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ON SEMI-INVARIANTS OF TILTED ALGEBRAS OF TYPE A_n

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Abstract. We prove that for algebras obtained by tilts from the path algebras of equioriented Dynkin diagrams of type A_n, the rings of semi-invariants are polynomial.

Introduction. Let $Q = (Q_0, Q_1)$ be a quiver with the set Q_0 of vertices and Q_1 of arrows. For every arrow $\alpha \in Q_1$, we denote by $t(\alpha)$ and $h(\alpha)$ the tail and head of α. Fix an algebraically closed field K of characteristic zero. Let $d = (d_x)_{x\in Q_0}$ be a dimension vector for Q and let $V_x = K^{d_x}$ for every $x \in Q_0$. The representation variety of the quiver Q in dimension d is the affine variety $R(Q, d) = \prod_{\alpha\in Q_0} \mathrm{Hom}(V_{t(\alpha)}, V_{h(\alpha)})$. The algebraic group $G(d) = \prod_{x\in Q_0} \mathrm{GL}(d_x)$ acts on the variety $R(Q, d)$ in a natural way and the classification problem for representations of Q in dimension d is equivalent to the classification of orbits of that action. The first approximation to the problem is to describe the invariants of $G(d)$ on regular functions on $R(Q, d)$, since the ring of invariants describes closed orbits. But the ring of invariants is trivial unless the quiver Q has oriented cycles. It turns out that one obtains more subtle information by taking regular functions which are invariant with respect to the subgroup $G'(d) = \prod_{x\in Q_0} \mathrm{SL}(d_x)$ of $G(d)$. The invariants of $G'(d)$ are called semi-invariants and we denote the ring of semi-invariants by $S(Q, d)$. In particular it was proven in [12] that one can read off the representation type of a quiver from the algebraic structure of the rings of its semi-invariants. Namely, a quiver Q is of tame representation type if and only if the ring $S(Q, d)$ is a complete intersection for every dimension vector d.

For quivers with relations one can repeat the same construction but the situation gets more complicated, since the varieties of representations are no more affine spaces. In that case the research seems to be on the stage of collecting examples. In [5], [11] some rings of semi-invariants were calculated for representation varieties of canonical algebras.

The purpose of this paper is to describe the rings of semi-invariants for tilted algebras of type A_n. Let Q be a quiver of type A_n, let $V_1, \ldots, V_n$ be the indecomposable pairwise nonisomorphic representations of Q, and let

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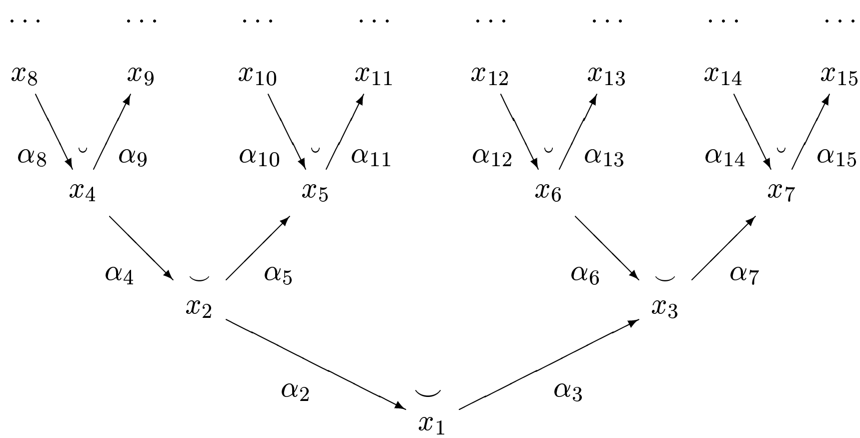
$T = V_1 \oplus \ldots \oplus V_n$. If $\mathrm{Ext}(T, T) = 0$, then the algebra $\mathrm{End}(T)$ is a tilted algebra of type A_n (see [1], [3], [6]).

In the first two sections we consider the semi-invariants of tilted algebras of type A_n obtained from equioriented quivers. It is convenient to view representations of all such algebras as representations of some infinite quiver Y. In Section 1 we define the quiver Y and describe varieties of its representations. Using Kempf's technique of vector bundle collapsing [9], we describe the coordinate rings of irreducible components of $R(Y, d)$ as representations of the algebraic group $G(d)$. This information is used in Section 2 to show that for every irreducible component in $R(Y, d)$ its ring of semi-invariants is a polynomial algebra.

In Section 3 we give two examples of algebras obtained by the tilting process from nonequioriented quivers of type A_n. They show that in general the ring of semi-invariants of such an algebra is much more complicated.

1. The quiver Y and its representations. Let $Y = (Y_0, Y_1)$ be an oriented quiver with zero relations obtained from an infinite binary tree in the following way. The set Y_0 of vertices is an infinite countable set and we identify it with $\{x_1, x_2, \ldots\}$. Every vertex $x_i \in Y_0$ is a parent for two children: left x_{2i} and right x_{2i+1}. The edges from parents to children are oriented in the following way. The left edge goes from a child to the parent, the right one from the parent to the other child, and the two arrows are subject to a zero relation.

We denote the arrow going from x_{2i} to x_i by α_{2i} and the one from x_i to x_{2i+1} by α_{2i+1}. We have $Y_1 = \{\alpha_2, \alpha_3, \ldots\}$. For a vertex $x = x_i$, we will also write $\alpha_{2i} = \alpha_1(x)$ and $\alpha_{2i+1} = \alpha_2(x)$.



Locally in the neighborhood of every vertex $x \in Y_0$ (except the root x_1 of the tree), the quiver Y looks like the central point in the letter Y with arms oriented compatibly from left to right and the zero relation. Orientation of the leg depends on whether x is a left or right child of its parent.

