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PRODUCT PRESERVING GAUGE BUNDLE FUNCTORS ON VECTOR BUNDLES

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Abstract. A complete description is given of all product preserving gauge bundle functors F on vector bundles in terms of pairs (A, V) consisting of a Weil algebra A and an A-module V with $\dim_{\mathbb{R}}(V) < \infty$. Some applications of this result are presented.

0. Let us recall the following definitions (see e.g. [4]).

Let $F : \mathcal{VB} \to \mathcal{FM}$ be a covariant functor from the category $\mathcal{VB}$ of all vector bundles and their vector bundle homomorphisms into the category $\mathcal{FM}$ of fibered manifolds and their fibered maps. Let $B_{\mathcal{VB}} : \mathcal{VB} \to \mathcal{M}f$ and $B_{\mathcal{FM}} \to \mathcal{M}f$ be the respective base functors.

A *gauge bundle functor* on $\mathcal{VB}$ is a functor F satisfying $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}}$ and the localization condition: for every inclusion of an open vector subbundle $i_{E|U} : E|U \to E$, $F(E|U)$ is the restriction $p_E^{-1}(U)$ of $p_E : FE \to B_{\mathcal{VB}}(E)$ over U and $Fi_{E|U}$ is the inclusion $p_E^{-1}(U) \to FE$.

Given two gauge bundle functors F_1, F_2 on $\mathcal{VB}$, by a *natural transformation* $\tau : F_1 \to F_2$ we shall mean a system of base preserving fibered maps $\tau_E : F_1E \to F_2E$ for every vector bundle E satisfying $F_2f \circ \tau_E = \tau_G \circ F_1f$ for every vector bundle homomorphism $f : E \to G$.

A gauge bundle functor F on $\mathcal{VB}$ is *product preserving* if for any product projections $E_1 \xleftarrow{\mathrm{pr}_1} E_1 \times E_2 \xrightarrow{\mathrm{pr}_2} E_2$ in the category $\mathcal{VB}$, $FE_1 \xleftarrow{F\mathrm{pr}_1} F(E_1 \times E_2) \xrightarrow{F\mathrm{pr}_2} FE_2$ are product projections in the category $\mathcal{FM}$. In other words, $F(E_1 \times E_2) = F(E_1) \times F(E_2)$ modulo $(F\mathrm{pr}_1, F\mathrm{pr}_2)$.

In this paper we prove that all product preserving gauge bundle functors F on $\mathcal{VB}$ are in bijection with the pairs (A, V) consisting of a Weil algebra A and an A-module V with $\dim_{\mathbb{R}}(V) < \infty$, and that the natural transformations between two product preserving gauge bundle functors on the category $\mathcal{VB}$ are in bijection with the morphisms between corresponding pairs.

Some applications of the above classification results are also presented.

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The product preserving and fiber product preserving bundle functors on some other categories on manifolds have been described by many authors [1]–[8].

All manifolds are assumed to be Hausdorff, finite-dimensional, without boundary and of class C^∞. All maps between manifolds are assumed to be of class C^∞.

1. Let $A = \mathbb{R} \oplus n_A$ be a Weil algebra and V be an A-module with $\dim_{\mathbb{R}}(V) < \infty$. We generalize the construction of bundles of infinitely near points [9].

