maximal function*:

$$F^{**}(x^0) := \sup_{\substack{(x,y)\in\gamma_\alpha(x^0)\\ y\in\Omega_0}} |F(x+iy)|, \qquad x^0 \in \mathbb{R}^n,\ \boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n), \tag{3.14}$$

where

$$\gamma_{\boldsymbol{\alpha}}(x^0) = \{(x,y) \in \mathbb{R}^{2n} : |x_j - x^0_j| < \alpha_j y_j,\ j = 1,\ldots,n\}$$

is a cartesian product of conical regions with apertures $\alpha_1,\ldots,\alpha_n > 0$. Indeed, in the case of classical Hardy spaces it is known that

$$F \in H^p(T_\Omega) \ \Rightarrow\ F^{**} \in L^p(\mathbb{R}^n)$$

with equivalence of norms (see, e.g., Chapter III of [13]). Therefore, replacing F^* by F^{**} in the previous proof we obtain the following refinement of Theorem 3.11:

Theorem 3.15. *Let Ω_0 be a proper subcone of Ω and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) > 0$. Let $F \in H^p_\mu(T_\Omega)$ and*

$$F^{**}(x^0 + it) := \sup_{\substack{(x,y)\in\gamma_\alpha(x^0)\\ y\in\Omega_0}} |F(x + i(t + y))|, \quad x^0 + it \in \mathbb{R}^n + i \operatorname{Supp}\mu.$$

Then $F^{**} \in L^p_\mu$ and there is a constant $c = c(\Omega_0, \boldsymbol{\alpha}) > 0$ such that

$$\|F\|_{H^p_\mu} \le \|F^{**}\|_{L^p_\mu} \le c\|F\|_{H^p_\mu}.$$

We now have all the tools to prove the existence of boundary limits for functions in $H^p_\mu(T_\Omega)$. We state the result separately as a slightly different version of Theorem 1.2.

Theorem 3.16. *Let Ω_0 be a proper subcone of Ω and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) > 0$. If $F \in H^p_\mu(T_\Omega)$, then there exists $F^{(\mathrm{b})} \in L^p_\mu$ so that*

$$\lim_{\substack{(x,y)\to(x^0,0)\\ (x,y)\in\gamma_\alpha(x^0),\, y\in\Omega_0}} F(x + i(y + t)) = F^{(\mathrm{b})}(x^0 + it) \tag{3.17}$$

for a.e. $x^0 + it \in \mathbb{R}^n + i \operatorname{Supp}\mu$.

Proof. By Theorem 3.15 we may find a μ-null set E so that for every $t \in \operatorname{Supp}\mu \setminus E$,

$$F^{**}(x + it) < \infty \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Here F^{**} denotes the non-tangential maximal function restricted to a proper subcone Ω_0. Therefore, $F_t = F(\cdot + it)$ will be a holomorphic (hence harmonic) function in the smaller tube T_{Ω_0}, and *non-tangentially bounded in each variable* at almost every $x \in \mathbb{R}^n$. Thus, we can invoke the Theorem of Calderón (2) (Theorem 3.24 in Chapter II of [13]) which asserts that F_t has a non-tangential limit $F_t^{(\mathrm{b})}$ in each set of variables at almost every $x \in \mathbb{R}^n$.

Note that we cannot say in principle that $F_t^{(\mathrm{b})}(x)$ is jointly measurable in $(x, t) \in \mathbb{R}^n + i \operatorname{Supp}\mu$, and consequently, that the limit in (3.17) exists almost everywhere. To bridge this problem we define the measurable set

$$A = \{(x^0, t) : \varlimsup_{z\to(x^0,0)} \operatorname{Re} F(z + it) > \varliminf_{z\to(x^0,0)} \operatorname{Re} F(z + it)\},$$

where the limits are in the same non-tangential sense as in the statement of the theorem. Note that if we can show that A has $dxd\mu(t)$-measure zero,

(2) Calderón's Theorem is originally stated for harmonic functions in the tube $\mathbb{R}^n + i(0,\infty)^n$ (i.e., in the cartesian product of upper half-planes). An appropriate change of variables makes it valid for tubes $\mathbb{R}^n + i\Omega_0$, where Ω_0 is any proper subcone of Ω (see Chapter III of [13]).

then (after a parallel argument with $\operatorname{Im} F$) we obtain the existence of non-tangential limits a.e., and the measurability of the limit function $F^{(\mathrm{b})}$. The L^p_μ-integrability will follow from the pointwise estimate: $|F^{(\mathrm{b})}| \le F^{**}$.

Let us therefore show that $\operatorname{meas}(A) = 0$. We define

$$A_t = \{x \in \mathbb{R}^n : (x,t) \in A\}, \qquad t \in \operatorname{Supp} \mu.$$

These are Lebesgue measurable sets in $\mathbb{R}^n$ and by Fubini's theorem

$$\operatorname{meas}(A) = \int |A_t| \, d\mu(t).$$

Therefore, it suffices to see that $|A_t| = 0$ for $t \in \operatorname{Supp}\mu \setminus E$. But A_t is contained in the set of points $x^0 \in \mathbb{R}^n$ for which $F_t(x+iy)$ does not converge (non-tangentially) to $F_t^{(\mathrm{b})}(x^0)$. By Calderón's Theorem, this last set has Lebesgue measure zero, and therefore also $|A_t| = 0$. This completes the proof of Theorem 3.16. ∎

The following immediate corollary gives non-tangential restricted convergence in norm, a bit less than was stated in Theorem 1.2.

COROLLARY 3.18. *If $F \in H^p_\mu(T_\Omega)$ and $F^{(\mathrm{b})}$ is its boundary value, then*

$$\|F\|_{H^p_\mu} = \|F^{(\mathrm{b})}\|_{L^p_\mu}.$$

Furthermore, for every proper subcone Ω_0 of Ω and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) > 0$ we have

$$\lim_{\substack{(x,y)\to 0\\ (x,y)\in\gamma_\alpha(0),\, y\in\Omega_0}} \|F(\cdot + x + iy) - F^{(\mathrm{b})}\|_{L^p_\mu} = 0. \tag{3.19}$$

Proof. By Corollary 3.9 it suffices to show the second equality. But this is a consequence of Theorem 3.16 and the Dominated Convergence Theorem. ∎

With not much more effort we can also show a converse of Theorem 3.16, which will have an interesting consequence.

THEOREM 3.20. *Let F be holomorphic in T_Ω and such that, for every proper subcone Ω_0 of Ω and some $\boldsymbol{\alpha} > 0$, we have $F^{**} \in L^p_\mu$. Then $F \in H^p_\mu(T_\Omega)$ and*

$$\|F\|_{H^p_\mu} \le \|F^{**}\|_{L^p_\mu}.$$

Proof. Exactly the same proof as in Theorem 3.16 gives us the existence of a function $F^{(\mathrm{b})} \in L^p_\mu$ which is the non-tangential limit of F in the same sense as in (3.17). Consequently, by the Dominated Convergence Theorem we will also have

$$\lim_{\substack{y\to 0\\ y\in\Omega_0}} \int \int_{\mathbb{R}^n} |F(x+i(y+t))|^p \, dx \, d\mu(t) = \int \int_{\mathbb{R}^n} |F^{(\mathrm{b})}(x+it)|^p \, dx \, d\mu(t).$$

Let us fix for the moment a proper subcone Ω_0 of Ω. Then

$$\sup_{y\in\Omega_0}\int_{\mathbb{R}^n}|F_t(x+iy)|^p\,dx\le\int_{\mathbb{R}^n}|F^{**}(x+it)|^p\,dx\qquad\forall t\in\operatorname{Supp}\mu.$$

Now, except for a set $E=E(\Omega_0)$ of μ-measure zero we know that the integral on the right hand side is finite. Thus, we conclude that $F_t\in H^p(T_{\Omega_0})$, and therefore

$$\sup_{y\in\Omega_0}\int_{\mathbb{R}^n}|F_t(x+iy)|^p\,dx=\lim_{\substack{y\to 0\\ y\in\Omega_0}}\int_{\mathbb{R}^n}|F_t(x+iy)|^p\,dx=\int_{\mathbb{R}^n}|F^{(\mathrm{b})}(x+it)|^p\,dx.\tag{3.21}$$

Taking an increasing sequence of proper subcones $\{\Omega_j\}_{j=0}^\infty$ covering Ω, we deduce that, except for t in a set $E=\bigcup_{j=0}^\infty E(\Omega_j)$ of μ-measure zero, the following equality holds:

$$\sup_{y\in\Omega}\int_{\mathbb{R}^n}|F_t(x+iy)|^p\,dx=\int_{\mathbb{R}^n}|F^{(\mathrm{b})}(x+it)|^p\,dx.$$

Thus, $F_t\in H^p(T_\Omega)$. Further,

$$\int\|F_t\|^p_{H^p(T_\Omega)}\,d\mu(t)=\|F^{(\mathrm{b})}\|^p_{L^p_\mu}.$$

We now claim that F must belong to $H^p_\mu(T_\Omega)$. Indeed, using (3.21), the fact that $y\to\|F_t(\cdot+iy)\|_{L^p(\mathbb{R}^n)}$ is decreasing (in the partial order of the cone), and the Monotone Convergence Theorem we obtain

$$\sup_{y\in\Omega_0}\int\int_{\mathbb{R}^n}|F_y(x+it)|^p\,dx\,d\mu(t)=\lim_{\substack{y\to 0\\ y\in\Omega_0}}\|F_y\|^p_{L^p_\mu}=\|F^{(\mathrm{b})}\|^p_{L^p_\mu}.$$

Since this holds for any arbitrary subcone the claim follows, completing the proof of the theorem. ∎

A corollary of the previous proof is the following "vector-valued" result:

Corollary 3.22. *If $F\in H^p_\mu(T_\Omega)$, then $F_t=F(\cdot+it)\in H^p(T_\Omega)$ for a.e. $t\in\operatorname{Supp}\mu$, and*

$$\Big[\int\|F_t\|^p_{H^p}\,d\mu(t)\Big]^{1/p}=\|F\|_{H^p_\mu}.$$

We conclude this section by proving a very general density result for the spaces $H^p_\mu(T_\Omega)$. First of all note that, as a consequence of (3.19),

$$\lim_{\varepsilon\to 0}\|F_{\varepsilon\mathbf{e}}-F\|_{H^p_\mu}=0.$$

Thus, by Corollary 3.7 the space $H^p_\mu(T_\Omega)\cap H^p(T_\Omega)$ is dense in $H^p_\mu(T_\Omega)$. The next result shows that a much smaller space is also dense.

Theorem 3.23. *Let $0<p,q<\infty$ and μ,μ' be a pair of quasi-invariant measures with respect to H. Then $H^p_\mu\cap H^q_{\mu'}$ is dense in $H^p_\mu(T_\Omega)$.*

Proof. It will suffice to show that for N large we have

$$G(z)=\frac{e^{i(z|\mathbf{e})}}{\Delta((z+i\mathbf{e})/i)^N}\in H^q_{\mu'}(T_\Omega). \tag{3.24}$$

Indeed, in this case

$$F^\varepsilon(z):=G(\varepsilon z)F(z+i\varepsilon\mathbf{e})\in H^p_\mu\cap H^q_{\mu'},$$

since both $F_{\varepsilon\mathbf{e}}$ and G are bounded (3). Further, the pointwise limit of $F^\varepsilon(z)$ is $F(z)$ when $\varepsilon\to 0$. Since $F^*\in L^p_\mu$, we may use the Dominated Convergence Theorem and our previous results to obtain

$$\lim_{\varepsilon\to 0}\|F^\varepsilon-F\|_{H^p_\mu}=\lim_{\varepsilon\to 0}\|(F^\varepsilon)^{(\mathrm{b})}-F^{(\mathrm{b})}\|_{L^p_\mu}=0.$$

Our claim in (3.24) will be a consequence of the following lemma, which we prove in a more general setting in the appendix.

