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n-FUNCTIONALITY OF GRAPHS

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Abstract. We first characterize in a simple combinatorial way all finite graphs whose edges can be directed to form an n-functional digraph, for a fixed positive integer n. Next, we prove that the possibility of directing the edges of an infinite graph to form an n-functional digraph depends on its finite subgraphs only. These results generalize Ore's result for functional digraphs.

It is a classical result due to O. Ore (see e.g. [1], Chapter 3, Theorem 17) that all edges of an (undirected) graph can be directed to form a functional digraph iff each of its connected components contains at most one undirected cycle (a single loop is also a cycle here). In the present paper we generalize this result to digraphs which can be decomposed into n functional digraphs, where n is a given positive integer. We start with a simple combinatorial characterization of finite graphs that can be directed to have such a form. Next, we show that all the edges of an infinite graph can be directed in such a way iff each of its finite subgraphs can be turned into such a digraph. Note that we admit loops and multiple edges in the definition of a graph (such graphs are often called "multigraphs with loops").

If at most n edges start from each vertex of a digraph, then the digraph can be clearly decomposed into n edge-disjoint functional digraphs, and conversely. Therefore we introduce

DEFINITION 1. (a) A digraph D is said to be *n-functional*, where n is a positive integer, if for each vertex v, its outdegree $d(v)$ is not greater than n, where $d(v)$ is the number of edges starting from v.

(b) An n-functional digraph D is *total* if $d(v) = n$ for each vertex v.

The concept of n-functional digraph is quite natural. For example, such digraphs are obtained from unary partial algebras. Moreover, for any positive integer n, there are many graphs whose edges cannot be directed to form an n-functional digraph, e.g. each graph containing a vertex with at least $n+1$ loops. Therefore it is interesting to know when the edges of a graph can

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be directed to form an n-functional digraph. In the next paper [2] we will show, in particular, that any finite total n-functional digraph D is uniquely determined (up to the orientation of some cycles) in the class of all n-functional digraphs by its undirected graph.

The first aim of this paper is to prove the following result (for $n = 1$ this is an easy consequence of Ore's theorem).

THEOREM 2. *All the edges of a finite graph G can be directed to form an n-functional digraph*, *for some positive integer n*, *iff for any subgraph H*,

$$(*) \qquad m_{\mathrm{e}} \leq n m_{\mathrm{v}},$$

where m_{v} and m_{e} are the numbers of vertices and edges of H, *respectively.*

Proof. $\Rightarrow$ follows from the fact (see e.g. [3]) that for any finite digraph D,

$$\text{(Eq)} \qquad m_{\mathrm{e}} = \sum_{i=1}^{m_{\mathrm{v}}} d(v_i),$$

where $v_1, \ldots, v_{m_{\mathrm{v}}}$ are all its vertices, and m_{e} is the number of its edges.

$\Leftarrow$. We induct on the number of regular edges of G (an edge is *regular* if its endpoints are distinct). If G contains only loops, then G can be regarded as a digraph. Obviously $(*)$ implies that there are at most n loops at each vertex.

Assume that G has at least one regular edge f, and let u_1, u_2 be the endpoints of f. The graph obtained from G by omitting f also satisfies $(*)$. Thus, by the induction hypothesis, all edges different from f can be directed to form an n-functional digraph D'.

If the outdegree of u_1 (resp. u_2) in D' is less than n, then we direct f from u_1 to u_2 (resp. from u_2 to u_1). The digraph so obtained is n-functional, so we can assume

$$\text{(A)} \qquad d(u_1) = d(u_2) = n.$$

It is sufficient to show that the orientation of some edges of D' can be inverted in such a way that the new digraph is still n-functional, but one of these two outdegrees is less than n.

Take all (directed) chains in D' starting from u_1 or u_2. Next, let $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_l\}$ be the sets of vertices and of edges, respectively, of all these chains.