It has been shown in [7] that the class of basic tilted algebras obtained from equioriented quivers of type A_n coincides with the class of the quiver algebras for connected subquivers of Y containing x_1 and having n vertices.

A *dimension vector* for Y is an infinite sequence $d = (d(x))_{x \in Y_0}$ of nonnegative integers such that its *support*

$$\mathrm{supp}(d) = \{x \in Y_0 \mid d(x) \neq 0\}$$

is finite and connected. Let K be an algebraically closed field of characteristic zero. A *representation* of Y over K with dimension vector d is a collection $(V_x)_{x \in Y_0}$ of vector spaces, where $\dim V_x = d(x)$ for every vertex x, together with a collection f of linear maps $f(\alpha) : V_{t(\alpha)} \to V_{h(\alpha)}$, $\alpha \in Y_1$, such that the composition $f(\alpha_{2i+1}) \circ f(\alpha_{2i})$ is zero for every i. In this way one can view a representation of Y as a collection of short complexes indexed by vertices of Y. It is obvious that every representation of Y is in fact a representation of a finite subquiver of Y but regarding it as a representation of Y relieves us of describing exceptions on the boundary of finite quivers.

Let d be a dimension vector for Y and let $\mathcal{R}(d)$ be the representation variety of Y. It is a closed algebraic subvariety in $\bigoplus_{\alpha \in Y_1} \mathrm{Hom}(V_{t(\alpha)}, V_{h(\alpha)})$. The algebraic group $G(d) = \prod_{x \in Y_0} \mathrm{GL}(V_x)$ acts on $\mathcal{R}(d)$ in the standard way: if $g = (g_x)_{x \in Y_0}$ is an element of $G(d)$ and $f = (f(\alpha))_{\alpha \in Y_1}$ is an element of $\mathcal{R}(d)$ then $g \cdot f = (g_{t(\alpha)} \circ f(\alpha) \circ g_{h(\alpha)}^{-1})_{\alpha \in Y_1}$.

In general, the representation varieties $\mathcal{R}(d)$ are reducible. To describe their irreducible components we introduce more notation. Let d be a dimension vector for Y and let $r = (r(\alpha))_{\alpha \in Y_1}$ be a sequence of nonnegative integers. We define subsets of $\mathcal{R}(d)$ by imposing conditions on the ranks of the maps $f(\alpha)$ in the following way:

$$C(d, r) = \{(f(\alpha)) \in \mathcal{R}(d) \mid \mathrm{rk}(f(\alpha)) = r(\alpha) \text{ for every } \alpha \in Y_1\},$$
$$\overline{C}(d, r) = \{(f(\alpha)) \in \mathcal{R}(d) \mid \mathrm{rk}(f(\alpha)) \leq r(\alpha) \text{ for every } \alpha \in Y_1\}.$$

The subset $C(d, r)$ is nonempty if and only if the following two conditions are satisfied:

1. $r(\alpha_i) \leq \min\{d(x_i), d(x_{[i/2]})\}$ for every $i \geq 2$.
2. $r(\alpha_{2i}) + r(\alpha_{2i+1}) \leq d(x_i)$ for every $i \geq 1$.

We will say that r is an *admissible rank vector* for d if it satisfies the above two conditions. If r is admissible then $\overline{C}(d, r)$ is the closure of $C(d, r)$. For admissible rank vectors r and r', we have $\overline{C}(d, r) \subset \overline{C}(d, r')$ if and only if $r(\alpha) \leq r'(\alpha)$ for every $\alpha \in Y_1$. In this case we will say that r' *majorizes* r and we will write $r \leq r'$.

We describe the varieties $\overline{C}(d,r)$ in more detail. Recall that for a linear space V of dimension n over K, the rational irreducible representations of the group $G = \mathrm{GL}(V)$ are parameterized by the dominant weights λ for G. A dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a nonincreasing sequence of integers of length n; let $S_\lambda(V)$ denote the corresponding irreducible representation of $\mathrm{GL}(V)$. If all λ_i's are nonnegative, then λ is called a *partition* and the number of nonzero λ_i's is the *length* of λ.

PROPOSITION 1. *Let d be a dimension vector for Y and let r be an admissible rank vector for d. Then $\overline{C}(d,r)$ is a $G(d)$-invariant irreducible normal affine variety of dimension*

$$\sum_{i\geq 2} d(x_i)r(\alpha_i) + \sum_{i\geq 1}(r(\alpha_{2i}) + r(\alpha_{2i+i}))d(x_i) - \sum_{i\geq 1} r(\alpha_{2i})r(\alpha_{2i+i}) - \sum_{i\geq 2} r(\alpha_i)^2. \tag{1}$$

As a $G(d)$-module, the coordinate ring of $\overline{C}(d,r)$ is isomorphic to

$$K[\overline{C}(d,r)] = \bigotimes_{i\geq 1} \bigoplus_{\kappa,\mu} S_\kappa(V_{x_{2i}}) \otimes S_{(\mu|\kappa^*)}(V_{x_i}) \otimes S_\mu(V^*_{x_{2i+1}}), \tag{2}$$

where for given i the summation runs over all partitions κ and μ of length not greater than $r(\alpha_{2i})$ and $r(\alpha_{2i+1})$, respectively, and

$$(\mu \mid \kappa^*) = (\mu_1, \mu_2, \ldots, \mu_{r(\alpha_{2i+1})}, 0, \ldots, 0, -\kappa_{r(\alpha_{2i})}, \ldots, -\kappa_2, -\kappa_1)$$

is a sequence of length $d(x_i)$.

Proof. It is clear from definition that $\overline{C}(d,r)$ is closed and $G(d)$-invariant. To prove the proposition we use the technique of homogeneous vector bundle collapsing [9]. Kempf used this technique to prove the normality of varieties of complexes [8], which implies normality of $\overline{C}(d,r)$; but to prove the remaining results we have to present Kempf's construction.