Lemma 3.25. *If $\alpha>2n/r-1$, then there is a constant $c(\alpha)>0$ so that*

$$\int\limits_{\mathbb{R}^n}|\Delta(y+ix)|^{-\alpha}\,dx=c(\alpha)\Delta(y)^{-\alpha+n/r}\qquad\forall y\in\Omega. \tag{3.26}$$

Indeed, assume for a moment that the lemma holds. Then taking any integer N so that $Nq>2n/r-1$ we obtain

$$\begin{aligned}\int\int\limits_{\mathbb{R}^n}|G(x+it)|^q\,dx\,d\mu'(t)&=\int e^{-q(t|\mathbf{e})}\int\limits_{\mathbb{R}^n}|\Delta(\mathbf{e}+t+ix)|^{-Nq}\,dx\,d\mu'(t)\\&=c(Nq)\int e^{-q(t|\mathbf{e})}\Delta(\mathbf{e}+t)^{-Nq+n/r}\,d\mu'(t)\\&\le c(Nq)\int e^{-q(t|\mathbf{e})}\,d\mu'(t)=c'\mathcal{L}\mu'(\mathbf{e})<\infty,\end{aligned}$$

where in the inequality we have used Lemma 3.8. This shows (3.24) and establishes the theorem. ∎

4. Reproducing kernels on $H^p_\mu(T_\Omega)$. In this section we give some reconstruction formulas from the boundary values of functions in $H^p_\mu(T_\Omega)$, $p\ge 1$. These will be obtained from positive kernels of Poisson–Szegő type. Among the consequences, we shall prove unrestricted L^p_μ-convergence of F_y to $F^{(\mathrm{b})}$, as stated in Theorem 1.2. We start with the case $p=2$, which has a simpler characterization in terms of a Paley–Wiener theorem.

4.1. *A characterization of $H^2_\mu(T_\Omega)$.* In this section we shall assume that $\mu=\mu_{\mathbf{s}}$ for some fixed $\mathbf{s}\in\Xi$. We also let

$$L^2_{\mathbf{s}^*}(\Omega)=L^2(\Omega;\Delta^*_{\mathbf{s}^*}(2\xi)d\xi)=L^2(\Omega;\Delta_{\mathbf{s}}((2\xi)^{-1})^{-1}d\xi).$$

(3) That $G(z)$ is holomorphic and bounded follows directly from the definition and an elementary estimation on $\Delta(z)$ for complex z. We present these facts in some more detail in the appendix.

The following theorem gives a Paley–Wiener characterization for the space $H^2_\mu(T_\Omega)$. We point out that this type of result, for some of the spaces mentioned in the introduction, has been previously obtained by different authors [4], [10], [2], ... Here we include an elementary proof that covers the whole range of spaces.

THEOREM 4.1. *For every* $F \in H^2_\mu(T_\Omega)$ *there exists* $f \in L^2_{\mathbf{s}^*}(\Omega)$ *such that*

$$F(z) = \frac{1}{(2\pi)^{n/2}} \int_\Omega e^{i(z|\xi)} f(\xi) \Delta^*_{\mathbf{s}^*}(2\xi)\, d\xi, \qquad z \in T_\Omega. \tag{4.2}$$

Conversely, if $f \in L^2_{\mathbf{s}^*}(\Omega)$ *then the integral above converges absolutely to a function* $F \in H^2_\mu(T_\Omega)$. *In this case,*

$$\|F\|_{H^2_\mu} = \|f\|_{L^2_{\mathbf{s}^*}}.$$

Proof. Suppose first that $F \in H^2_\mu(T_\Omega) \cap H^2(T_\Omega)$. Then, by classical results on Hardy spaces, there exists a function $g \in L^2(\Omega)$ such that

$$F(z) = \frac{1}{(2\pi)^{n/2}} \int_\Omega e^{i(z|\xi)} g(\xi)\, d\xi, \qquad z \in T_\Omega \tag{4.3}$$

(see Chapter III of [13]). Thus, for $f(\xi) = g(\xi)\Delta^*_{-\mathbf{s}^*}(2\xi)$ the identity in (4.2) holds. We shall show that $f \in L^2_{\mathbf{s}^*}$. Using the Plancherel theorem in (4.3) we get

$$\begin{aligned}\int_{\mathbb{R}^n} |F(x+i(y+t))|^2\, dx &= \int_\Omega |e^{-(y+t|\xi)} g(\xi)|^2\, d\xi \\ &= \int_\Omega e^{-2(y+t|\xi)} |f(\xi)|^2 \Delta^*_{\mathbf{s}^*}(2\xi)^2\, d\xi.\end{aligned}$$

Integrating with respect to $d\mu(t)$, using Fubini's Theorem, and the identity $\mathcal{L}\mu(\xi) = \Delta^*_{\mathbf{s}^*}(\xi)^{-1}$ in Lemma 2.6, we conclude that

$$\int_{\overline{\Omega}} \int_{\mathbb{R}^n} |F(x+i(y+t))|^2\, dx\, d\mu(t) = \int_\Omega |f(\xi)|^2 e^{-2(y|\xi)} \Delta^*_{\mathbf{s}^*}(2\xi)\, d\xi.$$

Thus, by Corollary 3.9 and the Monotone Convergence Theorem,

$$\|F\|_{H^2_\mu} = \lim_{y\to 0} \|F(\cdot + iy)\|_{L^2_\mu} = \|f\|_{L^2_{\mathbf{s}^*}}.$$

The density result in Theorem 3.23 extends this isometry to all functions $F \in H^2_\mu(T_\Omega)$, establishing the direct part of the theorem.

To see that the isometry is surjective, we take an arbitrary $f \in L^2_{\mathbf{s}^*}(\Omega)$, and show that the integral in (4.2) converges absolutely to a holomorphic function $F(z)$ in T_Ω. This will suffice for our assertion, since exactly the same computations as above will give $F \in H^2_\mu(T_\Omega)$.

To prove the absolute convergence of the integral, it is enough to look at $z = i\mathbf{e}$. Now, using Hölder's inequality and Lemma 2.6, we get

$$\int_{\Omega} e^{-(\mathbf{e}|\xi)} |f(\xi)| \Delta^*_{\mathbf{s}^*}(2\xi)\, d\xi \le \|f\|_{L^2_{\mathbf{s}^*}} \Big(\int_{\Omega} e^{-2(\mathbf{e}|\xi)} \Delta^*_{2\mathbf{s}^*}(2\xi)\, d\xi \Big)^{1/2}$$
$$= \|f\|_{L^2_{\mathbf{s}^*}} 2^{-n/2} \Gamma_{\Omega}(2\mathbf{s}^* + n/r)^{1/2},$$

which is a finite quantity because $s_j \ge 0$, $j = 1, \dots, r$. ∎

In the next corollary we show the relation between the boundary limit $F^{(\mathrm{b})}$, and the function f in (4.2).

Corollary 4.4. *Let $F \in H^2_{\mu}(T_{\Omega})$, and denote by $F^{(\mathrm{b})}$ its boundary limit and f the function in* (4.2). *Then for a.e. $t \in \operatorname{Supp}\mu$ we have*

$$F^{(\mathrm{b})}(x+it) = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{i(x|\xi)} e^{-(t|\xi)} f(\xi) \Delta^*_{\mathbf{s}^*}(2\xi)\, d\xi, \tag{4.5}$$

where the equality is interpreted in the Fourier–Plancherel sense. Moreover,

$$\lim_{\substack{y\to 0\\ y\in\Omega}} \|F(\cdot + iy) - F^{(\mathrm{b})}\|_{L^2_{\mu}} = 0.$$

Proof. From (4.2) we know that

$$F(x + i(y+t)) = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} e^{i(x|\xi)} e^{-(y|\xi)} e^{-(t|\xi)} f(\xi) \Delta^*_{\mathbf{s}^*}(2\xi)\, d\xi \tag{4.6}$$

for all $x + it \in \mathbb{R}^n + \operatorname{Supp}\mu$ and $y \in \Omega$. Now, as we saw in the previous proof, the function $(\xi, t) \mapsto e^{-(t|\xi)} f(\xi) \Delta^*_{\mathbf{s}^*}(2\xi)$ belongs to $L^2(d\xi d\mu(t))$. Thus, by the Dominated Convergence Theorem we have

$$\lim_{y_k\to 0} \|(e^{-(y_k|\xi)} - 1) e^{-(t|\xi)} f(\xi) \Delta^*_{\mathbf{s}^*}(2\xi)\|_{L^2(d\xi\, d\mu(t))} = 0$$

for any sequence $y_k \to 0$ contained in Ω. We conclude that the limit as $y \to 0$ ($y \in \Omega$) of the right hand side of (4.6) exists (in L^2_{μ}), and equals the right hand side of (4.5). On the other hand, by Theorem 1.2 the left hand side of (4.6) converges to $F^{(\mathrm{b})}(x+it)$, giving us the identity in (4.5) and completing the proof of the corollary. ∎

A direct application of the isometric isomorphism in Theorem 4.1 provides an explicit formula for the *reproducing kernel* of the spaces $H^2_{\mu}(T_{\Omega})$. We shall write, with some abuse of notation, $d\mu(w) = du d\mu(t)$ whenever $w = u + it \in \mathbb{R}^n + i \operatorname{Supp}\mu$.

Corollary 4.7. *Let $\mathbf{s} \in \boldsymbol{\Xi}$ and $\mu = \mu_{\mathbf{s}}$. Then the reproducing kernel for $H^2_{\mu}(T_{\Omega})$ is given by*

$$K_{\mu}(z, w) = c(\mathbf{s}) \Big[\Delta_{\mathbf{s}+n/r} \Big(\frac{z - \overline{w}}{2i} \Big) \Big]^{-1}, \ z \in T_{\Omega},\ w \in \mathbb{R}^n + i \operatorname{Supp}\mu, \tag{4.8}$$

where $c(\mathbf{s}) = (4\pi)^{-n}\Gamma_\Omega(\mathbf{s}^* + n/r)$. *That is*, *if* $F \in H^2_\mu(T_\Omega)$ *and* $F^{(\mathrm{b})}$ *is its boundary value*, *then*

$$F(z) = \int\int_{\mathbb{R}^n} K_\mu(z,w)F^{(\mathrm{b})}(w)\,d\mu(w), \qquad z \in T_\Omega.$$

Proof. The isometry in Theorem 4.1 gives an abstract formula for the reproducing kernel of $H^2_\mu(T_\Omega)$:

$$K_\mu(z,w) = \frac{1}{(2\pi)^n}\int_\Omega e^{i(z-\overline{w}|\xi)}\Delta^*_{\mathbf{s}^*}(2\xi)\,d\xi$$

(see Proposition IX.3.4 in [3]). The final expression for K_μ in (4.8) follows by computing the integral above, which can be done explicitly using Lemma 2.6 (see also (7.7) in the appendix). ∎

4.2. *Poisson–Szegő kernels in* $H^p_\mu(T_\Omega)$. In this subsection we use well known techniques to construct a positive kernel that reproduces functions in $H^p_\mu(T_\Omega)$, for every $1 \le p < \infty$. We let

$$S_\mu(z,w) = \frac{|K_\mu(z,w)|^2}{K_\mu(z,z)}, \qquad z \in T_\Omega,\ w \in \mathbb{R}^n + i\,\mathrm{Supp}\,\mu. \tag{4.9}$$

Then the following properties are not difficult to verify:

(i) $S_\mu(z,w) > 0$ for all $z \in T_\Omega,\ w \in \mathbb{R}^n + i\,\mathrm{Supp}\,\mu$.