Example 1. Given a vector bundle $E = (E \xrightarrow{p} M)$ let $T^{A,V}E = \{(\varphi, \psi) \mid \varphi \in \mathrm{Hom}(C^\infty_z(M), A),\ \psi \in \mathrm{Hom}_\varphi(C^{\infty,\,\mathrm{f.l.}}_z(E), V),\ z \in M\}$, where $\mathrm{Hom}(C^\infty_z(M), A)$ is the set of all algebra homomorphisms φ from the (unital) algebra $C^\infty_z(M) = \{\mathrm{germ}_z(g) \mid g : M \to \mathbb{R}\}$ into A and where $\mathrm{Hom}_\varphi(C^{\infty,\,\mathrm{f.l.}}_z(E), V)$ is the set of all module homomorphisms ψ over φ from the $C^\infty_z(M)$-module $C^{\infty,\,\mathrm{f.l.}}_z(E) = \{\mathrm{germ}_z(h) \mid h : E \to \mathbb{R}$ is fiber linear$\}$ into V. Then $T^{A,V}E$ is a fibered manifold over M. A local vector bundle trivialization $(x^1 \circ p, \ldots, x^m \circ p, y^1, \ldots, y^k) : E|U \cong \mathbb{R}^m \times \mathbb{R}^k$ on E induces a fiber bundle trivialization $(\widetilde{x}^1, \ldots, \widetilde{x}^m, \widetilde{y}^1, \ldots, \widetilde{y}^k) : T^{A,V}E|U \tilde{=} A^m \times V^n = \mathbb{R}^m \times n_A^m \times V^n$ by $\widetilde{x}^i(\varphi, \psi) = \varphi(\mathrm{germ}_z(x^i)) \in A$, $\widetilde{y}^j(\varphi, \psi) = \psi(\mathrm{germ}_z(y^j)) \in V$, $(\varphi, \psi) \in T^{A,V}_z E$, $z \in U$, $i = 1, \ldots, m$, $j = 1, \ldots, k$. Given another vector bundle $G = (G \xrightarrow{q} N)$ and a vector bundle homomorphism $f : E \to G$ over $\underline{f} : M \to N$ let $T^{A,V}f : T^{A,V}E \to T^{A,V}G$, $T^{A,V}f(\varphi, \psi) = (\varphi \circ \underline{f}^*_z, \psi \circ f^*_z)$, $(\varphi, \psi) \in T^{A,V}_z E$, $z \in M$, where $\underline{f}^*_z : C^\infty_{\underline{f}(z)}(N) \to C^\infty_z(M)$ and $f^*_z : C^{\infty,\,\mathrm{f.l.}}_{\underline{f}(z)}(G) \to C^{\infty,\,\mathrm{f.l.}}_z(E)$ are given by the pull-back with respect to $\underline{f}$ and f. Then $T^{A,V}f$ is a fibered map over $\underline{f}$, and $T^{A,V}$ is a product preserving gauge bundle functor on $\mathcal{VB}$.

2. Let F be a product preserving gauge bundle functor on $\mathcal{VB}$.

Example 2. (i) Let $A^F = (G^F\mathbb{R}, G^F(+), G^F(\cdot), G^F(0), G^F(1))$, where $G^F : \mathcal{M}f \to \mathcal{FM}$, $G^F M = F(M \xrightarrow{\mathrm{id}_M} M)$, $G^F f = Ff : G^F M \to G^F N$, and where $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the sum map, $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the multiplication map, $0 : \mathbb{R} \to \mathbb{R}$ is the zero and $1 : \mathbb{R} \to \mathbb{R}$ is the unity. Then A^F is a Weil algebra.

(ii) Let $V^F = (F(\mathbb{R} \to \mathrm{pt}), F(+), F(\cdot), F(0))$, where pt is the one point manifold, $\mathbb{R} \to \mathrm{pt}$ is the vector bundle, $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the sum map, which is a vector bundle homomorphism $(\mathbb{R} \to \mathrm{pt}) \times (\mathbb{R} \to \mathrm{pt}) \to (\mathbb{R} \to \mathrm{pt})$ over $\mathrm{pt} \times \mathrm{pt} \to \mathrm{pt}$, $0 : \mathbb{R} \to \mathbb{R}$ is the zero map, which is a vector bundle homomorphism $(\mathbb{R} \to \mathrm{pt}) \to (\mathbb{R} \to \mathrm{pt})$ over $\mathrm{pt} \to \mathrm{pt}$, and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the multiplication map, which is a vector bundle homomorphism $(\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R}) \times$

$(\mathbb{R} \to \mathrm{pt}) \to (\mathbb{R} \to \mathrm{pt})$ over $\mathbb{R} \times \mathrm{pt} \to \mathrm{pt}$. Then V^F is an A^F-module with $\dim_{\mathbb{R}}(V^F) < \infty$.

3. Let F be a product preserving gauge bundle functor on $\mathcal{VB}$ and (A^F, V^F) be the corresponding pair. Let T^{A^F,V^F} be the product preserving gauge bundle functor for (A^F, V^F). We prove $F \cong T^{A^F,V^F}$.

For every vector bundle $E = (E \xrightarrow{p} M)$ we construct a fibered map $\Theta_E : FE \to T^{A^F,V^F}E$ covering id_M as follows. If $y \in F_zE$, $z \in M$, we define $\varphi_y : C^\infty_z(M) \to A^F$, $\varphi_y(\mathrm{germ}_z(g)) = F(g \circ p)(y) \in A^F = F(\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R})$, $g : M \to \mathbb{R}$, where $g \circ p : E \to \mathbb{R}$ is considered as a vector bundle homomorphism $(E \xrightarrow{p} M) \to (\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R})$ over $g : M \to \mathbb{R}$. Then φ_y is an algebra homomorphism. If $y \in F_zE$, $z \in M$, we define $\psi_y : C^{\infty,\mathrm{f.l}}_z(E) \to V^F$, $\psi_y(\mathrm{germ}_z(f)) = F(f)(y)$, $f : E \to \mathbb{R}$ is fiber linear, where f is considered as a vector bundle map $(E \xrightarrow{p} M) \to (\mathbb{R} \to \mathrm{pt})$ over $M \to \mathrm{pt}$. Then ψ_y is a module homomorphism over φ_y. We put $\Theta_E(y) = (\varphi_y, \psi_y) \in T^{A^F,V^F}_z E$, $y \in F_zE$, $z \in M$.