By (A), $u_1, u_2 \in V$. Thus V and $E \cup \{f\}$ form a subgraph of G. Hence,

$$l + 1 \leq nm,$$

since G satisfies $(*)$ and f does not belong to D'.

Now take the subdigraph K of D' consisting of V and E. Then

$$d^K(v_1) + \ldots + d^K(v_m) = l \leq nm - 1,$$

where $d^K(v_i)$ is the outdegree of v_i in the digraph K.

Since K is n-functional, this fact and (A) yield

(1) $$d^K(w) \leq n-1 \quad \text{for some } w \in V \setminus \{u_1, u_2\}.$$

Now we show

(2) $$d^{D'}(v) = d^K(v) \quad \text{for any } v \in V.$$

Take an edge e starting from v. If $v \in \{u_1, u_2\}$, then the one-edge chain (e) starts from u_1 or u_2. Thus e belongs to K. If $v \notin \{u_1, u_2\}$, then there is a chain going from $\{u_1, u_2\}$ to v. This chain together with e forms a new chain starting from u_1 or u_2. Obviously the new chain, and thus in particular e, belongs to K.

The above two facts (1) and (2) give

(3) $$d^{D'}(w) \leq n-1 \quad \text{for some } w \in V \setminus \{u_1, u_2\}.$$

Since w is neither u_1 nor u_2, there is a chain p going from $\{u_1, u_2\}$ to w. We can assume that this chain contains pairwise different vertices (in particular, pairwise different regular edges, too). Next, we can assume that p starts from u_1, since the second case is analogous.

Let D'' be the digraph obtained from D' by inverting the orientation of p (i.e. of all its edges). Then by (3), D'' is also n-functional. Further,

$$d^{D''}(u_1) = d^{D'}(u_1) - 1 = n - 1.$$

Thus we can add to D'' the edge f so that u_1 becomes its initial vertex and u_2 becomes its final vertex. This completes the proof of the induction step, and consequently, the proof of the second implication. ∎

With this result and the equation (Eq) we obtain the following characterization of total n-functional digraphs.

COROLLARY 3. *Let a finite graph G have m_{V} vertices and m_{E} edges. Then all the edges of G can be directed to form a total n-functional digraph iff*

$$m_{\mathrm{E}} = n m_{\mathrm{V}},$$

and each subgraph of G satisfies $(*)$ of Theorem 2.

Note also that $(*)$ in Theorem 2 can be replaced by a weaker condition.

COROLLARY 4. *All the edges of a finite graph G can be directed to form an n-functional digraph iff for any subset $W = \{v_1, \ldots, v_m\}$ of vertices, the number of edges with endpoints in W is not greater than nm.*

Proof. The implication $\Rightarrow$ follows from the equation (Eq). To prove the converse implication it is sufficient to observe that for any subgraph H, the number of edges of H is not greater than the number of edges of G with endpoints in H. ∎

Obviously the second condition in Corollary 3 can also be replaced by the right hand side of the equivalence in Corollary 4.

We illustrate our results by the following example. Take a positive integer n and a simple and complete graph G (i.e. G has no loops and there is exactly one edge between any two different vertices) with k vertices, where k is not less than $2n+2$. Then the number l of edges of G is $k(k-1)/2$. Hence, $l \geq k(2n+1)/2 > 2nk/2 = nk$. Thus by Theorem 2, the edges of G cannot be directed to form an n-functional digraph.

Now take a simple complete graph G with exactly $2n+2$ vertices. Then the edges of G cannot be directed to form an n-functional digraph, but the edges of each of its proper subgraphs can. Indeed, take a subgraph H with at most $2n+1$ vertices. Let k and l be the numbers of vertices and edges of H, respectively. Then $l \leq k(k-1)/2$, because H is a simple graph. Hence, $l \leq k(2n+1-1)/2 = nk$. Analogously, any subgraph K of H also satisfies such an inequality. Thus by Theorem 2, the edges of H can be directed to form an n-functional digraph.