(ii) $\int\int S_\mu(z,w)\,d\mu(w) = 1$ for all $z \in T_\Omega$.

(iii) If $1 \le p \le \infty$ and $z = x + iy \in T_\Omega$, then $S_\mu(z,\cdot) \in L^p_\mu$ and

$$\|S_\mu(z,\cdot)\|_{L^p_\mu} \le \left(\frac{c(\mathbf{s})}{\Delta_{\mathbf{s}+n/r}(y/4)}\right)^{1/p'}, \quad \text{where} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

(To check the last inequality, one can interpolate between the simpler cases $p = 1$ and $p = \infty$. These follow from (ii), and elementary estimates in Δ; see Lemmas 3.8 and 7.5.)

PROPOSITION 4.10. *Let* $1 \le p < \infty$. *If* $F \in H^p_\mu(T_\Omega)$ *and* $F^{(\mathrm{b})}$ *is its boundary limit*, *then*

$$F(z) = \int\int_{\mathbb{R}^n} S_\mu(z,w)F^{(\mathrm{b})}(w)\,d\mu(w), \qquad z \in T_\Omega. \tag{4.11}$$

Proof. Let $\varepsilon > 0$. Then, for every fixed $z \in T_\Omega$, we have $K_\mu(w,z)F_{\varepsilon\mathbf{e}}(w) \in H^2_\mu(T_\Omega)$ (since $F_{\varepsilon\mathbf{e}}$ is bounded). Thus, from the previous corollary we see that

$$\begin{aligned} F_{\varepsilon\mathbf{e}}(z) &= \int\int K_\mu(z,w)\,\frac{K_\mu(w,z)}{K_\mu(z,z)}F_{\varepsilon\mathbf{e}}(w)\,d\mu(w) \\ &= \int\int S_\mu(z,w)F_{\varepsilon\mathbf{e}}(w)\,d\mu(w). \end{aligned}$$

Now, since $S_\mu(z,\cdot)\in L^{p'}_\mu$ and $F_{\varepsilon\mathbf{e}}\to F^{(\mathrm{b})}$ in L^p_μ, as $\varepsilon\to 0$, the identity in (4.11) follows. ∎

Observe that, when $\mu=\delta_0$, $S_\mu(z,w)$ is the Poisson–Szegő kernel of the tube domain. In this case we use the classical notation:

$$P_y(x-u)=S_{\delta_0}(x+iy,u),\qquad x+iy\in T_\Omega,\ u\in\mathbb{R}^n.$$

Now, P_y is known to be an approximation of the identity. That is, in addition to (i)–(iii) above it has the crucial property

$$\text{(iv)}\qquad \lim_{\substack{y\to 0\\ y\in\Omega}}\int\limits_{|x|>\delta} P_y(x)\,dx=0\qquad\text{for every } \delta>0.$$

Using this, we obtain the following unrestricted limit in norm when $p\ge 1$, which establishes the last part of Theorem 1.2.

THEOREM 4.12. *Let $1\le p<\infty$ and $F\in H^p_\mu(T_\Omega)$, with boundary limit $F^{(\mathrm{b})}$. Then*

$$\lim_{\substack{y\to 0\\ y\in\Omega}}\|F(\cdot+iy)-F^{(\mathrm{b})}\|_{L^p_\mu}=0. \tag{4.13}$$

Proof. We use Corollary 3.22 so that, except for t in a set of μ-measure 0, $F_t=F(\cdot+it)\in H^p(T_\Omega)$, and hence

$$F_t(x+iy)=\int\limits_{\mathbb{R}^n} F_t^{(\mathrm{b})}(x-u)\,P_y(u)\,du,\qquad x+iy\in T_\Omega. \tag{4.14}$$

In particular, we can write

$$F_t(x+iy)-F_t^{(\mathrm{b})}(x)=\int\limits_{\mathbb{R}^n}(F_t^{(\mathrm{b})}(x-u)-F_t^{(\mathrm{b})}(x))\,P_y(u)\,du.$$

Thus, taking L^p_μ-norms in the above expression, for every $\delta>0$ we have

$$\begin{aligned}&\|F(\cdot+iy)-F^{(\mathrm{b})}\|_{L^p_\mu}\\ &\quad\le\int\limits_{|u|<\delta}\|F^{(\mathrm{b})}(\cdot-u)-F^{(\mathrm{b})}\|_{L^p_\mu}\,P_y(u)\,du+\int\limits_{|u|\ge\delta}2\|F^{(\mathrm{b})}\|_{L^p_\mu}\,P_y(u)\,du.\end{aligned}$$

Now, using the continuity of the L^p_μ-norm and (iv) above, one can easily show (4.13) by an (ε,δ)-argument. ∎

5. The boundary of a symmetric cone. In this section we go back to the geometry of the cone. We prove Theorem 1.6 in detail, and set the basis to study the tangential CR equations in the next section. We also provide an explicit expression for the measures $\mu_{\boldsymbol{\nu}}$ in terms of *Gauss coordinates*.

Let $\{c_1,\dots,c_r\}$ be a fixed Jordan frame in V, and H the triangular subgroup of G from §2.1. We begin by describing the orbits of H on $\overline{\Omega}$.

Since the group acts simply transitively on the cone, one can write $\Omega=H\mathbf{e}$ as a single orbit. However, the action of H on the boundary is no longer transitive, and consequently, $\partial\Omega$ will consist of many disjoint orbits. In the next proposition we show that these are determined by the action on the following idempotents:

$$c_{\boldsymbol{\varepsilon}} = \varepsilon_1 c_1 + \ldots + \varepsilon_r c_r, \quad \text{where } \boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r.$$

PROPOSITION 5.1. *Define $I = \{0,1\}^r$ and $I^* = I \setminus \{(1,\ldots,1)\}$. Then*

$$\overline{\Omega} = \bigcup_{\boldsymbol{\varepsilon}\in I} Hc_{\boldsymbol{\varepsilon}} \quad \text{and} \quad \partial\Omega = \bigcup_{\boldsymbol{\varepsilon}\in I^*} Hc_{\boldsymbol{\varepsilon}}, \tag{5.2}$$

where the unions are pairwise disjoint.

For the proof of this proposition, and other results in this section, we proceed by induction on the rank r of Ω. The usual technique goes as follows: write each $x \in V$ as

$$x = x_1 + x_{1/2} + x_0,$$

in terms of the Peirce decomposition $V(c_1,1) \oplus V(c_1,1/2) \oplus V(c_1,0)$. Then consider $V_0 := V(c_1,0)$ as a Jordan algebra of rank $r-1$, with Jordan frame $\{c_2,\ldots,c_r\}$ and associated cone Ω_0. The restriction of H to V_0, denoted by H_0, consists of lower triangular matrices with 1 in their first entry and zeros in the rest of the first column. That is, in the notation of (2.2), we can write every $h_0 \in H_0$ as

$$h_0 = \tau^{(2)}(z_2)\ldots\tau^{(r-1)}(z_{r-1})P\Big(c_1 + \sum_{j=2}^{r} a_j c_j\Big), \quad a_j > 0,\ z_j \in \bigoplus_{k=j+1}^{r} V_{j,k}.$$

Then one uses the following lemma, which gives the *Gauss decomposition* of $x \in \overline{\Omega}$ with respect to the idempotent c_1.

LEMMA 5.3 (see Proposition VI.3.2 in [3]). *If $x = x_1 + x_{1/2} + x_0 \in \overline{\Omega}$ and $x_1 \neq 0$, then there exist unique $v \in V(c_1,1/2)$ and $u > 0$ so that*

$$x = \tau^{(1)}(v/u)(u^2 c_1 + y) \quad \text{for some } y \in V(c_1,0).$$

In this case, $y \in \overline{\Omega}_0$ and

$$x_1 = u^2 c_1, \quad x_{1/2} = uv \quad \text{and} \quad x_0 = (v^2)_0 + y.$$

Proof of Proposition 5.1. It suffices to show the statement about $\overline{\Omega}$, since the only orbit which intersects the open cone Ω is $H\mathbf{e}$. The proof is by induction on the rank r of the cone. The case $r = 1$ is obvious (since $\partial\Omega = \{0\}$). We now assume the result holds for cones of rank $\leq r-1$.

Take any $x = x_1 + x_{1/2} + x_0 \in \overline{\Omega}$. Since $x_1 = (x|c_1)c_1$, if this term is zero we have $x = x_0 \in V(c_1,0) = V_0$. Further, $x \in \overline{\Omega}$ implies $x \in \overline{\Omega}_0$, so the induction hypothesis applies.

Assume now $x_1 \neq 0$, and consider the decomposition in Lemma 5.3. Since $y \in \overline{\Omega}_0$, by the induction hypothesis there exists $h_0 \in H_0$ so that $y = h_0 c_{\boldsymbol{\varepsilon}'}$ for some $\boldsymbol{\varepsilon}' = (0,\boldsymbol{\varepsilon}_0)$, $\boldsymbol{\varepsilon}_0 \in \{0,1\}^{r-1}$. Using $h_0 c_1 = c_1$, we can write $x = \tau^{(1)}(v/u)(u^2c_1 + h_0c_{\varepsilon'}) = hc_{(1,\varepsilon_0)}$ for some $h \in H$. This establishes the equality in (5.2).

It remains to prove that $Hc_{\varepsilon} \cap Hc_{\varepsilon'} = \emptyset$ when $\varepsilon \neq \varepsilon'$. That is, if $c_{\varepsilon} = hc_{\varepsilon'}$ for some $h \in H$, we want to show $\varepsilon = \varepsilon'$. Using the decomposition in (2.2), we shall write h as

$$h = \tau^{(1)}(v/u)P(uc_1 + c_2 + \ldots + c_r)h_0 \tag{5.4}$$

for some $u > 0$, $v \in V(c_1, 1/2)$ and $h_0 \in H_0$. Now, set $\varepsilon = (\varepsilon_1, \varepsilon_0)$ and $\varepsilon' = (\varepsilon_1', \varepsilon_0')$, and observe that necessarily $\varepsilon_1 = \varepsilon_1'$, since H is lower triangular.

In the case $\varepsilon_1 = \varepsilon_1' = 0$, $c_{\varepsilon} = hc_{\varepsilon'}$ is equivalent to $c_{\varepsilon_0} = h_0c_{\varepsilon_0'}$ (the terms of h involving c_1 play no role). Thus, the induction hypothesis gives $\varepsilon_0 = \varepsilon_0'$. In the other case, i.e. $\varepsilon_1 = \varepsilon_1' = 1$, the uniqueness in Lemma 5.3 applied to $x = c_{\varepsilon}$ implies $u = 1$, $v = 0$ and $c_{\varepsilon_0} = y = h_0c_{\varepsilon_0'}$. Again, the induction hypothesis gives $\varepsilon_0 = \varepsilon_0'$, establishing the proposition. ∎

Next we define a differentiable structure on each orbit $M_{\varepsilon} := Hc_{\varepsilon}$ for $\varepsilon \in I$. This is done by identifying M_{ε} with the homogeneous space $H/H_{c_{\varepsilon}}$, where $H_{c_{\varepsilon}} = \{h \in H : hc_{\varepsilon} = c_{\varepsilon}\}$ is the stabilizer of c_{ε}. Since H is a Lie group acting analytically on $\mathbb{R}^n$, M_{ε} becomes a submanifold of $\mathbb{R}^n$ (see, e.g., Theorem 2.9.7 of [14]). In fact, from the following proposition we see that M_{ε} is actually a *regular submanifold* of $\mathbb{R}^n$.