PROPOSITION 1. *$\Theta : F \to T^{A^F,V^F}$ is a natural isomorphism.*

Proof. It is sufficient to show that Θ_E is a diffeomorphism for any vector bundle E. Applying vector bundle trivializations, we can assume that $E = \mathbb{R}^m \times \mathbb{R}^k$ is a trivial vector bundle over $\mathbb{R}^m$. Since F and T^{A^F,V^F} are product preserving and E is a (multi) product of $\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R}$ and $\mathbb{R} \to \mathrm{pt}$, we can assume that E is either $\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R}$ or $\mathbb{R} \to \mathrm{pt}$.

(I) $E = (\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R})$. Consider $G^F\mathbb{R} \xrightarrow{\Theta_E} T^{A^F,V^F}(\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R}) \xrightarrow{\widetilde{x}^1} A^F$, where $\widetilde{x}^1$ is induced by $x^1 = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ (see Example 1). This composition is the identity map $G^F\mathbb{R} = A^F$. Hence Θ_E is a diffeomorphism.

(II) $E = (\mathbb{R} \to \mathrm{pt})$. Consider $F(\mathbb{R} \to \mathrm{pt}) \xrightarrow{\Theta_E} T^{A^F,V^F}(\mathbb{R} \to \mathrm{pt}) \xrightarrow{\widetilde{y}^1} V^F$, where $\widetilde{y}^1$ is induced by $y^1 = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$. This composition is the identity map $F(\mathbb{R} \to \mathrm{pt}) = V^F$. Hence Θ_E is a diffeomorphism. ∎

4. Let (A, V) be a pair, where A is a Weil algebra and V is an A-module with $\dim_{\mathbb{R}}(V) < \infty$. Let $T^{A,V}$ be the corresponding gauge bundle functor on $\mathcal{VB}$. Let $(\widetilde{A}, \widetilde{V})$ be the pair corresponding to $T^{A,V}$.

PROPOSITION 2. $(A, V) \cong (\widetilde{A}, \widetilde{V})$.

Proof. Clearly, $\widetilde{A} = T^{A,V}(\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R})$ and $\widetilde{V} = T^{A,V}(\mathbb{R} \to \mathrm{pt})$. Let $\mathcal{O} = \widetilde{x}^1 : T^{A,V}(\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R}) \to A$ and $\Pi = \widetilde{y}^1 : T^{A,V}(\mathbb{R} \to \mathrm{pt}) \to V$, where $\widetilde{x}^1$ is induced by $x^1 = \mathrm{id}_{\mathbb{R}}$ and $\widetilde{y}^1$ is induced by $y^1 = \mathrm{id}_{\mathbb{R}}$ (see Example 1).

Then $\mathcal{O} : \widetilde{A} \to A$ is an algebra isomorphism and $\Pi : \widetilde{V} \to V$ is a module isomorphism over $\mathcal{O}$. ∎

5. Let (A_1, V_1) and (A_2, V_2) be pairs, where A_i is a Weil algebra and V_i is an A_i-module with $\dim_{\mathbb{R}}(V_i) < \infty$, $i = 1, 2$. Let (μ, ν) be a morphism from (A_1, V_1) into (A_2, V_2), i.e. $\mu : A_1 \to A_2$ is an algebra homomorphism and $\nu : V_1 \to V_2$ is a module homomorphism over μ.

EXAMPLE 3. Let $E \to M$ be a vector bundle. We define $\tau_E^{\mu,\nu} : T^{A_1,V_1}E \to T^{A_2,V_2}E$, $\tau_E^{\mu,\nu}(\varphi, \psi) = (\mu \circ \varphi, \nu \circ \psi)$, $(\varphi, \psi) \in T_z^{A_1,V_1}E$, $z \in M$. Then $\tau^{\mu,\nu} : T^{A_1,V_1} \to T^{A_2,V_2}$ is a natural transformation.

6. Let $\tau : F_1 \to F_2$ be a natural transformation between product preserving gauge bundle functors on $\mathcal{VB}$. Let (A^{F_1}, V^{F_1}) and (A^{F_2}, V^{F_2}) be the pairs corresponding to F_1 and F_2.

