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THE NORM OF THE POLYNOMIAL TRUNCATION OPERATOR ON THE UNIT DISK AND ON $[-1,1]$

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Abstract. Let D and ∂D denote the open unit disk and the unit circle of the complex plane, respectively. We denote by $\mathcal{P}_n$ (resp. $\mathcal{P}_n^{\mathrm{c}}$) the set of all polynomials of degree at most n with real (resp. complex) coefficients. We define the truncation operators S_n for polynomials $P_n \in \mathcal{P}_n^{\mathrm{c}}$ of the form $P_n(z) := \sum_{j=0}^n a_j z^j$, $a_j \in \mathbb{C}$, by

$$S_n(P_n)(z) := \sum_{j=0}^n \widetilde{a}_j z^j, \qquad \widetilde{a}_j := \frac{a_j}{|a_j|} \min\{|a_j|, 1\}$$

(here $0/0$ is interpreted as 1). We define the norms of the truncation operators by

$$\|S_n\|_{\infty,\partial D}^{\mathrm{real}} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{z\in\partial D} |S_n(P_n)(z)|}{\max_{z\in\partial D} |P_n(z)|},$$
$$\|S_n\|_{\infty,\partial D}^{\mathrm{comp}} := \sup_{P_n \in \mathcal{P}_n^{\mathrm{c}}} \frac{\max_{z\in\partial D} |S_n(P_n)(z)|}{\max_{z\in\partial D} |P_n(z)|}.$$

Our main theorem establishes the right order of magnitude of the above norms: there is an absolute constant $c_1 > 0$ such that

$$c_1\sqrt{2n+1} \le \|S_n\|_{\infty,\partial D}^{\mathrm{real}} \le \|S_n\|_{\infty,\partial D}^{\mathrm{comp}} \le \sqrt{2n+1}.$$

This settles a question asked by S. Kwapień. Moreover, an analogous result in $L_p(\partial D)$ for $p \in [2,\infty]$ is established and the case when the unit circle ∂D is replaced by the interval $[-1,1]$ is studied.

1. New result. Let D and ∂D denote the open unit disk and the unit circle of the complex plane, respectively. We denote by $\mathcal{P}_n$ (resp. $\mathcal{P}_n^{\mathrm{c}}$) the set of all polynomials of degree at most n with real (resp. complex) coefficients. We define the truncation operators S_n for polynomials $P_n \in \mathcal{P}_n^{\mathrm{c}}$ of the form

$$P_n(z) := \sum_{j=0}^n a_j z^j, \qquad a_j \in \mathbb{C},$$

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by

$$S_n(P_n)(z) := \sum_{j=0}^{n} \widetilde{a}_j z^j, \qquad \widetilde{a}_j := \frac{a_j}{|a_j|} \min\{|a_j|, 1\} \tag{1.1}$$

(here 0/0 is interpreted as 1). In other words, we leave a coefficient a_j unchanged if $|a_j| < 1$, while we replace it by $a_j/|a_j|$ if $|a_j| \geq 1$. We define the norms of the truncation operators by

$$\|S_n\|^{\text{real}}_{\infty,\partial D} := \sup_{P_n \in \mathcal{P}_n} \frac{\max_{z\in\partial D} |S_n(P_n)(z)|}{\max_{z\in\partial D} |P_n(z)|},$$
$$\|S_n\|^{\text{comp}}_{\infty,\partial D} := \sup_{P_n \in \mathcal{P}^c_n} \frac{\max_{z\in\partial D} |S_n(P_n)(z)|}{\max_{z\in\partial D} |P_n(z)|}.$$

Our main theorem establishes the right order of magnitude of the above norms. This settles a question asked by S. Kwapień.