PROPOSITION 5.5. *Let $c_{\varepsilon} = c_{j_1} + \ldots + c_{j_s}$ for $1 \leq j_1 < \ldots < j_s \leq r$, and $M_{\varepsilon} = Hc_{\varepsilon}$. Then every $x \in M_{\varepsilon}$ can be written uniquely as*

$$x = \tau^{(j_1)}(z_{j_1}) \ldots \tau^{(j_s)}(z_{j_s})P(a_{j_1}c_{j_1} + \ldots + a_{j_s}c_{j_s} + (\mathbf{e} - c_{\varepsilon}))c_{\varepsilon}, \tag{5.6}$$

where $z^{(j_i)} \in \bigoplus_{k=j_i+1}^{r} V_{j_i,k}$ and $a_{j_i} > 0$, $i = 1, \ldots, s$. Moreover, M_{ε} is a regular submanifold of $\mathbb{R}^n$ of dimension $m_{\varepsilon} = s + d\sum_{i=1}^{s}(r - j_i)$.

Proof. The proof is again by induction on r. Let $x \in M_{\varepsilon}$, and write it as $x = hc_{\varepsilon}$, where $h \in H$ has been decomposed as in (5.4). If $j_1 \neq 1$, then we may take $u = 1$ and $v = 0$ in (5.4), and therefore apply the induction hypothesis to $x = h_0c_{\varepsilon} \in V_0$.

Suppose $j_1 = 1$ instead, and let $c_{\varepsilon'} = c_{j_2} + \ldots + c_{j_s}$. Then, using (5.4), we can write

$$x = \tau^{(1)}(v/u)(u^2c_1 + y),$$

where $y = h_0c_{\varepsilon'} \in H_0c_{\varepsilon'} \subset V_0$. One more application of the induction hypothesis gives

$$y = \tau^{(j_2)}(z_{j_2}) \ldots \tau^{(j_s)}(z_{j_s})P(a_{j_2}c_{j_2} + \ldots + a_{j_s}c_{j_s} + (\mathbf{e} - c_{\varepsilon}))c_{\varepsilon'}$$

for appropriate z_j and a_j. Hence, combining the last two formulas, and using the commutativity relation in (2.3), we obtain an expression like in (5.6). The uniqueness of the decomposition is also a consequence of the induction hypothesis, and the uniqueness in Lemma 5.3.

Finally, we show that M_ε is a regular submanifold of $\mathbb{R}^n$. First note that the correspondence in (5.4),

$$\mathbb{R}_+ \times V(c_1, 1/2) \times H_0 \to H, \quad (u, v, h_0) \mapsto h = \tau^{(1)}(v/u)P(uc_1 + (\mathbf{e} - c_1))h_0,$$

is actually a diffeomorphism (use again the induction hypothesis and Lemma 5.3). Therefore, it suffices to show that the mapping

$$(5.7) \qquad (u, v, y) \in \mathbb{R}_+ \times V(c_1, 1/2) \times H_0 c_{\varepsilon'} \mapsto x = \tau^{(1)}(v/u)(u^2 c_1 + y) \in M_\varepsilon$$

is open when the image space M_ε has the relative topology of $\mathbb{R}^n$. One more time, the induction hypothesis gives us the openness of the inclusion map

$$y \in H_0 c_{\varepsilon'} \mapsto y \in M_{\varepsilon'} \subset V_0$$

when the image space has the topology of $V_0 \equiv \mathbb{R}^{n_0}$. Thus, the openness of (5.7) follows by writing

$$x = x_1 + x_{1/2} + x_0 = u^2 c_1 + uv + ((v^2)_0 + y),$$

and looking at the projection in each variable separately. ∎

A consequence of the preceding proposition is that each M_ε is a Borel set in $\mathbb{R}^n$. To continue the proof of Theorem 1.6, we shall obtain an explicit expression for the measures $\mu_{\mathbf{s}}$, and compute $\mu_{\mathbf{s}}(M_\varepsilon)$ for each $\mathbf{s} \in \Xi$, $\boldsymbol{\varepsilon} \in I$.

PROPOSITION 5.8. *Let* $\mathbf{s} \in \Xi$. *Then there exists a unique* $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{s}) \in I$ *so that*

$$\mu_{\mathbf{s}}(M_\varepsilon) > 0 \quad \textit{and} \quad \mu_{\mathbf{s}}(\overline{\Omega} \setminus M_\varepsilon) = 0.$$

Proof. The proof is again by induction on r. The case $r = 1$ is simple, since the only manifolds are $M_1 = \Omega = (0, \infty)$ and $M_0 = \{0\}$, and the measures $\mu_{\mathbf{s}}$ are given by

$$\int_{\overline{\Omega}} f(x)\, d\mu_{\mathbf{s}}(x) = \begin{cases} \int_0^\infty f(x) \frac{x^s}{s} \frac{dx}{x} & \text{if } s > 0, \\ f(0) & \text{if } s = 0. \end{cases}$$

Suppose now the result holds for cones of rank $\leq r - 1$, and let Ω be a cone of rank r. Given $\mathbf{s} = (s_1, \mathbf{s}') \in \Xi$, we write $\mu_{\mathbf{s}}$ using the coordinates of the Peirce decomposition

$$x = x_1 + x_{1/2} + x_0 = u^2 c_1 + uv + (y + (v^2)_0)$$

with $u > 0$, $v \in V(c_1, 1/2)$, and $y \in \overline{\Omega}_0$. After an appropriate substitution of the variables one obtains

$$(5.9) \qquad \int_{\overline{\Omega}} f(x)\, d\mu_{\mathbf{s}}(x) = \int_{\overline{\Omega}_0} f(0, 0, y)\, d\mu^0_{\mathbf{s}'}(y) \quad \text{if } s_1 = 0,$$

$$\int_{\overline{\Omega}} f(x)\,d\mu_{\mathbf{s}}(x) = c_{\mathbf{s}} \int_{\overline{\Omega}_0} \int_{V(c_1,1/2)} \int_0^\infty f(u^2c_1, uv, y + (v^2)_0) \times u^{2s_1}\,\frac{du}{u}\,dv\,d\mu^0_{\mathbf{s}'-d/2}(y) \quad \text{if } s_1 > 0, \tag{5.10}$$

with the constant $c_{\mathbf{s}} = 2(2\pi)^{-(n/r-1)}/\Gamma(s_1)$. These formulas are proved, e.g., in Lemma VII.3.3 of [3].

Now, given $\boldsymbol{\varepsilon} = (\varepsilon_1, \boldsymbol{\varepsilon}') \in I$, we can write the manifold $M_{\boldsymbol{\varepsilon}}$ in Gauss coordinates as

$$\begin{aligned} M_{(0,\boldsymbol{\varepsilon}')} &= \{(0,0,y) : y \in M^{(0)}_{\boldsymbol{\varepsilon}'} = H_0 c_{\boldsymbol{\varepsilon}'}\}, \\ M_{(1,\boldsymbol{\varepsilon}')} &= \{(u,v,y) : u > 0,\ v \in V(c_1,1/2),\ y \in M^{(0)}_{\boldsymbol{\varepsilon}'}\}. \end{aligned} \tag{5.11}$$

Suppose first $s_1 = 0$. Then it is clear from (5.9) and (5.11) that

$$\mu_{\mathbf{s}}(M_{(1,\boldsymbol{\eta}')}) = 0 \quad \text{ whenever } \boldsymbol{\eta}' \in \{0,1\}^{r-1}.$$

Also, by the induction hypothesis, there exists a unique $\boldsymbol{\varepsilon}' \in \{0,1\}^{r-1}$ such that $\mu^0_{\mathbf{s}'}(M^{(0)}_{\boldsymbol{\varepsilon}'}) > 0$. Thus, if $\boldsymbol{\eta} = (0, \boldsymbol{\eta}')$ we have

$$\mu_{\mathbf{s}}(M_{\boldsymbol{\eta}}) = \mu^0_{\mathbf{s}'}(M^{(0)}_{\boldsymbol{\eta}'}) > 0 \quad \text{iff} \quad \boldsymbol{\eta}' = \boldsymbol{\varepsilon}'.$$

Suppose now $s_1 > 0$. Again from (5.10) and (5.11) it follows that

$$\mu_{\mathbf{s}}(M_{(0,\boldsymbol{\eta}')}) = 0 \quad \text{ for all } \boldsymbol{\eta}' \in \{0,1\}^{r-1}.$$

Also, by the induction hypothesis, there exists a unique $\boldsymbol{\varepsilon}' \in \{0,1\}^{r-1}$ such that $\mu^0_{\mathbf{s}'-d/2}(M^{(0)}_{\boldsymbol{\varepsilon}'}) > 0$. Thus, if $\boldsymbol{\eta} = (1, \boldsymbol{\eta}')$ we have

$$\mu_{\mathbf{s}}(M_{\boldsymbol{\eta}}) = c_{\mathbf{s}} \lim_{N\to\infty} \Big(\int_{|v|\le N} \int_{1/N}^{N} u^{2s_1}\,\frac{du}{u}\,dv \Big) \mu^0_{\mathbf{s}'-d/2}(M^{(0)}_{\boldsymbol{\eta}'}),$$

which is non-zero if and only if $\mu^0_{\mathbf{s}'-d/2}(M^{(0)}_{\boldsymbol{\eta}'}) > 0$, or equivalently, $\boldsymbol{\eta}' = \boldsymbol{\varepsilon}'$. ∎

For the last statement of Theorem 1.6, we need the following result.

LEMMA 5.12. *In the conditions of this section,* $\overline{\mathrm{co}}\,(Hc_1) = \overline{\Omega}$.

Proof. It suffices to show that

$$c_j \in \overline{Hc_1}, \quad j = 2, \ldots, r. \tag{5.13}$$

Indeed, in this case $\bigcup_{j=2}^r Hc_j \subset \overline{Hc_1}$, and therefore,

$$\overline{\mathrm{co}}\,(Hc_1) = \overline{\mathrm{co}}\,(Hc_1 + Hc_2 + \ldots + Hc_r) = \overline{\mathrm{co}}\,(H\mathbf{e}) = \overline{\Omega}.$$

To see (5.13), let $v \in V_{1,j}$ with $|v| = \sqrt{2}$, and $u_n \searrow 0$. Then, by Proposition 5.3,

$$x_n := \tau^{(1)}(v/u_n)(u_n^2 c_1) = u_n^2 c_1 + u_n v + (v^2)_0 \in Hc_1.$$

Thus, $\lim_{n\to\infty} x_n = (v^2)_0 \in \overline{Hc_1}$. But $v^2 \in V_{1,1} \oplus V_{j,j}$, and therefore,

$$(v^2)_0 = (v^2|c_j)c_j = \tfrac{1}{2}|v|^2 c_j = c_j. \ \blacksquare$$

The last statement of Theorem 1.6 follows immediately from the previous lemma. Indeed, if $c_\varepsilon = c_{j_1} + \ldots + c_{j_s}$ with $1 \le j_1 < \ldots < j_s \le r$, then $\overline{\mathrm{co}}\,(Hc_\varepsilon) = \overline{\Omega}_{(j_1)}$, which is the cone of squares associated with the Jordan algebra $V_{(j_1)} = \bigoplus_{j_1 \le j \le k \le r} V_{j,k}$. Thus, our claim follows from a fact we saw above: for $\mathbf{s} \in \Xi$, $s_1 = 0$ if and only if $\boldsymbol{\varepsilon}(\mathbf{s}) = (0, \boldsymbol{\varepsilon}')$ for some $\boldsymbol{\varepsilon}' \in I'$.

With the preceding propositions we have essentially completed the proof of Theorem 1.6. The fact that the measures $\mu_{\mathbf{s}}$ are *smooth volume forms* in M_ε is contained in the proofs presented above. Indeed, it follows from the parametrization of both, manifold and measure, in terms of the Gauss coordinates (see (5.9)–(5.11)).