EXAMPLE 4. Let $(\mu^\tau, \nu^\tau) = (\tau_{\mathrm{id}_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}}, \tau_{\mathbb{R}\to\mathrm{pt}}) : (A^{F_1}, V^{F_1}) \to (A^{F_2}, V^{F_2})$. Then (μ^τ, ν^τ) is a morphism of pairs.

7. We are now in a position to prove the following theorem.

THEOREM 1. *The correspondence $F \mapsto (A^F, V^F)$ induces a bijective correspondence between the equivalence classes of product preserving gauge bundle functors F on $\mathcal{VB}$ and the equivalence classes of pairs (A, V) consisting of a Weil algebra A and an A-module V with $\dim_{\mathbb{R}}(V) < \infty$. The inverse correspondence is induced by the correspondence $(A, V) \mapsto T^{A,V}$.*

Proof. The correspondence $[F] \mapsto [(A^F, V^F)]$ is well defined. For, if $\tau : F_1 \to F_2$ is an isomorphism, then so is $(\mu^\tau, \nu^\tau) : (A^{F_1}, V^{F_1}) \to (A^{F_2}, V^{F_2})$.

The correspondence $[(A, V)] \mapsto [T^{A,V}]$ is well defined. For, if $(\mu, \nu) : (A_1, V_1) \to (A_2, V_2)$ is an isomorphism, then so is $\tau^{\mu,\nu} : T^{A_1,V_1} \to T^{A_2,V_2}$.

From Proposition 1 it follows that $[F] = [T^{A^F,V^F}]$. From Proposition 2 it follows that $[(A, V)] = [(A^F, V^F)]$ if $F = T^{A,V}$. ∎

8. Let F_1 and F_2 be two product preserving gauge bundle functors on $\mathcal{VB}$. Let (A^{F_1}, V^{F_1}) and (A^{F_2}, V^{F_2}) be the corresponding pairs.

PROPOSITION 3. *Let $(\mu, \nu) : (A^{F_1}, V^{F_1}) \to (A^{F_2}, V^{F_2})$ be a morphism. Let $\tau^{[\mu,\nu]} : F_1 \to F_2$ be a natural transformation given by the composition $F_1 \xrightarrow{\Theta} T^{A^{F_1},V^{F_1}} \xrightarrow{\tau^{\mu,\nu}} T^{A^{F_2},V^{F_2}} \xrightarrow{\Theta^{-1}} F_2$, where Θ is as in Proposition 1 and $\tau^{\mu,\nu}$ is described in Example 3. Then $\tau = \tau^{[\mu,\nu]}$ is the unique natural transformation $F_1 \to F_2$ such that $(\mu^\tau, \nu^\tau) = (\mu, \nu)$, where (μ^τ, ν^τ) is as in Example 4.*

Proof. First we prove the uniqueness part. Suppose $\overline{\tau} : F_1 \to F_2$ is another natural transformation such that $(\mu^{\overline{\tau}}, \nu^{\overline{\tau}}) = (\mu, \nu)$. Then $\overline{\tau}$ coincides with τ on the vector bundles $\mathbb{R} \xrightarrow{\mathrm{id}_{\mathbb{R}}} \mathbb{R}$ and $\mathbb{R} \to \mathrm{pt}$ because of the definition of (μ^τ, ν^τ). Hence $\overline{\tau} = \tau$ by the same argument as in the proof of Proposition 1.

The existence part follows from the easily verified equalities $\Theta^{-1}_{\mathbb{R}\to\mathrm{pt}} \circ \tau^{\mu,\nu}_{\mathbb{R}\to\mathrm{pt}} \circ \Theta_{\mathbb{R}\to\mathrm{pt}} = \nu$ and $\Theta^{-1}_{\mathrm{id}_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}} \circ \tau^{\mu,\nu}_{\mathrm{id}_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}} \circ \Theta_{\mathrm{id}_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}} = \mu$. ■

Now, the following theorem is clear.

THEOREM 2. *Let F_1 and F_2 be two product preserving gauge bundle functors on $\mathcal{VB}$. The correspondence $\tau \mapsto (\mu^\tau, \nu^\tau)$ is a bijection between the natural transformations $F_1 \to F_2$ and the morphisms $(A^{F_1}, V^{F_1}) \to (A^{F_2}, V^{F_2})$ between corresponding pairs. The inverse correspondence is $(\mu, \nu) \mapsto \tau^{[\mu,\nu]}$.*

9. As an application of Theorems 1 and 2 we describe all the so-called *excellent pairs*, i.e. pairs (F, π) where F is a product preserving gauge bundle functor on $\mathcal{VB}$ and $\pi : F \to \mathrm{id}_{\mathcal{VB}}$ is a natural epimorphism (i.e. π is a natural transformation such that $\pi_E : FE \to E$ is a surjective submersion for any vector bundle E).