Now, in the case of infinite graphs this difference between the graph and its subgraphs disappears.

THEOREM 5. *Let G be an infinite graph, and n a positive integer. Then the following conditions are equivalent*:

(a) *The edges of G can be directed to form an n-functional digraph.*

(b) *For any finite subgraph of G, its edges can be directed to form a finite n-functional digraph.*

(c) *For any finite set W of vertices, all the edges with endpoints in W can be directed to form a finite n-functional digraph.*

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Proof. The implications (a)⇒(b) and (b)⇒(c) are obvious.

(c)⇒(a). Let $\mathcal{I}$ be the family of all finite non-empty subsets of the vertex set of G. For $I \in \mathcal{I}$, let $F_I = \{J \in \mathcal{I} : I \subseteq J\}$. Then the family $\{F_I : I \in \mathcal{I}\}$ has the finite intersection property and therefore can be extended to an ultrafilter $\mathcal{U}$ on $\mathcal{P}(\mathcal{I})$.

For each $I \in \mathcal{I}$, let G_I be the subgraph of G spanned on I. Since G_I is finite, all edges of G_I can be directed to form an n-functional digraph D_I. It is easy to see that D_I can be regarded as a finite partial unary algebra with n functions. Next, since D_I is non-empty, it can be extended to a (total) unary algebra A_I with n (total) functions $f_1^I, \ldots, f_n^I$. Conversely, with any partial (total) unary algebra with n functions we can associate, in a natural way, a (total) n-functional digraph.

Given the family $\{A_I = \langle I, f_1^I, \ldots, f_n^I\rangle : I \in \mathcal{I}\}$ of unary algebras of the same type $(f_1, \ldots, f_n)$, we take the product $\mathbf{A} = \prod_{I\in\mathcal{I}} A_I$, and next the ultraproduct $\mathbf{A}/\mathcal{U}$. Obviously $\mathbf{A}$, and consequently $\mathbf{A}/\mathcal{U}$, is a unary algebra with n functions. Thus to complete the proof of the implication (c)⇒(a), it is sufficient to show that G can be embedded in $\mathbf{A}/\mathcal{U}$.

Recall that the canonical embedding $\varphi = (\varphi_I)_{I\in\mathcal{I}}$ of the vertex set of G into the universe $\left(\prod_{I\in\mathcal{I}} I\right)/\mathcal{U}$ of $\mathbf{A}/\mathcal{U}$ is the composition $\pi \circ \overline{\varphi}$, where π is the natural homomorphism from $\mathbf{A}$ onto $\mathbf{A}/\mathcal{U}$, and $\overline{\varphi} = (\overline{\varphi})_{I\in\mathcal{I}}$ is defined by

$$\overline{\varphi}_I(v) = \begin{cases} v & \text{if } v \in I, \\ x_I & \text{otherwise,} \end{cases}$$

where for each $I \in \mathcal{I}$, x_I is an arbitrary fixed vertex in I.

Let v and w be vertices of G. Let $e_1, \ldots, e_k$ be all the (undirected) edges of G between v and w (finitely many by (c)). Note that

$$U(v,w) = \{I \in \mathcal{I} : v, w \in I\} \in \mathcal{U}.$$

For each $I \in U(v,w)$, $e_1, \ldots, e_k$ are also edges of G_I between v and w. Hence, D_I, and thus also A_I, contains their directed versions. More precisely, there are two subsets F_{vw}^I, F_{wv}^I (not necessarily disjoint) of $\{f_1, \ldots, f_n\}$ such that $|F_{vw}^I| + |F_{wv}^I| = k$ and $f^I(v) = w$, $g^I(w) = v$ for any $f \in F_{vw}^I$, $g \in F_{wv}^I$.