To conclude completely the proof of Theorem 1.6 it only remains to show that the orbits $M_\varepsilon = Hc_\varepsilon$ can actually be written as $M_\varepsilon = H_\varepsilon c_\varepsilon$ for a Lie subgroup H_ε of H. In view of Proposition 5.5, if $c_\varepsilon = c_{j_1} + \ldots + c_{j_s}$, we let H_ε be the set of $h \in H$ of the form

$$h = \tau^{(j_1)}(z_{j_1}) \ldots \tau^{(j_s)}(z_{j_s}) P(a_{j_1} c_{j_1} + \ldots + a_{j_s} c_{j_s} + (\mathbf{e} - c_\varepsilon)) \tag{5.14}$$

for z_{j_i}, a_{j_i} as in (5.6). Then H_ε is a closed subset of H acting simply transitively on M_ε. We further claim that H_ε is a subgroup of H. To see this, recall from §2.2 that

$$\tau^{(i)}(z) = \exp(2z \,\square\, c_i), \qquad z \in V(c_i, 1/2),$$

where $z \,\square\, w = L(zw) + [L(z), L(w)]$ (see also Chapter VI of [3]). Let

$$\mathfrak{g}_{i,j} = \{X = z \,\square\, c_i : z \in V_{i,j}\} \subset \mathfrak{g}.$$

Then our claim follows from the next lemma:

Lemma 5.15. *Let $1 \le i < k \le r$, $1 \le j < l \le r$, and suppose $i \le j$. Then, if $z \in V_{i,k}$ and $w \in V_{j,l}$, we have*

$$[z \,\square\, c_i, w \,\square\, c_j] = -\delta_{j,k}(zw) \,\square\, c_i = -\tfrac{1}{2}\delta_{j,k}\, L(zw). \tag{5.16}$$

Indeed, as $V_{i,j}V_{j,l} \subset V_{i,l}$ (see Chapter IV of [3]), the lemma implies that $\sum_{k=i+1}^r \mathfrak{g}_{i,k} \oplus \sum_{l=j+1}^r \mathfrak{g}_{j,l}$ is a Lie subalgebra of $\mathfrak{g}$ for every $i < j$. Thus, by the Baker–Campbell formula (see, e.g., Theorem 2.15.4 of [14]), the set

$$\Big\{\tau^{(i)}(z)\tau^{(j)}(w) : z \in \bigoplus_{k=i+1}^r V_{i,k},\ w \in \bigoplus_{l=j+1}^r V_{j,l}\Big\}$$

is a closed subgroup of H. After iteration, and by (2.3), it follows that H_ε is also a closed subgroup of H. Note that the inverse of $h \in H_\varepsilon$ in (5.14) is given by

$$h^{-1} = P(a_{j_1}^{-1} c_{j_1} + \ldots + a_{j_s}^{-1} c_{j_s} + (\mathbf{e} - c_\varepsilon))\tau^{(j_s)}(-z_{j_s}) \ldots \tau^{(j_1)}(-z_{j_1}).$$

The proof of Lemma 5.15 is elementary, and will be presented in the appendix.

6. The characterization of the spaces $H^p_\mu(T_\Omega)$. In this section we present a proof for the last two theorems of the paper: 1.3 and 1.4. This requires the use of results developed in previous sections, but also some new techniques involving CR equations in complex manifolds. For the last part we shall follow the presentation in [10], where, as we pointed out, the spaces $H^2_\mu(T_\Omega)$ were already characterized.

6.1. *The tangential Cauchy–Riemann equations.* Throughout this subsection, M will be a regular submanifold of $\mathbb{R}^n$, and H a Lie group acting simply transitively on M. That is, there exists a fixed $t_0 \in M$ such that every $x \in M$ can be written uniquely as $x = ht_0$ for some $h \in H$. We suppose $m = \dim M > 0$. A particular case of this situation are the orbits described in §5: $M_\varepsilon = H_\varepsilon c_\varepsilon$ for $\varepsilon \in I \setminus \{0\}$.

Consider the (real) manifold $T_M = \mathbb{R}^n + iM \subset \mathbb{R}^n + i\mathbb{R}^n$. We denote by $\mathcal{T}_p(T_M)$ the tangent space of T_M at p, and by $\mathcal{T}_p(T_M)_\mathbb{C}$ its complexification. A *tangential CR vector field in T_M* is a smooth vector field $Z : p \mapsto Z_p \in \mathcal{T}_p(T_M)_\mathbb{C}$ which is antiholomorphic in $\mathbb{C}^n \equiv \mathbb{R}^n + i\mathbb{R}^n$. That is,

$$Z_p \in \operatorname{span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}_1}\bigg|_p, \dots, \frac{\partial}{\partial \bar{z}_n}\bigg|_p \right\}, \quad \text{where} \quad \frac{\partial}{\partial \bar{z}_j}\bigg|_p = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\bigg|_p + i\frac{\partial}{\partial y_j}\bigg|_p \right).$$

DEFINITION 6.1. We say that a function $f \in C^1(T_M)$ satisfies the *tangential Cauchy–Riemann* (*CR*) *equations* if $Zf = 0$ for every tangential CR vector field Z in T_M.

A careful description of these equations related to manifolds of the form T_M (and even more general Siegel domains) can be found in [10]. For completeness, we give a more explicit form here, in terms of the action of the group H.

Suppose $\{e_1, \dots, e_n\}$ is a fixed basis of $\mathbb{R}^n$, for which the tangent plane at t_0 of M is given by

$$\mathcal{T}_{t_0}(M) = \operatorname{span}_\mathbb{R}\{D_{e_1|t_0}, \dots, D_{e_m|t_0}\},$$

and where D_v, $v \in \mathbb{R}^n \setminus \{0\}$, denotes the directional derivative vector field:

$$D_{v|p}[f] = \lim_{\varepsilon \to 0} \frac{f(p + \varepsilon v) - f(p)}{\varepsilon} \quad \text{if } f \in C^1(p).$$

Let $\{e_1, \dots, e_n, ie_1, \dots, ie_n\}$ be the corresponding basis of the complexification $\mathbb{R}^n + i\mathbb{R}^n \equiv \mathbb{R}^{2n}$. Then the tangent space of $T_M = \mathbb{R}^n + M$ at it_0 has the form

$$\mathcal{T}_{it_0}(T_M) = \operatorname{span}_\mathbb{R}\{D_{e_1|it_0}, \dots, D_{e_n|it_0}, D_{ie_1|it_0}, \dots, D_{ie_m|it_0}\}.$$

Further, if $p = x_0 + iht_0 \in T_M$ for unique $x_0 \in \mathbb{R}^n$ and $h \in H$, then the group action gives:

$$\mathcal{T}_p(T_M) = \operatorname{span}_{\mathbb{R}}\{D_{he_1|p}, \ldots, D_{he_n|p}, D_{ihe_1|p}, \ldots, D_{ihe_m|p}\}.$$

Consider now the following H-invariant vector fields:

$$X_{j|p} := D_{he_j|p} \quad \text{and} \quad Y_{j|p} := D_{ihe_j|p} \quad \text{for } p = x_0 + iht_0,\ j = 1, \ldots, n.$$

The span of $\{X_{j|p}, Y_{j|p}\}_{j=1}^m$ generates the largest "complex space" contained in each $\mathcal{T}_p(T_M)$. Thus, a basis of antiholomorphic vector fields in $\mathcal{T}(T_M)_{\mathbb{C}}$ is given by

$$\overline{Z}_j := \tfrac{1}{2}(X_j + iY_j), \qquad j = 1, \ldots, m.$$

In particular, $f \in C^1(T_M)$ satisfies the tangential Cauchy–Riemann equations if and only if

$$\overline{Z}_j f = 0 \quad \text{for all } j = 1, \ldots, m.$$

These vector fields can also be written as

$$(\overline{Z}_1, \ldots, \overline{Z}_n) = \overline{\nabla} \cdot A,$$

where A is the matrix-valued function $p = x_0 + iht_0 \in T_M \mapsto h \in H$, and $\overline{\nabla}$ denotes the "antiholomorphic gradient":

$$\overline{\nabla} = \left(\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right) \quad \text{for} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(D_{e_j} + iD_{ie_j}).$$

In particular, for every holomorphic function F in a neighborhood of T_M, we must have $\overline{Z}_j F = 0$, $j = 1, \ldots, n$.

Below, we shall apply the tangential Cauchy–Riemann equations to functions in $L^p_\mu = L^p(T_M; dxd\mu(t))$, and therefore, a definition of "weak derivative" is also needed (see §2.1 of [10]):

DEFINITION 6.2. We say that $f \in L^1_{\mathrm{loc}}(T_M)$ satisfies *weakly* the tangential CR equations (in symbols $f \in \mathrm{CR}(T_M)$) when

$$\int_{T_M} f \bar{\partial} \omega = 0 \quad \forall \omega \in \Lambda_{\mathrm{c}}^{(n,m-1)}(\mathbb{C}^n). \tag{6.3}$$

In this definition $\Lambda_{\mathrm{c}}^{(k,l)}(\mathbb{C}^n)$ denotes the set of all smooth compactly supported (k,l)-forms in $\mathbb{C}^n$. That is,

$$\omega = \sum_{\substack{i_1 < \ldots < i_k \\ j_1 < \ldots < j_l}} \varphi^{i_1, \ldots, i_k}_{j_1, \ldots, j_l} dz_{i_1} \wedge \ldots \wedge dz_{i_k} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_l},$$

where $\varphi^{i_1, \ldots, i_k}_{j_1, \ldots, j_l} \in C^\infty_{\mathrm{c}}(\mathbb{C}^n)$ and

$$dz_j = dx_j + idy_j, \qquad d\bar{z}_j = dx_j - idy_j.$$

Also, $\bar{\partial}$ is the chain complex mapping of $\Lambda^{(k,l)}$ into $\Lambda^{(k,l+1)}$, defined on smooth functions as

$$\bar{\partial}\varphi = \sum_{j=1}^{n} \frac{\partial \varphi}{\partial \bar{z}_j}\, d\bar{z}_j, \qquad \varphi \in C^1(\mathbb{C}^n).$$

In our particular situation, (6.3) is equivalent to the simpler expression

$$\int_{T_M} f\overline{Z}_j\varphi = 0 \quad \forall\, j = 1,\dots,m,\ \varphi \in C_{\mathrm{c}}^\infty(T_M) \tag{6.4}$$

(see 2.1.5 in [10]). The equivalence of Definitions 6.1 and 6.2 for $C^1(T_M)$ functions, as well as other properties of CR equations, can be found in §§1, 2 of [10].

6.2. *The spaces $A_\mu^p(T_M)$.* We start by recalling the definition of the spaces $A_\mu^p(T_M)$:

DEFINITION 6.5. Let μ be a positive volume form in M and $1 \le p < \infty$. We define $A_\mu^p(T_M) := L^p(T_M; dxd\mu) \cap \mathrm{CR}(T_M)$.

It is immediate to verify that $L_\mu^p = L^p(T_M; dxd\mu(t)) \subset L_{\mathrm{loc}}^1(T_M)$, and $A_\mu^p(T_M)$ is a closed subspace of L_μ^p. In particular, $A_\mu^p(T_M)$ is a Banach space.