Thanks to our previous considerations we have:

(a) Let (F, π) be an excellent pair. Then we have (A^F, V^F) and a morphism $(\mu^\pi, \nu^\pi) : (A^F, V^F) \to (A^{\mathrm{id}_{\mathcal{VB}}}, V^{\mathrm{id}_{\mathcal{VB}}}) = (\mathbb{R}, \mathbb{R})$. In other words, we have a triple $(A^{F,\pi}, V^{F,\pi}, \varrho^{F,\pi})$, where $A^{F,\pi} = A^F$, $V^{F,\pi} = V^F$ and $\varrho^{F,\pi} = \nu^\pi : V^{F,\pi} \to \mathbb{R}$. Of course, $A^{F,\pi}$ is a Weil algebra, $V^{F,\pi}$ is an A^F-module with $\dim_{\mathbb{R}}(V^F) < \infty$ and $\varrho^{F,\pi}$ is a non-zero module homomorphism over the algebra homomorphism $A^{F,\pi} \to \mathbb{R}$.

(b) Conversely, let (A, V, ϱ) be a triple, where A is a Weil algebra, V is an A-module with $\dim_{\mathbb{R}}(V) < \infty$ and $\varrho : V \to \mathbb{R}$ is a non-zero module homomorphism over the unique algebra homomorphism $\kappa : A \to \mathbb{R}$. Then $\tau^{\kappa,\varrho} : T^{A,V} \to T^{\mathbb{R},\mathbb{R}} \cong \mathrm{id}_{\mathcal{VB}}$ is a natural epimorphism. In other words, we have an excellent pair $(T^{A,V,\varrho}, \pi^{A,V,\varrho}) := (T^{A,V}, \Theta^{-1} \circ \tau^{\kappa,\varrho})$, where $\Theta : \mathrm{id}_{\mathcal{VB}} \to T^{\mathbb{R},\mathbb{R}}$.

(c) Let (F, π) be an excellent pair. Then $\Theta : F \to T^{A^F, V^F}$ is an isomorphism of the excellent pairs (F, π) and $(T^{A^{F,\pi}, V^{F,\pi}, \varrho^{F,\pi}}, \pi^{A^{F,\pi}, V^{F,\pi}, \varrho^{F,\pi}})$, i.e. we have $\pi^{A^{F,\pi}, V^{F,\pi}, \varrho^{F,\pi}} \circ \Theta = \pi$.

(d) Let (A, V, ϱ) be a triple as above. Let $(T^{A,V,\varrho}, \pi^{A,V,\varrho})$ be the corresponding excellent pair. Let $(\widetilde{A}, \widetilde{V}, \widetilde{\varrho})$ be the triple corresponding to $(T^{A,V,\varrho}, \pi^{A,V,\varrho})$. Then $(\mathcal{O}, \varPi) : (\widetilde{A}, \widetilde{V}) \to (A, V)$ is an isomorphism of the triples $(\widetilde{A}, \widetilde{V}, \widetilde{\varrho})$ and (A, V, ϱ), i.e. we have $\varrho \circ \varPi = \widetilde{\varrho}$.

(e) Let $(\mu, \nu) : (A_1, V_1, \varrho_1) \to (A_2, V_2, \varrho_2)$ be a morphism between triples, where A_i is a Weil algebra, V_i is a V_i-module with $\dim_{\mathbb{R}}(V_i) < \infty$ and $\varrho_i : V_i \to \mathbb{R}$ is a non-zero module homomorphism over the algebra homomorphism $A_i \to \mathbb{R}$, $i = 1, 2$. This means that $(\mu, \nu) : (A_1, V_1) \to (A_2, V_2)$

is a morphism between pairs such that $\varrho_2 \circ \nu = \varrho_1$. Then $\tau^{\mu,\nu} : T^{A_1,V_1} \to T^{A_2,V_2}$ is a morphism between the excellent pairs $(T^{A_1,V_1,\varrho_1}, \pi^{A_1,V_1,\varrho_1})$ and $(T^{A_2,V_2,\varrho_2}, \pi^{A_2,V_2,\varrho_2})$, i.e. we have $\pi^{A_2,V_2,\varrho_2} \circ \tau^{\mu,\nu} = \pi^{A_1,V_1,\varrho_1}$.