We will construct some vector bundle over an irreducible projective variety and a projection from it onto $\overline{C}(d,r)$ which is a birational isomorphism. The existence of such a projection proves the irreducibility of $\overline{C}(d,r)$ and allows us to determine its dimension and the coordinate ring.

Fix a dimension vector d and let r be an admissible rank vector for d. For every vertex $x = x_i \in Y_0$, denote by $r_1(x)$ and $r_2(x)$ the ranks $r(\alpha_{2i})$ and $r(\alpha_{2i+1})$, respectively. Let $\mathcal{F}_x(d,r)$ be the variety of flags of linear spaces $0 \subset V'_x \subset V''_x \subset V_x$ with $\dim V'_x = r_1(x)$ and $\dim V_x/V''_x = r_2(x)$. A flag $0 \subset V'_x \subset V''_x \subset V_x$ in $\mathcal{F}_x(d,r)$ is said to be *compatible* with a representation $f \in \overline{C}(d,r)$ if

$$\mathrm{Im}\, f(\alpha_1(x)) \subset V'_x \quad \text{and} \quad V''_x \subset \mathrm{Ker}\, f(\alpha_2(x)). \tag{3}$$

We need several bundles on $\mathcal{F}_x(d,r)$. Let $\mathcal{V}_x$ be the trivial bundle $V_x \times \mathcal{F}_x(d,r)$. The bundle $\mathcal{R}_x$ is a tautological subbundle in $\mathcal{V}_x$ whose fiber over a flag $0 \subset V'_x \subset V''_x \subset V_x$ is V'_x. The bundle $\mathcal{Q}_x$ is a quotient bundle of $\mathcal{V}_x$ with fiber V_x/V''_x. Now let $\mathcal{F}(d,r) = \prod_x \mathcal{F}_x(d,r)$. Since almost all flag varieties consist of a single point, $\mathcal{F}(d,r)$ is a projective variety. We use the same symbols $\mathcal{V}_x$, $\mathcal{R}_x$ or $\mathcal{Q}_x$ for the pullbacks of the respective bundles from $\mathcal{F}_x(d,r)$ to $\mathcal{F}(d,r)$.

Let $\widetilde{C}(r,d)$ be the subset of all pairs $(f,(0 \subset V'_x \subset V''_x \subset V_x)_{x\in Y_0})$ in $\overline{C}(r,d) \times \mathcal{F}(r,d)$ for which every flag is compatible with f. Obviously $\widetilde{C}(r,d)$ is a vector bundle over $\mathcal{F}(r,d)$ and it is isomorphic to

$$\bigoplus_{i\geq 1}(\mathcal{V}^*_{x_{2i}} \otimes \mathcal{R}_{x_i} \oplus \mathcal{Q}^*_{x_i} \otimes \mathcal{V}_{x_{2i+1}}). \tag{4}$$

Moreover, its rank equals $\sum_{i\geq 2} d(x_i)r(\alpha_i)$.

The natural projection q from $\widetilde{C}(d,r)$ onto $\overline{C}(d,r)$ is invertible over $C(d,r)$, so it is a birational isomorphism. Due to normality of $\overline{C}(d,r)$, it follows from the Kempf theorem that the structure sheaf $\mathcal{O}_{\overline{C}(d,r)}$ is isomorphic to the direct image $q_*\mathcal{O}_{\widetilde{C}(d,r)}$, and in particular we have $K[X] = H^0(\widetilde{C}(d,r),\mathcal{O}_{\widetilde{C}(d,r)})$. Let q be the projection of $\widetilde{C}(d,r)$ onto $\mathcal{F}(r,d)$. It is an affine morphism, so the global sections of $\mathcal{O}_{\widetilde{C}(d,r)}$ and of $q_*\mathcal{O}_{\widetilde{C}(d,r)}$ are the same and as a consequence we have

$$K[\overline{C}(d,r)] = H^0\Big(\mathcal{F}(r,d), \bigotimes_{i\geq 1} \mathrm{Sym}(\mathcal{V}_{x_{2i}} \otimes \mathcal{R}^*_{x_i}) \otimes \mathrm{Sym}(\mathcal{Q}_{x_i} \otimes \mathcal{V}^*_{x_{2i+1}})\Big). \tag{5}$$

Now the result on the structure of the coordinate ring $K[\overline{C}(d,r)]$ follows from the Cauchy formula for symmetric powers of the tensor product and the Bott theorem. ∎

As a corollary of Proposition 1 we obtain the following description of irreducible components of the representation variety $\mathcal{R}(d)$.

PROPOSITION 2. *The varieties $\overline{C}(d,r)$, with r maximal with respect to majorizing order, are the irreducible components of $\mathcal{R}(d)$. In particular, $\mathcal{R}(d)$ is irreducible if and only if for every vertex $x_i \in Y_0$ one of the following conditions holds*: (a) $d(x_i) \geq d(x_{2i}) + d(x_{2i+1})$ *or* (b) *at least one of the dimensions $d(x_{2i})$ or $d(x_{2i+1})$ is zero.*

Proof. The first statement is clear since the varieties $\overline{C}(d,r)$ are irreducible and $\overline{C}(d,r') \subset \overline{C}(d,r)$ for $r' \leq r$. Now assume that, for some $x_i \in Y_0$, we have $d(x_i) < d(x_{2i}) + d(x_{2i+1})$, $d(x_{2i}) > 0$ and $d(x_{2i+1}) > 0$. Let r be a maximal admissible rank vector for d. Then $r(\alpha_{2i}) + r(\alpha_{2i+1}) = d(x_i)$, and therefore either $d(x_{2i}) > r(\alpha_{2i})$ or $d(x_{2i+1}) > r(\alpha_{2i+1})$. In any case we can obtain another maximal rank vector by increasing by 1 one of the ranks $r(\alpha_{2i})$ or $r(\alpha_{2i+1})$ and decreasing the other one by 1. ∎