Next, we quote the following Paley–Wiener characterization of the Hilbert space $A_\mu^2(T_M)$. Below, we shall use the notation

$$I_\mu(\xi) := \int_M e^{-2(\xi|t)}\, d\mu(t), \qquad \xi \in \mathbb{R}^n.$$

THEOREM 6.6 (see Th. 2.2.1 in [10]). *For every $F \in A_\mu^2(T_M)$ there exists a function $\varphi \in L^2(\mathbb{R}^n; I_\mu(\xi)d\xi)$ such that*

$$F_t(x) = F(x+it) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(x+it|\xi)}\varphi(\xi)\, d\xi, \qquad x \in \mathbb{R}^n, \tag{6.7}$$

defined in the Fourier–Plancherel sense for μ-a.e. $t \in M$. Moreover, the correspondence

$$F \in A_\mu^2(T_M) \mapsto \varphi = e^{(\xi|t)}\widehat{F_t} \in L^2(\mathbb{R}^n; I_\mu(\xi)d\xi)$$

is an isometric isomorphism of Hilbert spaces.

Observe that this theorem, together with our results in §4, provides a characterization of the spaces $H_\mu^2(T_\Omega)$, at least when $\overline{\mathrm{co}}\,(\mathrm{Supp}\,\mu) = \overline{\Omega}$. To see this, suppose that $\mu = \mu_{\mathbf{s}}$ and $M = M_\varepsilon$ for some fixed $\mathbf{s} \in \mathbf{\Xi} \setminus \{0\}$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{s}) \in I \setminus \{0\}$. Then $\mu_{\mathbf{s}}$ is a positive volume form in M_ε, and we can identify $\mathrm{Supp}\,\mu_{\mathbf{s}} \equiv M$, $T_{\mu_{\mathbf{s}}} = \mathbb{R}^n + i\,\mathrm{Supp}\,\mu_{\mathbf{s}} \equiv T_M$. In this situation, and using the results in §2.2, we have

$$I_{\mu_{\mathbf{s}}}(\xi) = \begin{cases} \mathcal{L}\mu_{\mathbf{s}}(2\xi) = \Delta_{-\mathbf{s}^*}^*(2\xi), & \xi \in (M^\sharp)^\circ, \\ \infty, & \xi \notin M^\sharp, \end{cases} \tag{6.8}$$

where

$$M^\sharp := \{x \in \mathbb{R}^n : (x|y) \geq 0, \ \forall y \in M\}$$

(see also 2.3.1 in [10]). Now, it is immediate to verify that

$$M^\sharp = (\operatorname{Supp} \mu_{\mathbf{s}})^\sharp = [\overline{\operatorname{co}}\,(\operatorname{Supp} \mu_{\mathbf{s}})]^\sharp = \overline{\Omega},$$

from which

$$L^2(\mathbb{R}^n; I_{\mu_{\mathbf{s}}}(\xi) d\xi) = L^2(\Omega; \Delta^*_{-\mathbf{s}^*}(2\xi) d\xi) = L^2_{-\mathbf{s}^*}(\Omega).$$

Thus, using the results in §4.1 we easily conclude that

$$F \in H^2_{\mu_{\mathbf{s}}}(T_\Omega) \ \Leftrightarrow \ F^{(\mathrm{b})} \in A^2_{\mu_{\mathbf{s}}}(T_M), \tag{6.9}$$

with equality of norms: $\|F\|_{H^2_\mu(T_\Omega)} = \|F^{(\mathrm{b})}\|_{A^2_\mu(T_M)}$.

To extend this characterization to all values of $p \geq 1$, as stated in Theorem 1.3, we shall need some lemmas concerning the spaces $A^p_\mu(T_M)$. The first is an elementary density property, which *a priori* only holds for $p \geq 2$.

LEMMA 6.10. *Let $2 \leq p < \infty$ and μ be be a measure as above. Then $A^2_\mu(T_M) \cap A^p_\mu(T_M)$ is dense in $A^p_\mu(T_M)$.*

Proof. Let $F \in A^p_\mu(T_M)$ and $r = p/2 > 1$. We choose a function G as in (3.24) belonging to the space $L^{2r'}_\mu(T_M)$, and let

$$F^\varepsilon(z) = G(\varepsilon z)\, F(z), \qquad z \in T_M.$$

Since G is holomorphic in a neighborhood of T_Ω we have $F^\varepsilon \in \mathrm{CR}(T_M)$. Further, since G is bounded in $\overline{T}_\Omega$, we also have $F^\varepsilon \in A^p_\mu(T_M)$. Now, by Hölder's inequality,

$$\|F^\varepsilon\|_{L^2_\mu} \leq \|F\|_{L^p_\mu} \|G(\varepsilon\,\cdot)\|_{L^{2r'}_\mu} < \infty,$$

and therefore $F^\varepsilon \in A^2_\mu(T_M)$. Finally, the limit $\|F^\varepsilon - F\|_{L^p_\mu} \to 0$ as $\varepsilon \to 0$ follows by the Dominated Convergence Theorem. ∎

The second lemma is an extension of Theorem 6.6 to the range $1 \leq p \leq 2$. As usual, if $G \in A^p_\mu(T_M)$, we let $G_t(x) = G(x + it)$ as a function in $\mathbb{R}^n$, and

$$\widehat{G}_t(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi|x)} G_t(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

Observe that $G_t \in L^p(\mathbb{R}^n)$ for a.e. $t \in M$, and therefore, $\widehat{G}_t \in L^{p'}(\mathbb{R}^n)$, where $1/p + 1/p' = 1$. The following lemma translates the CR condition in G into an explicit form for $\widehat{G}_t$.

LEMMA 6.11. *Let $1 \leq p \leq 2$. For every $G \in A^p_\mu(T_M)$ there exists a function φ in $\mathbb{R}^n$ such that $I_\mu(p\xi)^{1/p}\varphi(\xi) \in L^{p'}(\mathbb{R}^n)$, and*

$$\widehat{G}_t(\xi) = e^{-(\xi|t)}\varphi(\xi) \qquad \text{a.e. } \xi + it \in T_M. \tag{6.12}$$

In particular, if $\overline{\operatorname{co}}\,(\operatorname{Supp} \mu) = \overline{\Omega}$, the function φ is supported in $\overline{\Omega}$.

Proof. The identity in (6.12) can be obtained following essentially the same steps as in the proof of Theorem 6.6 (see [10]). More precisely, one just needs to show that the function $\varphi(\xi,t) := e^{(\xi|t)}\widehat{G}_t(\xi)$ is independent of the variable t. Then, one proceeds as in Théorème 2.2.1 in [10], except that the assumption $\widehat{G}_t \in L^2_\mu(T_M)$ has to be replaced by $\widehat{G}_t \in L^{p'}_\mu(T_M)$. A brief sketch, adapted to this situation, is the following:

1. By a Weyl type lemma for the manifold T_M, a locally integrable function $\varphi(\xi,t)$ is independent of t if and only if

$$\iint_{T_M} \varphi(\xi,t)\,(Y_j\psi)(\xi,t) = 0 \qquad \forall j = 1,\dots,m,\ \psi \in C^\infty_{\mathrm{c}}(T_M) \tag{6.13}$$

(see Lemma 2.2.2 in [10]).

2. Consider the identity

$$Y_j(e^{(\xi|t)}\psi(\xi,t)) = e^{(\xi|t)}\,(Y_j\psi) + (he_j|\xi)\,e^{(\xi|t)}\psi,$$

where we have written $t = ht_0$, $h \in H$. Then, calling $\theta(\xi,t) := \psi(\xi,t)e^{(\xi|t)}$, (6.13) is equivalent to

$$\iint_{T_M} \widehat{G}_t(\xi)\,Y_j[\theta(\xi,t)] = \iint_{T_M} \widehat{G}_t(\xi)(he_j|\xi)\theta(\xi,t). \tag{6.14}$$

Further, using integration by parts we can write

$$\begin{aligned} i(he_j|\xi)\,\theta(\xi,t) &= \frac{i(he_j|\xi)}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(x|\xi)}\,\check{\theta}_t(x)\,dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(x|\xi)} X_j[\check{\theta}_t(x)]\,dx. \end{aligned}$$

Thus, by applying the Plancherel Theorem in $\mathbb{R}^n$, (6.14) becomes equivalent to

$$\int_M\int_{\mathbb{R}^n} G_t(x)\,Y_j[\check{\theta}_t(x)] = \int_M\int_{\mathbb{R}^n} G_t(x)\,iX_j[\check{\theta}_t(x)]. \tag{6.15}$$

3. Finally, observe that (6.15) is precisely the CR condition on G (see (6.4)). Note, however, that $\check{\theta}_t(x)$ is not compactly supported in $\mathbb{R}^n$, so one shows (6.15) with a limiting argument involving cut-off functions (see p. 46 of [10]).

Thus, assuming (6.12), we may turn to the last claim of the lemma. From (6.12) and Young's inequality, it follows that

$$\Big(\int_{\mathbb{R}^n} |e^{-(t|\xi)}\varphi(\xi)|^{p'}\,d\xi\Big)^{1/p'} \le \Big(\int_{\mathbb{R}^n} |G_t(x)|^p\,dx\Big)^{1/p} \qquad \text{a.e. } t \in M.$$

Using this and Minkowski's inequality ($p'/p \ge 1$) we obtain

$$\|I_\mu(p\,\cdot)\varphi\|_{p'} = \Big(\int\limits_{\mathbb{R}^n}\Big|\int\limits_M e^{-p(t|\xi)}\,d\mu(t)\Big|^{p'/p}|\varphi(\xi)|^{p'}\,d\xi\Big)^{1/p'}$$
$$\le \Big[\int\limits_M\Big(\int\limits_{\mathbb{R}^n} e^{-p'(t|\xi)}\,|\varphi(\xi)|^{p'}\,d\xi\Big)^{p/p'}\,d\mu(t)\Big]^{1/p} \le \|G\|_{L^p_\mu}.$$

This clearly implies $I_\mu(p\xi)^{1/p}\,\varphi(\xi)\in L^{p'}(\mathbb{R}^n)$, concluding the proof of the lemma. ∎

6.3. *The proof of Theorems 1.3 and 1.4.* We now turn to the proof of Theorem 1.3. The first part is actually a straightforward consequence of our results in §3, and the definition of CR equations. More precisely, suppose $F\in H^p_\mu(T_\Omega)$. Then, for all $y\in\Omega$, $F_y = F(\cdot + iy)$ is holomorphic in a neighborhood of $\overline{T}_\Omega$, and in particular belongs to $\mathrm{CR}(T_M)$. Also, by Theorem 1.2 we know that $F_y\to F^{(\mathrm{b})}$ in L^p_μ as $y\to 0$. Therefore, we also have convergence in $L^1_{\mathrm{loc}}(T_M)$ (since $p\ge 1$), and hence

$$0=\int\limits_{T_M} F_y\overline{Z}_j\varphi \to \int\limits_{T_M} F^{(\mathrm{b})}\overline{Z}_j\varphi \qquad \forall j=1,\dots,m,\ \varphi\in C^\infty_{\mathrm{c}}(T_M).$$

This shows that $F^{(\mathrm{b})}$ belongs to $\mathrm{CR}(T_M)$, and gives us $F^{(\mathrm{b})}\in A^p_\mu(T_M)$. Note that we actually have $F^{(\mathrm{b})}\in A^p_\mu(T_M;\Omega)$. Indeed, this is true for $F\in H^2_\mu(T_\Omega)\cap H^p_\mu(T_\Omega)$ (see §4), and extends to all $F\in H^p_\mu(T_\Omega)$ by density.