(f) Let $\tau : (F_1, \pi_1) \to (F_2, \pi_2)$ be a morphism between excellent pairs, i.e. $\tau : F_1 \to F_2$ is a natural transformation such that $\pi_2 \circ \tau = \pi_1$. Then $(\mu^\tau, \nu^\tau) : (A^{F_1}, V^{F_1}) \to (A^{F_2}, V^{F_2})$ is a morphism between the triples $(A^{F_1,\pi_1}, V^{F_1,\pi_1}, \varrho^{F_1,\pi_1})$ and $(A^{F_2,\pi_2}, V^{F_2,\pi_2}, \varrho^{F_2,\pi_2})$, i.e. $\varrho^{F_2,\pi_2} \circ \nu^\tau = \varrho^{F_1,\pi_1}$.

Thus we have the following theorem corresponding to Theorem 1.

THEOREM 1′. *The correspondence $(F, \pi) \mapsto (A^{F,\pi}, V^{F,\pi}, \varrho^{F,\pi})$ induces a bijection between the equivalence classes of excellent pairs (F, π) and the equivalence classes of triples (A, V, ϱ) consisting of a Weil algebra A, an A-module V with $\dim_{\mathbb{R}}(V) < \infty$ and a non-zero module homomorphism $\varrho : V \to \mathbb{R}$ over the algebra homomorphism $A \to \mathbb{R}$. The inverse bijection is induced by $(A, V, \varrho) \mapsto (T^{A,V,\varrho}, \pi^{A,V,\varrho})$.*

REMARK 1. Let $A = \mathbb{R} \oplus n_A$ be a Weil algebra and V be an A-module. If $\varrho : V \to \mathbb{R}$ is a module homomorphism over the algebra homomorphism $A \to \mathbb{R}$, then $\ker(\varrho) \supset n_A \cdot V$. Conversely, if $\varrho : V \to \mathbb{R}$ is a functional such that $\ker(\varrho) \supset n_A \cdot V$, then it is a module homomorphism over $A \to \mathbb{R}$.

(g) Let (F_1, π_1), (F_2, π_2) be excellent pairs. Let $(\mu, \nu) : (A^{F_1,\pi_1}, V^{F_1,\pi_1}, \varrho^{F_1,\pi_1}) \to (A^{F_2,\pi_2}, V^{F_2,\pi_2}, \varrho^{F_2,\pi_2})$ be a morphism between the corresponding triples. Then $\tau^{[\mu,\nu]} : F_1 \to F_2$ (see Proposition 3) is a morphism between the excellent pairs (F_1, π_1) and (F_2, π_2), i.e. $\pi_2 \circ \tau^{[\mu,\nu]} = \pi_1$.

Thus we have the following theorem corresponding to Theorem 2.

THEOREM 2′. *Let (F_1, π_1) and (F_2, π_2) be excellent pairs. The correspondence $\tau \mapsto (\mu^\tau, \nu^\tau)$ gives a bijection between the morphisms $(F_1, \pi_1) \to (F_2, \pi_2)$ between excellent pairs and the morphisms $(A^{F_1,\pi_1}, V^{F_1,\pi_1}, \varrho^{F_1,\pi_1}) \to (A^{F_2,\pi_2}, V^{F_2,\pi_2}, \varrho^{F_2,\pi_2})$ between the corresponding triples. The inverse bijection is $(\mu, \nu) \mapsto \tau^{[\mu,\nu]}$.*

10. As another application of Theorem 2 we solve the problem of when for a product preserving gauge bundle functor F there is an excellent pair (F, π).

COROLLARY 1. *Let F be a product preserving gauge bundle functor on $\mathcal{VB}$. Then there exists a natural epimorphism $F \to \mathrm{id}_{\mathcal{VB}}$ if and only if $V^F \neq \{0\}$.*

Proof. If $\pi : F \to \mathrm{id}_{\mathcal{VB}}$ is a natural epimorphism, then so is $(\mu^\pi, \nu^\pi) : (A^F, V^F) \to (\mathbb{R}, \mathbb{R})$. Hence, $V^F \neq \{0\}$.

Assume $V \neq \{0\}$. Then $n_A \cdot V \neq V$. (For, if $n_A \cdot V = V$, then $V = n_A \cdot V = n_A^2 \cdot V = \ldots = n_A^l \cdot V = 0$ for some l.) So there is a module epimorphism $\varrho : V \to \mathbb{R}$ over $A \to \mathbb{R}$. Next, we can apply Theorem 2. ■

11. As an application of Theorem 1′ we present two non-equivalent excellent pairs (F, π_1) and (F, π_2) for some product preserving gauge bundle functor F.