2. Semi-invariants of $\overline{C}(d,r)$. Let d be a dimension vector for Y and let r be an admissible rank vector for d. The group $G(d)$ acts on $\overline{C}(d,r)$ and hence on regular functions on this variety. Let $S(d,r)$ be the ring of semi-invariants of this action. Our goal is to prove that $S(d,r)$ is a polynomial ring. To state the result more precisely let us recall the description of the semi-invariants for quivers of type A_n. Let Q be a quiver of type A_n with vertices $0,1,\ldots,n$ and arrows $\alpha_{i,i+1}$ joining i and $i+1$ (we make no assumption on the directions of the arrows). Let d be a dimension vector for Q and let r be an admissible rank vector for d. Then the ring of semi-invariants of representations of Q in dimension d with ranks bounded by r is a polynomial ring with generators which can be described as follows.

Let δ be an unoriented path in Q. We divide δ into maximal compatibly oriented subpaths: $\delta=(\delta_1,,\ldots,\delta_k)$, where each δ_j joins i_{j-1} and i_j. We call δ an *elementary path* with respect to dimension d and rank r if the following conditions are satisfied:

$$\sum_{j=0}^{k}(-1)^j d(i_j)=0; \tag{6}$$

$$\sum_{j=0}^{k'}(-1)^{k'-j} d(i_j)>0 \tag{7}$$

for every $k'<k$;

$$d(s)-\sum_{j=0}^{k'}(-1)^{k'-j} d(i_j)>0 \tag{8}$$

for every intermediate vertex s on the path $\delta_{k'+1}$; and

$$r(\alpha)\ge\sum_{j=0}^{k'}(-1)^{k'-j} d(i_j) \tag{9}$$

for every arrow α in the path $\delta_{k'+1}$.

Let $\delta=(\delta_1,\ldots,\delta_k)$ be an elementary path and let f be a representation of Q. Define $\delta_i(f)$, $i=1,\ldots,k$, to be the composition of the maps $f(\alpha)$ for all the arrows α along the path δ_i. The maps $\delta_i(f)$ define the map $\delta(f)=\delta_1\oplus\ldots\oplus\delta_k$ from $V_{i_0}\oplus V_{i_2}\oplus\ldots$ to $V_{i_1}\oplus V_{i_3}\oplus\ldots$ (or in the opposite direction). Then the determinant of $\delta(f)$ is a semi-invariant. We denote it by $\det(\delta)$.

Let us remark that to define $\det(\delta)$ the condition (6) is only needed. But if (9) does not hold for some arrow α then $\det(\delta)$ vanishes on representations of rank $r' \leq r$. On the other hand if (8) is not satisfied by an intermediate vertex s, then s divides δ into two subpaths δ' and δ'' and $\det(\delta) = \det(\delta') \cdot \det(\delta'')$.

An *elementary path in* Y is an elementary path in any subquiver of Y of type A_n without relations.

THEOREM 1. *Let d be a dimension vector for the quiver Y and let r be an admissible rank vector for d. The ring $S(d,r)$ of semi-invariants is a polynomial ring in the algebraically independent determinantal semi-invariants* $\det(\delta)$, *where δ runs over all paths in Y which are elementary with respect to dimension d and rank r.*

Proof. In order to find the semi-invariants, we reformulate the description of the coordinate ring of $\overline{C}(d,r)$ given in Proposition 1. We can exchange the order of $\otimes$ and $\oplus$ in the formula (2) of the proposition in the following way. Denote by Λ the set of all sequences $\lambda = (\lambda(\alpha))_{\alpha \in Y_1}$ where every $\lambda(\alpha)$ is a partition of length not greater than $r(\alpha)$. Then, as a $G(d)$-representation, the coordinate ring of $\overline{C}(d,r)$ is isomorphic to

$$(10) \qquad \bigoplus_{\lambda \in \Lambda} \bigotimes_{i \geq 1} S_{\lambda(\alpha_{2i})} V_{x_{2i}} \otimes S_{(\lambda(\alpha_{2i+1})|\lambda(\alpha_{2i})^*)} V_{x_i} \otimes S_{\lambda(\alpha_{2i+1})} V^*_{x_{2i+1}}.$$

If $\lambda = (\lambda(\alpha))$ is an element of Λ, then for every arrow α_i we can treat $\lambda(\alpha_i)$ as a dominant weight of $\mathrm{GL}(V_{x_i})$. In this way, Λ is a semigroup contained in the group of dominant weights of $G(d)$.

Now we fix λ and assume that the tensor product in (10) contains a semi-invariant. In this product, the space V_{x_1} appears only once as a Schur module $S_{(\lambda(\alpha_3)|\lambda(\alpha_2)^*)} V_{x_1}$. It follows immediately that at most one of the partitions $\lambda(\alpha_2)$ or $\lambda(\alpha_3)$ can be nonzero and if $\lambda(\alpha_i)$, where i is 2 or 3, is nonzero then $r(\alpha_i) = d(x_1)$ and

$$\lambda(\alpha_i) = \underbrace{(a, \ldots, a)}_{r(\alpha_i) \text{ times}}$$

for some positive integer a. For any other vertex x, the tensor product in (10) contains exactly two Schur modules of V_x. Recall that if V is a vector space of dimension t, then the tensor product $S_{(\mu_1,\ldots,\mu_t)}(V) \otimes S_{(\nu_1,\ldots,\nu_t)}(V)$ contains a semi-invariant of $\mathrm{GL}(V)$ if and only if