For the converse, we shall distinguish two cases: $p\le 2$ and $p\ge 2$, and indicate where the crucial hypothesis $\overline{\mathrm{co}}\,(\operatorname{Supp}\mu)=\overline{\Omega}$ is used. The simplest one is the latter, which can be reduced to the case $p=2$ from the following elementary lemma:

LEMMA 6.16. *Let $1\le p<\infty$. Suppose $F\in H^2_\mu(T_\Omega)$ and $F^{(\mathrm{b})}$ is its boundary limit. Then $F\in H^p_\mu(T_\Omega)$ if and only if $F^{(\mathrm{b})}\in L^p_\mu$.*

Proof. Since $F\in H^2_\mu(T_\Omega)$, we may reconstruct it from its boundary value using the classical Poisson–Szegő kernel (see (4.14)):

$$F_t(x+iy)=\int\limits_{\mathbb{R}^n} F_t^{(\mathrm{b})}(x-u)\,P_y(u)\,du, \qquad x+iy\in T_\Omega$$

(except perhaps for t in a set of μ-measure 0). Then, taking L^p_μ-norms for each $y\in\Omega$, we obtain

$$\|F_y\|_{L^p_\mu}\le \|F^{(\mathrm{b})}\|_{L^p_\mu}\int\limits_{\mathbb{R}^n} P_y(u)\,du = \|F^{(\mathrm{b})}\|_{L^p_\mu},$$

from which the result follows. ∎

The proof of the second part of Theorem 1.5 for $p\ge 2$ is now a simple consequence of the case $p=2$ in (6.9). Indeed, by Corollary 3.18 and the first part of the theorem, the correspondence

$$F\in H^p_\mu(T_\Omega)\mapsto F^{(\mathrm{b})}\in A^p_\mu(T_M)$$

is an isometry of Banach spaces. Also, by the case $p = 2$ and Lemma 6.16, every function G in $A^2_\mu(T_M) \cap A^p_\mu(T_M)$ is the boundary value of some $F \in H^p_\mu(T_\Omega)$. Thus, the density result in Lemma 6.10 immediately implies surjectivity, establishing our claim. Observe that in order to use the case $p = 2$, we are assuming $\overline{\mathrm{co}}\,(\operatorname{Supp}\mu) = \overline{\Omega}$.

Consider now the case $1 \le p \le 2$. The simple argument used above cannot be applied now, because we do not know (*a priori*) a density result as in Lemma 6.10. We proceed instead with classical Hardy space theory, for which we need the following proposition:

PROPOSITION 6.17. *Let* $1 \le p < \infty$ *and*

$$L^p_\Omega(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \operatorname{Supp}\widehat{f} \subset \overline{\Omega}\}.$$

Then

$$F \in H^p(T_\Omega) \mapsto F^{(\mathrm{b})} \in L^p_\Omega(\mathbb{R}^n)$$

is an isometric isomorphism of Banach spaces. When $1 \le p \le 2$, *the inverse mapping is given by*

$$f \in L^p_\Omega(\mathbb{R}^n) \mapsto F(z) = \int_\Omega e^{i(z|\xi)} \widehat{f}(\xi)\, d\xi, \tag{6.18}$$

where the integral converges absolutely for every $z \in T_\Omega$.

Proof. For $p = 2$ the result is well known (see IX.4 in [3]). For general p the proof is a simple modification of the ideas presented here. Indeed, by the classical theory, the correspondence

$$F \in H^p(T_\Omega) \mapsto F^{(\mathrm{b})} \in L^p(\mathbb{R}^n)$$

is an isometry of Banach spaces. By density of $H^2 \cap H^p$ in H^p, this mapping takes values in $L^p_\Omega(\mathbb{R}^n)$. To show surjectivity it suffices to see that $L^2 \cap L^p_\Omega$ is dense in L^p_Ω. When $p < 2$ one uses a standard approximation argument. Namely, given $g \in L^p_\Omega(\mathbb{R}^n)$ and a smooth approximation of the identity $\{\varphi_\varepsilon\}$, we have $\lim_{\varepsilon\to 0} \|g*\varphi_\varepsilon - g\|_p = 0$, while by Young's inequality $g*\varphi_\varepsilon \in L^2 \cap L^p_\Omega$. When $p > 2$, one should approach instead with $G(\varepsilon\cdot)(g * \varphi_\varepsilon)$, where G is a function as in the proof of Lemma 6.10 (note that $\widehat{g}$ is now a tempered distribution).

Finally, the equality in (6.18) also follows by density, together with the simple estimate

$$\begin{aligned}\int_\Omega e^{-(y|\xi)} |\widehat{f}(\xi)|\, d\xi &\le \|\widehat{f}\|_{p'} \Big(\int_\Omega e^{-p(y|\xi)} d\xi\Big)^{1/p} \\ &\le \|f\|_p \Delta(py)^{-n/r} < \infty, \qquad y \in \Omega. \ \blacksquare\end{aligned}$$

REMARK 6.19. When $2 < p < \infty$, there is a similar formula for the inverse mapping in (6.18): $F_y = \mathcal{F}^{-1}(e^{-(y|\cdot)}\widehat{f})$, $y \in \Omega$, which now has to be interpreted as a distributional Fourier–Laplace transform (see Chapter VII of [7]).

The proof of Theorem 1.5 now continues as follows. Let $G \in A^p_\mu(T_M)$ and $E \subset M$ be a set such that $\mu(M \setminus E) = 0$ and $G_t \in L^p(\mathbb{R}^n)$ for all $t \in E$. By Lemma 6.11, there is a function φ so that $\operatorname{Supp}\varphi \subset \overline{\Omega}$ (under the assumption $\overline{\mathrm{co}}\,(M) = \overline{\Omega}$), and $\widehat{G}_t(\xi) = e^{-(\xi|t)}\varphi(\xi)$. Further, the properties of φ imply that the integral

$$F(z) := \int_\Omega e^{i(z|\xi)}\varphi(\xi)\,d\xi \tag{6.20}$$

converges absolutely for every $z \in T_\Omega$. Indeed:

$$\begin{aligned}\int_\Omega e^{-(y|\xi)}|\varphi(\xi)|\,d\xi &\le \Big(\int_\Omega |\varphi(\xi)\, I_{\mu_{\mathbf{s}}}(\xi)^{1/p}|^{p'}\,d\xi\Big)^{1/p'}\Big(\int_\Omega e^{-p(y|\xi)}\Delta^*_{\mathbf{s}^*}(2\xi)\,d\xi\Big)^{1/p}\\ &= c\|\varphi I_\mu^{1/p}\|_{L^{p'}(\mathbb{R}^n)}\,\Delta_{\mathbf{s}+n/r}(y)^{-1/p} < \infty, \qquad y \in \Omega,\end{aligned}$$

where we have used the expression for $I_\mu = I_{\mu_{\mathbf{s}}}$ in (6.8). Therefore, F is a holomorphic function in T_Ω.

It remains to show that $F \in H^p_\mu(T_\Omega)$, and its boundary limit in T_M equals G. But this is an immediate consequence of Proposition 6.17. Indeed, from equalities (6.12) and (6.20), we see that

$$F_t(z) := F(z+it) = \int_\Omega e^{i(z|\xi)}\,\widehat{G}_t(\xi)\,d\xi$$

is a function in $H^p(T_\Omega)$ for every $t \in E$. In particular,

$$\lim_{y\to 0}\|F_t(\cdot + iy) - G_t\|_{L^p(\mathbb{R}^n)} = 0.$$

Furthermore,

$$\int_{\mathbb{R}^n} |F_t(x+iy)|^p\,dx \le \|F_t\|^p_{H^p(T_\Omega)} = \|G_t\|^p_{L^p(\mathbb{R}^n)}, \qquad y \in \Omega.$$

Integrating with respect to $d\mu(t)$ we immediately see that $F \in H^p_\mu(T_\Omega)$ and $\|F\|_{H^p_\mu} \le \|G\|_{L^p_\mu}$. Finally, to show that $\lim_{y\to 0}\|F_y - G\|_{L^p_\mu} = 0$, one uses Theorem 3.11 and dominated convergence. This completes the proof of Theorem 1.3. ■

To conclude this section, we indicate the very minor modifications required to establish as well Theorem 1.4.

Proof of Theorem 1.4. The fact that $F^{(\mathrm{b})} \in A^p_\mu(T_M;\Omega)$ for every $F \in H^p_\mu(T_\Omega)$ was already pointed out during the proof of the previous theorem. For the converse, if $1 \le p \le 2$, and if we assume $G \in A^p_\mu(T_M;\Omega)$, again the same proof as above is valid. Indeed, we only used the assumption $\overline{\mathrm{co}}\,(M) = \overline{\Omega}$ to guarantee that $\operatorname{Supp}\widehat{G}_t \subset \overline{\Omega}$, and properly define the holomorphic function F in (6.20). Thus, the same argument gives $F \in H^p_\mu(T_\Omega)$ and $F^{(\mathrm{b})} = G$.

We have proved, in particular, the case $p = 2$. The cases $p > 2$ are obtained from this one, and the density of $A^2_\mu(T_M) \cap A^p_\mu(T_M; \Omega)$ in $A^p_\mu(T_M; \Omega)$, exactly as we did above. This last density result can also be established with minor modifications of our proofs (see the proof of Proposition 6.17). ∎

7. Appendix. We present here some general facts on symmetric cones that were used at different stages of the paper, but whose proofs were postponed for the reader's convenience. As in §2, we use the notation and standard results from [3]. We begin with the proof of Lemma 3.8, whose idea will come again in subsequent lemmas.

Proof of Lemma 3.8. By the action of H it suffices to verify the lemma for $y = \mathbf{e}$. Now,

$$\Delta_{\mathbf{s}}(y+\mathbf{e}) = \Delta_1(y+\mathbf{e})^{s_1} \left(\frac{\Delta_2(y+\mathbf{e})}{\Delta_1(y+\mathbf{e})}\right)^{s_2} \cdots \left(\frac{\Delta_r(y+\mathbf{e})}{\Delta_{r-1}(y+\mathbf{e})}\right)^{s_r}.$$

Since by hypothesis $s_1, \ldots, s_r \geq 0$, it suffices to see that

$$\frac{\Delta_k(y+\mathbf{e})}{\Delta_{k-1}(y+\mathbf{e})} \geq 1 \qquad y \in \Omega,\ k = 1, \ldots, r. \tag{7.1}$$

Now, $\Delta_1(y+\mathbf{e}) = (y+\mathbf{e}|c_1) = (y|c_1) + 1 \geq 1$, so (7.1) follows for $k = 1$. For $k = r$ we may use the identity

$$\frac{\Delta_r(\xi)}{\Delta_{r-1}(\xi)} = \frac{1}{\Delta_1^*(\xi^{-1})} \qquad \forall \xi \in \Omega, \tag{7.2}$$

where Δ_1^* denotes the first principal minor with respect to the rotated Jordan frame $\{c_r, \ldots, c_1\}$ (see Chapter VII of [3]). But if $\xi = y+\mathbf{e}$, then the spectral theorem tells us that there is a Jordan frame $\{d_1, \ldots, d_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r \geq 1$ so that $\xi = \lambda_1 d_1 + \ldots + \lambda_r d_r$. Thus, $\xi^{-1} = \lambda_1^{-1} d_1 + \ldots + \lambda_r^{-1} d_r$ and

$$\Delta_1^*(\xi^{-1}) = (\xi^{-1}|c_r) = \sum_{j=1}^r \lambda_j^{-1}(d_j|c_r) \leq (\mathbf{e}|c_r) = 1.$$

This shows (7.1) for $k = r$.