Example 5. Let $A = C_0^\infty(\mathbb{R}^2)/m^3$ be the Weil algebra where m is the maximal ideal in $C_0^\infty(\mathbb{R}^2)$. Let $t^i = [\mathrm{germ}_0(x^i)] \in A$ for $i = 1, 2$, where x^1, x^2 are the usual coordinates on $\mathbb{R}^2$. Then $1, t^1, t^2, (t^1)^2, (t^2)^2, t^1t^2$ form a basis (over $\mathbb{R}$) of A and $t^1, t^2, (t^1)^2, (t^2)^2, t^1t^2$ form a basis (over $\mathbb{R}$) of the maximal nilpotent ideal $n_A \subset A$. Define $V \subset A$ to be the vector subspace generated by $t^1, (t^1)^2, (t^2)^2, t^1t^2$. Then V is an ideal in A, and hence V is a module over A. Moreover, $n_A \cdot V$ is spanned by $t^1t^2, (t^1)^2$. Define two functionals $\varrho_1, \varrho_2 : V \to \mathbb{R}$ by $\varrho_1(t^1) = \varrho_1((t^1)^2) = \varrho_1(t^1t^2) = 0$, $\varrho_1((t^2)^2) = 1$, $\varrho_2((t^1)^2) = \varrho_2(t^1t^2) = \varrho_2((t^2)^2) = 0$ and $\varrho_2(t^1) = 1$. Then ϱ_1, ϱ_2 are module homomorphisms over the algebra homomorphism $A \to \mathbb{R}$ because $\ker(\varrho_i) \supset n_A \cdot V$ for $i = 1, 2$. The triples (A, V, ϱ_1) and (A, V, ϱ_2) are not equivalent. (For, suppose that there exist an algebra isomorphism $\mu : A \to A$ and a module isomorphism $\nu : V \to V$ over μ such that $\varrho_2 \circ \nu = \varrho_1$. We have $1 = \varrho_1((t^2)^2) = \varrho_2(\nu((t^2)^2))$. Then $\nu((t^2)^2) = t^1 + \alpha(t^1)^2 + \beta t^1t^2 + \gamma(t^2)^2$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Since $\mu^{-1}(t^1) \in N$, $\mu^{-1}(t^1) \cdot (t^2)^2 = 0$. Hence $0 = \nu(\mu^{-1}(t^1) \cdot (t^2)^2) = \mu(\mu^{-1}(t^1)) \cdot \nu((t^2)^2) = t^1 \cdot \nu((t^2)^2) = (t^1)^2$, a contradiction.) Then (by Theorem 1′) the corresponding pairs $(T^{A,V,\varrho_1}, \pi^{A,V,\varrho_1}) = (T^{A,V}, \pi^{A,V,\varrho_1})$ and $(T^{A,V,\varrho_2}, \pi^{A,V,\varrho_2}) = (T^{A,V}, \pi^{A,V,\varrho_2})$ are not equivalent.

12. As an application of Proposition 1 we have:

Corollary 2. *Let F be a product preserving gauge bundle functor on $\mathcal{VB}$. For every vector bundle $p : E \to M$ we have a canonical vector bundle stucture (and a canonical A^F-module bundle structure) on $Fp : FE \to FM$, where M is the vector bundle $\mathrm{id}_M : M \to M$ and $p : E \to M$ is the vector bundle map covering id_M. For every vector bundle map $f : E \to G$ over $\underline{f} : M \to N$ the map $Ff : FE \to FG$ is a vector bundle map (and an A^F-module bundle map) over $F\underline{f} : FM \to FN$.*

Proof. Using the isomorphism Θ from Proposition 1 we can assume that $F = T^{A,V}$, where A is a Weil algebra and V is an A-module with $\dim_{\mathbb{R}}(V) < \infty$. Now, the statements follow from Example 1. ■

13. Using Corollary 2 one can define the composition $F_2 \circ F_1$ of product preserving gauge bundle functors F_1 and F_2 on $\mathcal{VB}$.

EXAMPLE 6. Let $p : E \to M$ be a vector bundle. Then $F_1p : F_1E \to F_1M$ is also a vector bundle (Corollary 2). Applying F_2, we define a fibered manifold $F_2 \circ F_1(E) := F_2(F_1E \xrightarrow{F_1p} F_1M)$ over M, where the projection $F_2 \circ F_1(E) \to M$ is the composition $F_2 \circ F_1(E) \to F_1M \to M$ of projections for F_2 and F_1. Let $f : E \to G$ be a vector bundle homomorphism covering $\underline{f} : M \to N$. Then $F_1f : F_1E \to F_2E$ is a vector bundle homomorphism over $F_1\underline{f}$ (Corollary 2). We put $F_2 \circ F_1(f) := F_2(F_1f) : F_2 \circ F_1(E) \to F_2 \circ F_1(G)$ and get a fibered map covering $\underline{f}$. It follows that $F_2 \circ F_1$ is a product preserving gauge bundle functor on $\mathcal{VB}$.