$$(11) \qquad \mu_1 + \nu_t = \mu_2 + \nu_{t-1} = \ldots = \mu_t + \nu_1$$

and, if this is the case, it contains a one-dimensional space of semi-invariants of weight $\mu_1 + \nu_t$. Fix a vertex $x = x_i \in Y_0$, $i > 1$, and let $t = \dim V_{x_i}$, $r_1 = r(\alpha_{2i})$, $r_2 = r(\alpha_{2i+1})$, $\nu = \lambda(\alpha_i)$, $\kappa = \lambda(\alpha_{2i})$, $\mu = \lambda(\alpha_{2i+1})$. Then the condition (11) for the existence of a semi-invariant reads as follows. If $t > r_1 + r_2$, then

$$
(12)\qquad
\begin{aligned}
\mu_1-\mu_2&=\nu_{t-1}-\nu_t,\\
&\;\vdots\\
\mu_{r_2-1}-\mu_{r_2}&=\nu_{t-r_2+1}-\nu_{t-r_2+2},\\
\mu_{r_2}&=\nu_{t-r_2}-\nu_{t-r_2+1},\\
0&=\nu_{t-r_2-1}-\nu_{t-r_2},\\
&\;\vdots\\
0&=\nu_{r_1+1}-\nu_{r_1+2},\\
\kappa_{r_1}&=\nu_{r_1}-\nu_{r_1+1},\\
\kappa_{r_1-1}-\kappa_{r_1}&=\nu_{r_1-1}-\nu_{r_1},\\
&\;\vdots\\
\kappa_1-\kappa_2&=\nu_1-\nu_2,
\end{aligned}
\qquad\text{or}\qquad
\begin{aligned}
\mu_1-\mu_2&=\nu_1-\nu_2,\\
&\;\vdots\\
\mu_{r_2-1}-\mu_{r_2}&=\nu_{r_2-1}-\nu_{r_2},\\
\mu_{r_2}&=\nu_{r_2}-\nu_{r_2+1},\\
0&=\nu_{r_2+1}-\nu_{r_2+2},\\
&\;\vdots\\
0&=\nu_{t-r_1-1}-\nu_{t-r_1},\\
\kappa_{r_1}&=\nu_{t-r_1}-\nu_{t-r_1+1},\\
\kappa_{r_1-1}-\kappa_{r_1}&=\nu_{t-r_1+1}-\nu_{t-r_1+2},\\
&\;\vdots\\
\kappa_1-\kappa_2&=\nu_{t-1}-\nu_t,
\end{aligned}
$$

depending on the direction of α_x (or equivalently on the parity of i): the left system corresponds to the case of $\begin{smallmatrix}\searrow & & \nearrow\\ & \cdot & \\ & \downarrow & \end{smallmatrix}$ while the right system corresponds to the case of $\begin{smallmatrix}\searrow & & \nearrow\\ & \cdot & \\ & \uparrow & \end{smallmatrix}$.

Similarly, for $t=r_1+r_2$ we have

$$
(13)\qquad
\begin{aligned}
\mu_1-\mu_2&=\nu_{t-1}-\nu_t,\\
&\;\vdots\\
\mu_{r_2-1}-\mu_{r_2}&=\nu_{r_1+1}-\nu_{r_1+2},\\
\mu_{r_2}+\kappa_{r_1}&=\nu_{r_1}-\nu_{r_1+1},\\
\kappa_{r_1-1}-\kappa_{r_1}&=\nu_{r_1-1}-\nu_{r_1},\\
&\;\vdots\\
\kappa_1-\kappa_2&=\nu_1-\nu_2,
\end{aligned}
\qquad\text{or}\qquad
\begin{aligned}
\mu_1-\mu_2&=\nu_1-\nu_2,\\
&\;\vdots\\
\mu_{r_2-1}-\mu_{r_2}&=\nu_{r_2-1}-\nu_{r_2},\\
\mu_{r_2}+\kappa_{r_1}&=\nu_{r_2}-\nu_{r_2+1},\\
\kappa_{r_1-1}-\kappa_{r_1}&=\nu_{r_2+1}-\nu_{r_2+2},\\
&\;\vdots\\
\kappa_1-\kappa_2&=\nu_{t-1}-\nu_t.
\end{aligned}
$$

In this way, the characters of semi-invariants are in one-to-one correspondence with the sequences $\lambda=(\lambda(\alpha))_{\alpha\in Y_1}\in\Lambda$ satisfying for every arrow α an appropriate system of equations of type (12) or (13). Denote by Λ_0 the set of all such λ's.

We want to treat (12) and (13) as equations for κ and μ. If for given ν a system of equations of the form (12) has a solution then the solution is unique. On the contrary, a system of type (13) always has $a+1$ solutions, where $a=\nu_{r_1}-\nu_{r_1+1}$ for the left system and $a=\nu_{r_2}-\nu_{r_2+1}$ for the right one. Every such solution is uniquely determined by possible choices of μ_{r_2} and κ_{r_1}, and if there are nonzero solutions, then they are linear combinations of two basic solutions obtained for $\mu_{r_2}=0$ and $\kappa_{r_1}=0$, respectively.

In any case, if $\nu = \nu' + \nu''$ is a componentwise sum of two (possibly empty) partitions, then the system has a solution for ν if and only if the corresponding systems for ν' and ν'' have solutions, and every solution for ν is a sum of solutions for ν' and ν''. So we can restrict our discussion to the case when $\nu = (1, \ldots, 1, 0, \ldots, 0)$ is a fundamental weight and denote by ϱ the number of 1's in it. Then it is easy to see that the empty partitions $\kappa = (0, \ldots, 0)$ and $\mu = (0, \ldots, 0)$ are solutions if and only if $\varrho = t$, and if a nonzero solution exists, then one of the partitions κ or μ is empty, while the other one is a fundamental weight. In the last case, the nonempty partition has either ϱ ones (when the arrows corresponding to ν and the nonempty partition are compatibly oriented) or $t - \varrho$ ones (for the arrows oriented in a noncompatible way).