In particular, to any $I \in U(v,w)$ we assign a pair $\langle F_{vw}^I, F_{wv}^I\rangle$ of subsets of $\{f_1, \ldots, f_n\}$. There are only finitely many such pairs. Thus this assignment divides $U(v,w)$ into pairwise disjoint subfamilies $U_1, \ldots, U_l$ (one such subfamily contains all the elements of $U(v,w)$ with the same pair of sets). Since $U_1 \cup \ldots \cup U_l = U(v,w) \in \mathcal{U}$, for some $1 \le i \le l$ we have

$$U = U_i \in \mathcal{U}.$$

Let $\langle F_{vw}, F_{wv}\rangle$ be the pair of sets corresponding to U. Then for $f \in F_{vw}$ and $g \in F_{wv}$,

$$U \subseteq \{I \in \mathcal{I} : v, w \in I \text{ and } f^I(v) = w\}$$

and

$$U \subseteq \{I \in \mathcal{I} : v, w \in I \text{ and } g^I(w) = v\}.$$

Thus these sets belong to $\mathcal{U}$. Hence, for each $f \in F_{vw}$ and $g \in F_{wv}$,

$$f^{\mathbf{A}}(\varphi(v)) \equiv_{\mathcal{U}} \varphi(w) \quad \text{and} \quad g^{\mathbf{A}}(\varphi(w)) \equiv_{\mathcal{U}} \varphi(v).$$

Summarizing, since v and w were arbitrarily chosen vertices of G, we embed G into the unary algebra $\mathbf{A}/\mathcal{U}$ with n functions, or equivalently, in the n-functional digraph. Now, transporting the orientation of edges from $\mathbf{A}/\mathcal{U}$, we get some orientation of edges of G, forming an n-functional digraph. ■

Using Theorems 2 and 5, and also Corollary 4, we immediately get

COROLLARY 6. *Let G be an infinite graph, and n a positive integer. Then the following conditions are equivalent*:

(a) *The edges of G can be directed to form an n-functional digraph.*

(b) *For any finite subgraph H, $m_{\mathrm{e}} \le n m_{\mathrm{v}}$, where m_{v} and m_{e} are the numbers of vertices and edges of H, respectively.*

(c) *For any finite set W of vertices, there are at most $n|W|$ edges with endpoints in W.*

Finally, we construct a graph G whose edges cannot be directed to form an $\aleph_0$-functional digraph (analogously to Definition 1, a digraph is said to be $\aleph_0$-*functional* if for each vertex v, the cardinality of the set of edges starting from v is not greater than $\aleph_0$). However, each subgraph with vertex set of cardinality less than the cardinality of G can be directed to form such a digraph. This shows that the assumption of the finiteness of n is essential in Theorem 5.

Take a set X of cardinality $\aleph_2$, and a simple complete graph G with X as vertex set.

Take any subgraph H of G such that the cardinality of the vertex set of H is less than $\aleph_2$. Then all the vertices of H can be arranged in an injective sequence $(v_\alpha)_{\alpha<\xi}$ of order type ξ, where $\xi = \aleph_0$ or $\xi = \aleph_1$. For any edge e, we take its endpoint with greater index to be the initial vertex of e, and, of course, the other to be the final vertex of e. Observe that the resulting digraph is $\aleph_0$-functional.

Now we show that the edges of G cannot be directed to form an $\aleph_0$-functional digraph. Assume otherwise, and let D be such an $\aleph_0$-functional digraph.

Take a subset Y_0 of X with cardinality $\aleph_1$. Define $Y_{m+1} = Y_m \cup \overline{Y}_m$, where $\overline{Y}_m$ is the set of target vertices of the edges in D that have the source in Y_m. Clearly $|Y_m| = \aleph_1$ for all m and consequently for the set $Y = Y_1 \cup Y_2 \cup \ldots$ we have $|Y| = \aleph_1$. Now, for any vertex u in the obviously non-empty set $X \setminus Y$, we know that the edge connecting u with an arbitrary vertex v in Y has to start at u, as otherwise the definition of Y would give $u \in Y$, a contradiction. Consequently $d^D(u) \ge |Y| = \aleph_1$.

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