14. We now compute the pair $(A^{F_2\circ F_1}, V^{F_2\circ F_1})$ corresponding to the composition $F_2 \circ F_1$ of product preserving gauge bundle functors F_1 and F_2 on $\mathcal{VB}$.

By tensoring A^{F_1} and A^{F_2} we obtain the Weil algebra $A^{F_1} \otimes_{\mathbb{R}} A^{F_2}$. By tensoring V^{F_1} and V^{F_2} we obtain the module $V^{F_1} \otimes_{\mathbb{R}} V^{F_2}$ over $A^{F_1} \otimes_{\mathbb{R}} A^{F_2}$.

PROPOSITION 4. $(A^{F_2\circ F_1}, V^{F_2\circ F_1}) \cong (A^{F_1} \otimes_{\mathbb{R}} A^{F_2}, V^{F_1} \otimes_{\mathbb{R}} V^{F_2})$.

Proof. We have to construct an algebra isomorphism $\widetilde{\mu} : A^{F_1} \otimes_{\mathbb{R}} A^{F_2} \to A^{F_2\circ F_1}$ and a module isomorphism $\widetilde{\nu} : V^{F_1} \otimes_{\mathbb{R}} V^{F_2} \to V^{F_2\circ F_1}$ over $\widetilde{\mu}$.

For any point $a \in A^{F_1}$ the map $i_a : \mathbb{R} \to A^{F_1}$, $i_a(t) = ta$, $t \in \mathbb{R}$, is a homomorphism between vector bundles $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ and $\mathrm{id}_{A^{F_1}} : A^{F_1} \to A^{F_1}$. Applying F_2, we obtain $F_2(i_a) : A^{F_2} \to A^{F_2\circ F_1}$. Define $\widetilde{\mu} : A^{F_1} \times A^{F_2} \to A^{F_2\circ F_1}$, $\widetilde{\mu}(a, b) = F_2(i_a)(b)$, $a \in A^{F_1}$, $b \in A^{F_2}$. Using the definitions of the algebra operations, one can show that $\widetilde{\mu}$ is $\mathbb{R}$-bilinear. Then (by the universal factorization property) we have a linear map $\widetilde{\mu} : A^{F_1} \otimes_{\mathbb{R}} A^{F_2} \to A^{F_2\circ F_1}$, $\widetilde{\mu}(a \otimes b) = F_2(i_a)(b)$, $a \in A^{F_1}$, $b \in A^{F_2}$. Considering bases (over $\mathbb{R}$) of A^{F_1} and A^{F_2} and using the product property for F_2, one can prove that $\widetilde{\mu}$ is an isomorphism. Using again the definitions of the algebra operations, one can show that $\widetilde{\mu}$ is an algebra isomorphism.

For any point $u \in V^{F_1}$ the map $i_u : \mathbb{R} \to V^{F_1}$, $i_u(t) = tu$, $t \in \mathbb{R}$, is a homomorphism between the vector bundles $\mathbb{R} \to \mathrm{pt}$ and $V^{F_1} \to \mathrm{pt}$. Applying F_2, we obtain $F_2(i_u) : V^{F_2} \to V^{F_2\circ F_1}$. Define $\widetilde{\nu} : V^{F_1} \times V^{F_2} \to V^{F_2\circ F_1}$, $\widetilde{\nu}(u, w) = F_2(i_u)(w)$, $u \in V^{F_1}$, $w \in V^{F_2}$. Similarly to $\widetilde{\mu}$, $\widetilde{\nu}$ is also $\mathbb{R}$-bilinear. Then we have a linear map $\widetilde{\nu} : V^{F_1} \otimes_{\mathbb{R}} V^{F_2} \to V^{F_2\circ F_1}$, $\widetilde{\nu}(u \otimes w) = F_2(i_u)(w)$, $u \in V^{F_1}$, $w \in V^{F_2}$. Similarly to $\widetilde{\mu}$, $\widetilde{\nu}$ is a linear isomorphism. Using the definitions of the module operations, one can show that $\widetilde{\nu}$ is a module isomorphism over $\widetilde{\mu}$. ∎

COROLLARY 3. $F_2 \circ F_1 \cong F_1 \circ F_2$.

Proof. The exchange isomorphism $(A^{F_1} \otimes_{\mathbb{R}} A^{F_2}, V^{F_1} \otimes_{\mathbb{R}} V^{F_2}) \cong (A^{F_2} \otimes_{\mathbb{R}} A^{F_1}, V^{F_2} \otimes_{\mathbb{R}} V^{F_1})$ induces the natural isomorphism $F_2 \circ F_1 \cong F_1 \circ F_2$. ∎

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