Now let λ be an indecomposable element of the semigroup Λ_0. Denote by i'_1 the smallest i such that $\lambda(\alpha_i)$ is nonempty. The arrow $\alpha_{i'_1}$ joins the vertex $x_{i'_1}$ to its parent x_{i_0}, and applying to x_{i_0} the same arguments used above for x_1, we see that $\lambda(\alpha_{i'_1}) = (1^\varrho)$ (sequence of ϱ ones), where $\varrho = r(\alpha_{i'_1}) = d(x_{i_0})$. We will analyze λ upwards using the fact that in each step a system of equations of the type (12) or (13) must be satisfied. In particular, it follows that the partition associated to at most one arrow joining a given vertex to its children can be nonempty, and the nonempty one is still a fundamental weight. In this way all arrows α such that $\lambda(\alpha)$ is nonempty form an unoriented path δ starting at x_{i_0}. Let x_{i_k} be the ending point of δ and let $i_i < \ldots < i_{k-1}$ be the indices of those vertices along δ at which the arrows forming δ are noncompatibly oriented. As long as we move along a compatible segment of the path, the partition $\lambda(\alpha)$ remains constant while at every noncompatibility point the number of 1's changes, and on successive segments it is equal to

$$d(x_{i_0}), \quad d(x_{i_1}) - d(x_{i_0}), \quad d(x_{i_2}) - d(x_{i_1}) + d(x_{i_0}), \ \ldots;$$

in particular the inequality (9) is satisfied for every $k' = 0, 1, \ldots, k-1$ and every arrow α in the path $\delta_{k'+1}$. The system for the end point x_{i_k} has a zero solution so we have

$$d(x_{i_k}) = d(x_{i_{k-1}}) - d(x_{i_{k-2}}) + \ldots \pm d(x_{i_0}).$$

It follows from the indecomposability of λ that an inequality of the form (8) holds for every intermediate vertex x_s of δ, which proves that δ is an elementary path.

It remains to prove that the generators $\det(\delta)$ are algebraically independent or, equivalently, that the semigroup Λ_0 is freely generated by the sequences of partitions corresponding to the semi-invariants $\det(\delta)$.

Let $\delta = (\delta_1, \ldots, \delta_k)$ be an elementary path joining x_{i_0} and x_{i_k} and let x_{i_1}, $x_{i_2}, \ldots, x_{i_{k-1}}$ be the vertices at which the arrows of δ are noncompatible.

Denote by λ_δ the sequence of partitions corresponding to $\det(\delta)$. Recall that $\lambda_\delta(\alpha)$ is empty for every arrow α not in δ and $\lambda_\delta(\alpha) = (1^{\varrho_j})$ for every arrow α from the segment δ_j of δ joining $x_{i_{j-1}}$ and x_{i_j}, where

$$\varrho_j = d(x_{i_{j-1}}) - d(x_{i_{j-2}}) + \ldots \pm d(x_{i_0}). \tag{14}$$

Let $\sum_\delta n_\delta \lambda_\delta = \lambda = \sum_\delta n'_\delta \lambda_\delta$ be a relation among λ_δ. We can assume that the relation is minimal in the sense that if $n_\delta > 0$ then $n'_\delta = 0$ and vice versa. Let i'_0 be the smallest index i for which $\lambda(\alpha_i)$ is nonempty. Then there exist two different paths δ and δ', both starting with $\alpha_{i'_0}$, and such that $n_\delta > 0$ and $n'_{\delta'} > 0$. Let x_{i_0} be the starting point of δ and δ', and let x_s be a common vertex of both paths with highest index. We may choose δ and δ' in such a way that s is maximal possible. It follows from the local structure of the quiver Y that x_s is a compatibility point for exactly one of the paths δ or δ'. Assume that x_s is a compatibility vertex in δ. Let $x_{i_1}, \ldots, x_{i_j}$ be all the noncompatibility vertices of δ (and of δ') lying between x_{i_0} and x_s. Let α be the last common arrow of δ and δ' and let α' be the first arrow of δ' which does not belong to δ. Then $\lambda_\delta(\alpha) = \lambda_{\delta'}(\alpha) = (1^{\varrho_j})$, where ϱ_j is given by (14), while $\lambda_{\delta'}(\alpha') = (1^{d(x_s)-\varrho_j})$. Therefore, there exists an elementary path δ'' containing α' such that $n_{\delta''} > 0$ and $\lambda_{\delta''} = (1^{d(x_s)-\varrho_j})$. The starting point $x_{s'}$ of δ'' lies somewhere on the common beginning of δ and δ'. If $x_{s'}$ lies on the segment joining $x_{i_{j'}-1}$ and $x_{i_{j'}}$ and is different from $x_{i_{j'-1}}$, then applying the formula (14) to the path δ'' we obtain

$$d(x_s) - \varrho_j = d(x_s) - d(x_{i_j}) + d(x_{i_{j-1}}) - \ldots \pm d(x_{i_{j'}}) \mp d(x_{s'}).$$

Since ϱ_j is given by (14), we obtain

$$d(x_{i_0}) - d(x_{i_1}) + \ldots \pm d(x_{i_{j'-1}}) \mp d(x_s) = 0,$$

which contradicts the fact that δ is elementary. The only possibility left is $s' = i_0$. But then the paths δ'' and δ' have a longer common initial subpath than that of δ and δ', which contradicts the choice of the latter. ∎

Remark. For a quiver without relations, Schofield [10] defined a family of semi-invariants c^V. One can easily interpret the semi-invariants $\det(\delta)$ as restrictions of some special semi-invariants of Schofield.