THEOREM 1.1. *There is an absolute constant $c_1 > 0$ such that*

$$c_1\sqrt{2n+1} \leq \|S_n\|^{\text{real}}_{\infty,\partial D} \leq \|S_n\|^{\text{comp}}_{\infty,\partial D} \leq \sqrt{2n+1}.$$

In fact, we are able to establish an $L_p(\partial D)$ analogue of this as follows. For $p \in (0,\infty)$, let

$$\|S_n\|^{\text{real}}_{p,\partial D} := \sup_{P_n \in \mathcal{P}_n} \frac{\|S_n(P_n)\|_{L_p(\partial D)}}{\|P_n\|_{L_p(\partial D)}},$$
$$\|S_n\|^{\text{comp}}_{p,\partial D} := \sup_{P_n \in \mathcal{P}^c_n} \frac{\|S_n(P_n)\|_{L_p(\partial D)}}{\|P_n\|_{L_p(\partial D)}}.$$

THEOREM 1.2. *There is an absolute constant $c_1 > 0$ such that*

$$c_1(2n+1)^{1/2-1/p} \leq \|S_n\|^{\text{real}}_{p,\partial D} \leq \|S_n\|^{\text{comp}}_{p,\partial D} \leq (2n+1)^{1/2-1/p}$$

for every $p \in [2,\infty)$.

Note that it remains open what is the right order of magnitude of $\|S_n\|^{\text{real}}_{p,\partial D}$ and $\|S_n\|^{\text{comp}}_{p,\partial D}$ when $0 < p < 2$. In particular, it would be interesting to see if $\|S_n\|^{\text{comp}}_{p,\partial D} \leq c$ is possible for any $0 < p < 2$ with an absolute constant c. We record the following observation in this direction, due to S. Kwapień.

THEOREM 1.3. *There is an absolute constant $c > 0$ such that*

$$\|S_n\|^{\text{real}}_{1,\partial D} \geq c\sqrt{\log n}.$$

If the unit circle ∂D is replaced by the interval $[-1,1]$, we get a completely different order of magnitude of the polynomial truncation projector. In this case the norms of S_n are defined as before with $[-1,1]$ in place of ∂D.

THEOREM 1.4. *We have*

$$2^{n/2-4} \le \|S_n\|^{\mathrm{real}}_{\infty,[-1,1]} \le \|S_n\|^{\mathrm{comp}}_{\infty,[-1,1]} \le \sqrt{2n+1}\cdot 8^{n/2}.$$

2. Lemmas. To prove the lower bound of Theorem 1.1 we need two lemmas. The first one is from [LSV].

LEMMA 2.1 (Lovász, Spencer, Vesztergombi). *Let $a_{j,k}$, $j = 1,\dots,n_1$, $k = 1,\dots,n_2$, be such that $|a_{j,k}| \le 1$. Let also $p_1,\dots,p_{n_2} \in [0,1]$. Then there are choices*

$$\varepsilon_k \in \{-p_k, 1-p_k\}, \qquad k = 1,\dots,n_2,$$

such that for all j,

$$\Big|\sum_{k=1}^{n_2} \varepsilon_k a_{j,k}\Big| \le C\sqrt{n_1}$$

with an absolute constant C.

Our second lemma is a direct consequence of the well known Bernstein inequality (see Theorem 1.1 on p. 97 of [DL]) and the Mean Value Theorem.

LEMMA 2.2. *Suppose Q_n is a polynomial of degree n (with complex coefficients) and*

$$\theta_n := \exp\left(\frac{2\pi}{14n}\right),$$

$$z_j := \exp(ij\theta_n), \qquad |Q_n(z_j)| \le M, \qquad j = 1,\dots,3n.$$

Then

$$\max_{z\in\partial D} |Q_n(z)| \le 2M.$$

The inequalities below (see Theorem 2.6 on p. 102 of [DL]) will be needed to prove the upper bound of Theorem 1.1.

LEMMA 2.3 (Nikol'skiĭ Inequality). *Let $0 < q \le p \le \infty$. If P_n is a polynomial of degree at most n with complex coefficients then*

$$\|P_n\|_{L_p(\partial D)} \le \left(\frac{2nr+1}{2\pi}\right)^{1/q-1/p} \|P_n\|_{L_q(\partial D)},$$

where $r = r(q)$ is the smallest integer not less than $q/4$.

The next lemma may be found in [Ri] or [Er].