The other cases follow by induction on r, since

$$\frac{\Delta_k(y+\mathbf{e})}{\Delta_{k-1}(y+\mathbf{e})} = \frac{\Delta_k^{(k)}(P_k y + \mathbf{e}_k)}{\Delta_{k-1}^{(k)}(P_k y + \mathbf{e}_k)},$$

where now $P_k y$, $\mathbf{e}_k = c_1 + \ldots + c_k$, and $\Delta_j^{(k)}$, $j = 1, \ldots, k$, are objects associated with the Jordan algebra $V^{(k)}$ of rank k. ∎

We now briefly recall how to extend the definition of $\Delta_j(z)$, $j = 1, \ldots, r$, to values of z in the complexification $V + iV = V^{\mathbb{C}}$. Since $V^{\mathbb{C}}$ is a complex Jordan algebra (of the same rank as V), there is a natural definition of determinant, $\det_{V^{\mathbb{C}}}(z)$. This extends the original determinant of V in the sense that $\Delta(x) = \det_{V^{\mathbb{C}}}(x + i0)$, $x \in V$. In fact, $\det_{V^{\mathbb{C}}}(z)$ can be obtained from $\Delta(x)$ by just replacing the real coordinates of $x \in V$ by the complex coordinates of $z \in V + iV$. Indeed, since the determinant is always a polynomial in the coordinates of the vector space (see Chapter II of [3]), if we write $\Delta(x) = a_0(X_1, \ldots, X_n)$ for $x = (X_1 \ldots, X_n) \in V \equiv \mathbb{R}^n$, then we will have

$$\det_{V^{\mathbb{C}}}(z) = a_0(Z_1, \ldots, Z_n) \quad \text{for } z = (Z_1, \ldots, Z_n) \in V + iV \equiv \mathbb{C}^n.$$

Thus, with no further comment we shall write $\Delta(z) = \det_{V^{\mathbb{C}}}(z)$, obtaining in this case a holomorphic function in $z \in V^{\mathbb{C}} \equiv \mathbb{C}^n$ (in fact, a homogeneous polynomial of degree r). Exactly the same reasoning applies to the principal minors $\Delta_j(z)$, which are now determinants of the Jordan algebras $(V^{(j)})^{\mathbb{C}}$, $j = 1, \ldots, r$.

We are now interested in giving a sense to the generalized power $\Delta_{\mathbf{s}}(z)$, $\mathbf{s} \in \mathbb{C}^r$, but only when $z \in \Omega + iV$. The next lemma solves the problem of choosing an appropriate determination of the argument.

Lemma 7.3. *Let $x \in V$ and $y \in \Omega$. Then $\Delta_j(y + ix) \neq 0$ and*

$$\frac{\Delta_j(y + ix)}{\Delta_{j-1}(y + ix)} \in \Pi^+ = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}, \qquad j = 1, \ldots, r. \tag{7.4}$$

Proof. It suffices to show (7.4) for $y = \mathbf{e}$. When $j = 1$ this is almost immediate:

$$\Delta_1(\mathbf{e} + ix) = (\mathbf{e} + ix|c_1) = 1 + i(x|c_1) \in \Pi^+.$$

Suppose now $j = r$. On the one hand, every $z \in T_\Omega$ is invertible in $V^{\mathbb{C}}$, and $-z^{-1} \in T_\Omega$ (see Theorem X.1.1 in [3]). In particular, we have $\Delta_1^*((\mathbf{e} + ix)^{-1})\Delta_r(\mathbf{e} + ix) \neq 0$ for all $x \in V$. Thus, we may extend the identity in (7.2) to the complex algebra $V^{\mathbb{C}}$:

$$\Delta_1^*((\mathbf{e} + ix)^{-1})\Delta_r(\mathbf{e} + ix) = \Delta_{r-1}(\mathbf{e} + ix) \quad \forall x \in V,$$

and we obtain $\Delta_{r-1}(\mathbf{e} + ix) \neq 0$ as well. Now, by the spectral theorem there is a Jordan frame $\{d_1, \ldots, d_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ so that $x = \lambda_1 d_1 + \ldots + \lambda_r d_r$. Thus,

$$(e + ix)^{-1} = (1 + i\lambda_1)^{-1} d_1 + \ldots + (1 + i\lambda_r)^{-1} d_r,$$

and therefore,

$$\Delta_1^*((\mathbf{e} + ix)^{-1}) = \Big(c_r \Big| \sum_{j=1}^{r} (1 + i\lambda_j)^{-1} d_j\Big) = \sum_{j=1}^{r} \frac{1 - i\lambda_j}{1 + \lambda_j^2}\,(c_r|d_j).$$

Since $\sum_{j=1}^r (c_r|d_j) = (c_r|\mathbf{e}) = 1$, there must exist some j so that $(c_r|d_j) \neq 0$. Hence, we conclude that $[\Delta_1^*((\mathbf{e} + ix)^{-1})]^{-1}$ has positive real part, establishing our claim. The remaining cases $j = 2, \dots, r-1$ follow from an induction process on the rank, as was described at the end of the proof of Lemma 3.8. ∎

Using the previous lemma we shall define

$$\Delta_{\mathbf{s}}(z) = \Delta_1(z)^{s_1}\left(\frac{\Delta_2(z)}{\Delta_1(z)}\right)^{s_2}\dots\left(\frac{\Delta_r(z)}{\Delta_{r-1}(z)}\right)^{s_r}, \quad z \in \Omega + iV,$$

whenever $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$, and where the determination of the root is positive in the positive real axis. Note that $z \mapsto \Delta_{\mathbf{s}}(z)$ is a holomorphic function in $\Omega + iV$. We now show some general estimates that were previously used in the paper.

LEMMA 7.5. *If $x \in V$, $y \in \Omega$ and $\mathbf{s} \in \mathbb{C}^r$, then*

$$|\Delta_{\mathbf{s}}(y + ix)| \ge \Delta_{\operatorname{Re}\mathbf{s}}(y).$$

Proof. As usual, it suffices to prove that

$$\left|\frac{\Delta_j(\mathbf{e} + ix)}{\Delta_{j-1}(\mathbf{e} + ix)}\right| \ge 1, \quad j = 1, r. \tag{7.6}$$

Using the same computations as in the proof of the previous lemma we see that (7.6) holds trivially for $j = 1$, while for $j = r$ matters reduce to showing

$$|\Delta_1^*((\mathbf{e} + ix)^{-1})| \le 1.$$

But by using again the spectral decomposition of x this follows from

$$|\Delta_1^*((\mathbf{e} + ix)^{-1})| \le \sum_{j=1}^{r} \frac{(c_r|d_j)}{1 + \lambda_j^2}|1 - i\lambda_j| \le 1. \ ∎$$

The generalized powers $\Delta_{\mathbf{s}}(z)$ can also be defined via the Fourier–Laplace transform, at least for certain values of the parameter $\mathbf{s}$. Indeed, let us denote by $\mu_{\boldsymbol{\nu}}^*$ the same distribution as in (1.3), but with Δ_j replaced by Δ_j^* (i.e., $\mu_{\boldsymbol{\nu}}^*$ is the composition of $\mu_{\boldsymbol{\nu}}$ with a rotation $k \in K$ such that $kc_j = c_{r-j+1}$). Then, in view of Proposition 2.8, $\mu_{\boldsymbol{\nu}}^*$ is a positive measure if and only if $\boldsymbol{\nu} \in \Xi$. In this case, the following integral is absolutely convergent and defines a holomorphic function on the tube T_Ω:

$$F_{\boldsymbol{\nu}}(z) = \int_\Omega e^{i(z|\xi)}\, d\mu_{\boldsymbol{\nu}}^*(\xi), \quad z \in T_\Omega.$$

Now, for $\boldsymbol{\nu} = \mathbf{s}^*$, Lemma 2.6 tells us that $F_{\mathbf{s}^*}(iy) = [\Delta_{\mathbf{s}}(y)]^{-1}$ for all $y \in \Omega$. Therefore, by analytic continuation it follows that

$$[\Delta_{\mathbf{s}}(z/i)]^{-1} = \int_\Omega e^{i(z|\xi)}\, d\mu_{\mathbf{s}^*}^*(\xi), \quad z \in T_\Omega, \tag{7.7}$$

at least when $\mathbf{s}^* \in \Xi$. With this formulation we can easily compute integrals involving $\Delta_{\mathbf{s}}(z)$, as we illustrate in the next lemma (whose particular case is Lemma 3.25).

Lemma 7.8. *Let $y \in \Omega$ and $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$. Then the integral*

$$J_{\mathbf{s}}(y) = \int_{\mathbb{R}^n} \frac{dx}{|\Delta_{\mathbf{s}}(y+ix)|}$$

converges if and only if $s_j > (r-j)d/2 + n/r$, $j = 1, \ldots, r$. In this case,

$$J_{\mathbf{s}}(y) = c(\mathbf{s}) \Delta_{n/r - \mathbf{s}}(y),$$

where the constant equals $c(\mathbf{s}) = (4\pi)^n 2^{-|\mathbf{s}|} \Gamma_\Omega(\mathbf{s}^*/2)^{-2} \Gamma_\Omega(\mathbf{s}^* - n/r)$.

Proof. By using the invariance of $\Delta_{\mathbf{s}}$ under H it suffices to show the lemma for $y = \mathbf{e}$. Then from (7.7) and the Plancherel Theorem we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{dx}{|\Delta_{\mathbf{s}}(\mathbf{e}+ix)|} &= \|\Delta_{-\mathbf{s}/2}(\mathbf{e}+i\cdot)\|^2_{L^2(\mathbb{R}^n)} \\
&= \frac{(2\pi)^n}{\Gamma_\Omega(\mathbf{s}^*/2)^2} \int_\Omega e^{-2(\xi|\mathbf{e})} \Delta^*_{\mathbf{s}^*}(\xi) \frac{d\xi}{\Delta(\xi)^{2n/r}} \\
&= \frac{(4\pi)^n}{\Gamma_\Omega(\mathbf{s}^*/2)^2} 2^{-(s_1+\ldots+s_r)} \Gamma_\Omega\left(\mathbf{s}^* - \frac{n}{r}\right),
\end{aligned}$$

where the last integral is finite (and equal to the constant above) iff $s_j - n/r > (r-j)d/2$, $j = 1, \ldots, r$. ■

We conclude the paper with the proof of Lemma 5.15. We recall from Chapter IV of [3] the multiplicative relation between different entries of the Peirce decomposition:

$$(7.9) \qquad V_{i,k} \cdot V_{j,k} \subset V_{i,j} \quad \text{and} \quad V_{i,k} \cdot V_{j,l} = \{0\} \quad \text{if } \{i,k\} \cap \{j,l\} = \emptyset.$$

Proof of Lemma 5.15. Let

$$P(x,y) = L(x)L(y) + L(y)L(x) - L(xy), \qquad x, y \in V,$$

so that $(x \square y)z = P(x,z)y$ (see Chapter VI in [3]). We shall use the following equality:

$$(7.10) \qquad [X, a \square b] = (Xa) \square b - a \square (X^* b), \qquad X \in \mathfrak{g},\ a, b \in V$$

(see Lemma VI.3.4 in [3]). Thus, from (7.10) and $(x \square y)^* = y \square x$, it follows that

$$[z \square c_i, w \square c_j] = (P(z,w)c_i) \square c_j - w \square (P(c_i,c_j)z).$$

Suppose first that $i < j$. Now, using (7.9) we can write

$$\begin{aligned}
P(z,w)c_i &= L(z)L(w)c_i + L(w)L(z)c_i - L(zw)c_i \\
&= 0 + zw/2 - (\delta_{k,j} + \delta_{k,l}) zw/2 = 0.
\end{aligned}$$

For the second term we have

$$P(c_i,c_j)z = L(c_i)L(c_j)z + L(c_j)L(c_i)z - 0 = \delta_{j,k} z/2.$$

Thus,

$$[z \square c_i, w \square c_j] = -\tfrac{1}{2}\, \delta_{j,k}\, w \square z.$$

Assume now $j = k$. Then we must have $[L(w), L(z)] = 0$. Indeed, this follows from the equality

$$[L(w), L(z)] = 2[L(w), L(c_i z)] = 2[L(zw), L(c_i)] = \ldots = [L(z), L(w)],$$

for which one uses standard properties of L (see Proposition II.1.1 of [3]). The identity in (5.16) is then immediate. The case $i = j$ is similar, and left to the reader. ∎

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