Let d be a dimension vector for Y. Denote by Y' a quiver without relations whose set of vertices is equal to $\mathrm{supp}(d)$. The set of arrows in Y' consists of all arrows in Y joining points in $\mathrm{supp}(d)$. In general the quiver Y' is wild. The variety of representations of Y in dimension d and rank r is a closed subset of the variety of representations of Y'. Let $\delta = (\delta_1, \ldots, \delta_k)$ be a path in Y elementary with respect to d and r. Assume for simplicity that δ_j goes from $x_{i_{j-1}}$ to x_{i_j} for j odd and from $x_{i_{j+1}}$ to x_{i_j} for j even. Let V be a string module corresponding to the path δ. It is an indecomposable representation of the quiver Y' obtained in the following way. We put one-dimensional spaces at all the vertices along the path δ and identity on all arrows in δ, and put zero spaces at all other vertices and zero maps on all other arrows. Then $\det(\delta)$ is a restriction of c^V.

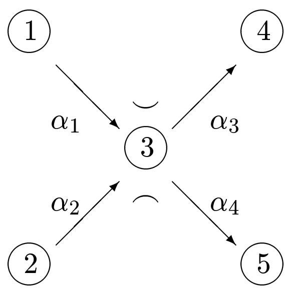
In fact, the module V has a projective resolution of the form

$$0 \to P' \to P \to V \to 0,$$

where $P = P_0 \oplus P_2 \oplus \ldots$, $P' = P_1 \oplus P_3 \oplus \ldots$, and P_j is a projective cover of a simple module with support in x_{i_j}. Then for a representation W of Y in dimension d, the value of c^V at W equals the determinant of the induced map from $\mathrm{Hom}(P, W)$ to $\mathrm{Hom}(P', W)$ and therefore $\det(\delta)$ and c^V are proportional.

3. Tilted algebras for A_n with any orientation. The rings of semi-invariants for algebras obtained by tilts of nonequioriented A_n quivers are no more polynomial rings. We present two examples.

Example 1. Let X be a quiver with vertices $X_0 = \{1, 2, 3, 4, 5\}$, arrows $X_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, and relations $\alpha_3\alpha_1 = \alpha_4\alpha_2 = 0$, as in the picture below.



The path algebra of X can be realized as a tilted algebra for a suitably oriented A_5 quiver (see [1]). Let d be a dimension vector for X of the form $d = (n, n, 2n, n, n)$. The variety $R(X, d)$ of representations of X in dimension d is a product of two complex varieties and hence it is irreducible. It follows from the collapsing technique that the coordinate ring of $R(X, d)$ is isomorphic, as a $G(d)$-module, to

$$\bigoplus S_\lambda(V_1) \otimes S_{(\mu|\lambda^*)}(V_3) \otimes S_\mu(V_4^*) \otimes S_\nu(V_2) \otimes S_{(\xi|\nu^*)}(V_3) \otimes S_\xi(V_5^*),$$

where the summation runs over all partitions λ, μ, ν, ξ with no more than n parts. If there exists a semi-invariant in such a summand then each of the partitions λ, μ, ν, ξ must consist of n equal parts. Assume that $\lambda = (l^n)$, $\mu = (m^n)$, $\nu = (p^n)$, $\xi = (k^n)$. Then the corresponding summand contains a semi-invariant if and only if $m + l = k + p$, and then it contains a one-dimensional space of semi-invariants. The ring of semi-invariants $S(X, d) =$

$\sum_s S(X,d)_s$ is graded by the total degree with respect to matrix elements of linear maps corresponding to the arrows. It follows that the Hilbert function $P(t) = \sum_s \dim(S(X,d)_s)t^s$ is of the form $P(t) = \sum_s (s+1)^2 t^{2ns}$.

Moreover, looking at the finer grading of $S(X,d)$ given by the characters of $G(d)$, we can identify its generators. Let δ_1, δ_2, δ_3, δ_4 be the semi-invariants which take the following values at a representation $V \in R(X,d)$:

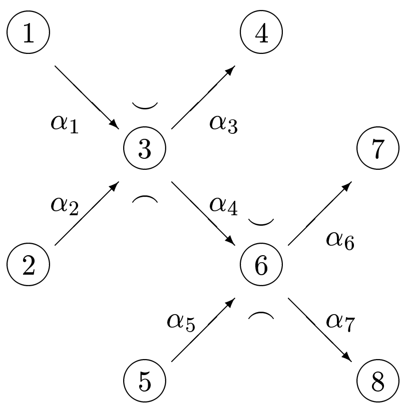
$$\delta_1(V) = \det(V(\alpha_1) + V(\alpha_2) : V(1) \oplus V(2) \to V(3)),$$
$$\delta_2(V) = \det(V(\alpha_3) + V(\alpha_4) : V(3) \to V(4) \oplus V(5)),$$
$$\delta_3(V) = \det(V(\alpha_4) \circ V(\alpha_1) : V(1) \to V(5)),$$
$$\delta_4(V) = \det(V(\alpha_3) \circ V(\alpha_2) : V(2) \to V(3)).$$

Now look at the linear map $V(\alpha_4)\circ V(\alpha_1) \oplus V(\alpha_3)\circ V(\alpha_2) :\ V(1)\oplus V(2) \to V(4) \oplus V(5)$. Its determinant is $\delta_3(V)\delta_4(V)$ but factoring the morphism through $V(3)$ one can see that the determinant is equal to $\delta_1(V)\delta_2(V)$ as well. Hence the semi-invariants δ_i satisfy the relation $\delta_1\delta_2 - \delta_3\delta_4 = 0$. A simple calculation shows that

$$P(t) = \frac{1 - t^{4n}}{(1-t^{2n})^4},$$

which coincides with the Hilbert function of an algebra generated by four elements of degree $2n$ with one relation of degree $4n$, and we conclude that $S(X,d)$ is the hypersurface $K[\overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_3, \overline{\delta}_4]/(\overline{\delta}_1\overline{\delta}_2 - \overline{\delta}_3\overline{\delta}_4)$.