LEMMA 2.4 (Erdős). *Suppose that $z_0 \in \mathbb{C}$ and $|z_0| \ge 1$. Then*

$$|P_n(z_0)| \le |T_{2n}(z_0)|^{1/2} \max_{x\in[-1,1]} |P_n(x)|, \qquad P_n \in \mathcal{P}^{\mathrm{c}}_n,$$

where $T_{2n} \in \mathcal{P}_{2n}$ defined by

$$T_{2n}(x) := \cos(2n \arccos x), \qquad x \in [-1,1],$$

is the Chebyshev polynomial of degree $2n$. *As a consequence*, *writing*

$$T_{2n}(z) = 2^{2n-1} \prod_{j=1}^{n} (z^2 - x_j^2), \qquad x_j \in (0,1),$$

we have

$$\max_{z \in \partial D} |P_n(z)| \le 8^{n/2} \max_{x \in [-1,1]} |P_n(x)|.$$

3. Proofs

Proof of Theorem 1.1. We apply Lemma 2.1 with $n_1 = 3n$, $n_2 = n$,

$$\theta_n := \exp(2\pi/(3n)), \qquad a_{j,k} := \exp(ijk\theta_n),$$

and $p_1 = \ldots = p_n = 1/3$; with the choices

$$\varepsilon_k \in \{-1/3, 2/3\}, \qquad k = 1, \ldots, n,$$

coming from Lemma 2.1, we define

$$Q_n(z) = 3 \sum_{j=1}^{n} \varepsilon_k z^k.$$

Then Q_n is a polynomial of degree n with each coefficient in $\{-1, 2\}$, and with the notation

$$z_j := \exp(ij\theta_n), \qquad j = 1, \ldots, 3n,$$

we have

$$|Q_n(z_j)| \le 3C\sqrt{3n}, \qquad j = 1, \ldots, 3n.$$

Hence Lemma 2.2 yields

$$\max_{z \in \partial D} |Q_n(z)| \le 12C\sqrt{n}. \tag{3.1}$$

In particular, if we denote by m the number of indices k for which $\varepsilon_k = 2/3$, then

$$|3m - n| = |2m - (n - m)| = |Q_n(1)| \le 12C\sqrt{n},$$

hence

$$|S_n(Q_n)(1)| = |m - (n - m)| = |2m - n| \ge n/3 - 8C\sqrt{n}. \tag{3.2}$$

Now (3.1) and (3.2) give the lower bound of the theorem.

To see the upper bound, observe that Lemma 2.3 implies

$$\max_{z \in \partial D} |S_n(P_n)(z)| \le \frac{\sqrt{2n+1}}{\sqrt{2\pi}} \|S_n(P_n)\|_{L_2(\partial D)} \le \frac{\sqrt{2n+1}}{\sqrt{2\pi}} \|P_n\|_{L_2(\partial D)}$$
$$\le \sqrt{2n+1} \max_{z \in \partial D} |P_n(z)|$$

for all polynomials P_n of degree at most n with complex coefficients. This proves the upper bound of the theorem. ∎

Proof of Theorem 1.2. Let $p \in [2, \infty)$. Using (3.2) and the Nikol'skiĭ-type inequality of Lemma 2.3, we obtain

$$\|S_n(Q_n)\|_{L_p(\partial D)} \geq c_1 n^{1-1/p} \tag{3.3}$$

with an absolute constant $c_1 > 0$. On the other hand, (3.1) implies

$$\|Q_n\|_{L_p(\partial D)} \leq c_2 n^{1/2} \tag{3.4}$$

with an absolute constant $c_2 > 0$, and the lower bound of the theorem follows.

To see the upper bound, observe that Lemma 2.3 implies

$$\begin{aligned}\|S_n(P_n)\|_{L_p(\partial D)} &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2-1/p} \|S_n(P_n)\|_{L_2(\partial D)} \\ &\leq \left(\frac{2n+1}{2\pi}\right)^{1/2-1/p} \|P_n\|_{L_2(\partial D)} \\ &\leq (2n+1)^{1/2-1/p} \|P_n\|_{L_p(\partial D)}\end{aligned}$$

for all polynomials P_n of degree at most n with complex coefficients. This proves the upper bound of the theorem. ∎