EXAMPLE 2. Let X' be the following quiver with zero relations:



Then the path algebra of X' is an iterated tilted algebra of type A_5 (see [2]).

Let d be a dimension vector for X' of the form $d = (n,n,2n,n,n,2n,n,n)$ and let $r = (n,n,n,n,n,n,n)$. The variety $R(X',d,r)$ of representations of X' in dimension d with ranks bounded by r is a product of three varieties of complexes, and it is an irreducible component of the variety of all representations in dimension d.

To calculate the ring $S(X', d, r)$ of semi-invariants we adopt the same method as in the previous example. First, by the collapsing method we find that, as a $G(d)$-module, the ring $S(X', d, r)$ is a direct sum of one-dimensional spaces

$$(\textstyle\bigwedge^n V_1)^a \otimes (\bigwedge^n V_2)^c \otimes (\bigwedge^{2n} V_3)^{b-c} \otimes (\bigwedge^n V_4)^{-b} \\ \otimes (\bigwedge^n V_5)^f \otimes (\bigwedge^{2n} V_6)^{e-f} \otimes (\bigwedge^n V_7)^{-e} \otimes (\bigwedge^n V_8)^{-g}$$

parameterized by the sequences (a, b, c, d, e, f, g) of natural numbers such that

$$\begin{cases} a + b = d + c, \\ d + e = f + g. \end{cases}$$

We can find generators of $S(X', d, r)$ by identifying the semi-invariants corresponding to indecomposable solutions of the above system of equations. They are as follows:

$$(15)\qquad \begin{aligned} &\delta_1(V) = \det(V_1 \oplus V_2 \to V_3), && \delta_5(V) = \det(V_1 \oplus V_5 \to V_6), \\ &\delta_2(V) = \det(V_2 \to V_4), && \delta_6(V) = \det(V_1 \to V_8), \\ &\delta_3(V) = \det(V_5 \to V_7), && \delta_7(V) = \det(V_3 \oplus V_5 \to V_4 \oplus V_6), \\ &\delta_4(V) = \det(V_6 \to V_7 \oplus V_8), && \delta_8(V) = \det(V_3 \to V_4 \oplus V_8). \end{aligned}$$

If we choose a grading in such a way that the matrix elements corresponding to α_4 have degree zero and all other matrix elements have degree one, then the semi-invariants $\delta_1, \ldots, \delta_8$ have degree $2n$ and the Hilbert function of $S(X', d, r)$ equals

$$(16)\qquad P'(t) = \sum_{c,d,e} (d + c + 1)(d + e + 1) t^{2n(c+d+e)}.$$

It is easy to see that the δ_i's satisfy the following relations:

$$(17)\qquad \begin{aligned} &\delta_4\delta_5 - \delta_3\delta_6 = 0, \quad \delta_1\delta_7 - \delta_2\delta_5 = 0, \quad \delta_2\delta_6 - \delta_1\delta_8 = 0, \\ &\delta_5\delta_8 - \delta_6\delta_7 = 0, \quad \delta_4\delta_7 - \delta_3\delta_8 = 0. \end{aligned}$$

Let $\overline{\delta}_1, \ldots, \overline{\delta}_8$ be independent variables of degree $2n$ and define $\overline{S} = K[\overline{\delta}_1, \ldots, \overline{\delta}_8]$. Denote by $\varrho_1, \ldots, \varrho_5$ the expressions in $\overline{\delta}_i$ corresponding to the left-hand sides of equations (17). We claim that the algebra $S(X', d, r)$ is generated by the δ_i's with relations (17), i.e. it is isomorphic to $S = \overline{S}/(\varrho_1, \ldots, \varrho_5)\overline{S}$.

To see this we interpret $\varrho_1, \ldots, \varrho_5$ as 4×4 Pfaffians of the antisymmetric matrix

$$A = \begin{bmatrix} 0 & -\overline{\delta}_8 & -\overline{\delta}_7 & -\overline{\delta}_2 & 0 \\ \overline{\delta}_8 & 0 & 0 & -\overline{\delta}_4 & -\overline{\delta}_6 \\ \overline{\delta}_7 & 0 & 0 & -\overline{\delta}_3 & -\overline{\delta}_5 \\ \overline{\delta}_2 & \overline{\delta}_4 & \overline{\delta}_3 & 0 & -\overline{\delta}_1 \\ 0 & \overline{\delta}_6 & \overline{\delta}_5 & \overline{\delta}_1 & 0 \end{bmatrix}.$$

Then the complex

$$0 \longrightarrow \overline{S} \xrightarrow{\varrho^t} \overline{S}^5 \xrightarrow{A} \overline{S}^5 \xrightarrow{\varrho} \overline{S} \longrightarrow 0, \tag{18}$$

where $\varrho = [\varrho_1, \ldots, \varrho_5]$, is a minimal free resolution of S over $\overline{S}$. Since the differentials in (18) are homogeneous of degree $4n$, $2n$ and $4n$ respectively, we can calculate the Hilbert function $P_S(t)$ of S to be

$$P_S(t) = \frac{1 - 5t^{4n} + 5t^{6n} - t^{10n}}{(1 - t^{2n})^8} = \frac{1 + 3t^{2n} + t^{4n}}{(1 - t^{2n})^5}.$$

By an elementary calculation, we see that the Hilbert functions $P'(t)$ and $P_S(t)$ coincide, which proves our claim.

In particular, it follows that the algebra of semi-invariants $S(X', d, r)$ is not a complete intersection. This means that the theorem of Skowroński and Weyman [12] on the rings of semi-invariants for tame quivers without relations is no longer valid for quivers with relations.

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