Proof of Theorem 1.3. Let $n = 2^{m+2} - 2$. Consider the polynomial

$$P_n(z) = 4z^{2^{m+1}-1} \prod_{k=0}^{m} \left(1 + \frac{z^{2^k} + z^{-2^k}}{2}\right).$$

Then, for $z \in \partial D$,

$$|P_n(z)| = 4 \prod_{k=0}^{m} \left(1 + \frac{z^{2^k} + z^{-2^k}}{2}\right),$$

and hence $\|P_n\|_{L_1(\partial D)} = 4$. Also,

$$P_n(z) - S_n(P_n)(z) = z^{2^{m+1}-1} \left(3 + \sum_{k=0}^{m} (z^{2^k} + z^{-2^k})\right).$$

Let

$$R_n(z) := 3 + \sum_{k=0}^{m} (z^{2^k} + z^{-2^k}).$$

Then

$$\|S_n(P_n)\|_{L_1(\partial D)} \geq \|S_n(P_n) - P_n\|_{L_1(\partial D)} - \|P_n\|_{L_1(\partial D)} = \|R_n\|_{L_1(\partial D)} - 4.$$

We will prove that $\|R_n\|_{L_1(\partial D)} \geq c\sqrt{m}$ for some absolute constant $c > 0$. It is easy to see that if $b, a_0, a_1, \ldots, a_m$ are complex numbers and

$$F(z) = b + \sum_{k=0}^{m} a_k (z^{2^k} + z^{-2^k}),$$

then

$$\|F\|_{L_4(\partial D)} \leq \sqrt[4]{3}\Big(|b|^2 + \sum_{k=0}^{m} |2a_k|^2\Big)^{1/2}.$$

Therefore

$$\|R_n\|_{L_4(\partial D)} \leq \sqrt[4]{3}\sqrt{9+4(m+1)}.$$

Moreover,

$$\|R_n\|_{L_2(\partial D)} = \sqrt{9+2(m+1)}.$$

By Hölder's inequality,

$$\|R_n\|_{L_4(\partial D)}^{2/3} \, \|R_n\|_{L_1(\partial D)}^{1/3} \geq \|R_n\|_{L_2(\partial D)}.$$

Hence we obtain

$$(\sqrt[4]{3}\sqrt{9+4(m+1)})^{2/3}\|R_n\|_{L_1(\partial D)}^{1/3} \geq \sqrt{9+2(m+1)},$$

and thus $\|R_n\|_{L_1(\partial D)} \geq c\sqrt{m}$. This gives

$$\frac{\|S_n(P_n)\|_{L_1(\partial D)}}{\|P_n\|_{L_1(\partial D)}} \geq c'\sqrt{m} \geq c''\sqrt{\log n}$$

with absolute constants $c' > 0$ and $c'' > 0$. ■

Proof of Theorem 1.4. First we prove the upper bound. Using Lemma 2.4 we obtain

$$\begin{aligned}
\max_{x\in[-1,1]} |S_n(P_n)(x)| &\leq \max_{z\in\partial D} |S_n(P_n)(z)| \\
&\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} \|S_n(P_n)\|_{L_2(\partial D)} \\
&\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} \|P_n\|_{L_2(\partial D)} \\
&\leq \left(\frac{2n+1}{2\pi}\right)^{1/2} 8^{n/2}\sqrt{2\pi} \max_{x\in[-1,1]} |P_n(x)|,
\end{aligned}$$

which proves the upper bound of the theorem.

Now we turn to the lower bound. We define $Q_n \in \mathcal{P}_{4n}$ by

$$Q_n(z) := z^{2n}(1-z^2)^n = z^{2n}\sum_{j=0}^{n} (-1)^j \binom{n}{j} z^{2j}.$$

Then

$$\max_{x\in[-1,1]} |Q_n(x)| = \left(\frac{1}{4}\right)^n. \tag{3.5}$$

Also,

$$S_n(Q_n)(z) = z^{2n} \sum_{j=0}^{n} (-1)^j z^{2j},$$

hence for every positive even n,

$$|S_n(Q_n)(1)| = 1. \tag{3.6}$$

Now we deduce the lower bound of the theorem by combining (3.5) and (3.6). ∎

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