COLLOQUIUM MATHEMATICUM

VOL. 90 2001 NO. 2

A VERSION OF THE LAW OF LARGE NUMBERS

BY

KATUSI FUKUYAMA (Kobe)

Abstract. By the method of Rio [10], for a locally square integrable periodic function f, we prove $(f(\mu_1^t x)+\ldots+f(\mu_n^t x))/n \to \int_0^1 f$ for almost every x and $t>0$.

1. Introduction. In this paper we are concerned with the law of large numbers

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(\lambda_k x)=\int_0^1 f(y)\,dy \quad \text{for a.e. } x, \tag{1}$$

where f is a function with period 1.

By the criterion of Weyl [12], (1) holds for any strictly increasing sequence $\{\lambda_k\}$ of real numbers satisfying $\lambda_{k+1}-\lambda_k\ge\delta>0$ and for all bounded Riemann integrable functions.

We are interested in extending this result to Lebesgue integrable functions. We now give a brief survey on studies in this direction.

Khintchin [4] conjectured that, in case $\lambda_k=k$, (1) is valid for all indicator functions of Borel sets.

In case $\lambda_k=2^k$, Raikov [7] proved (1) for all $f\in L^1$, where L^p denotes the class of locally p-integrable functions with period 1. Later, Riesz [8] pointed out that it is an example of ergodic theorem.

Kac, Salem and Zygmund [3] proved (1) assuming $\lambda_{k+1}/\lambda_k\ge q>1$, $f\in L^2$ and $\|f-s(n,f)\|_2=O((\log n)^{-\varepsilon})$ for some $\varepsilon>0$. Here $s(n,f)$ denotes nth sub-sum of the Fourier series of f and $\|\cdot\|_2$ the L^2-norm. Erdős [2] relaxed the condition to $\|f-s(n,f)\|_2=O((\log\log n)^{-1-\varepsilon})$. Moreover, he constructed a sequence with $\lambda_{k+1}/\lambda_k\ge q>1$ and a square integrable function for which (1) fails to hold. This example shows that (1) cannot be expected to be valid for all integrable functions and sequences.

Assuming $\sum_n\|f-s(n,f)\|_2^2/n<\infty$, Koksma [5] proved (1) for a sequence $\{\lambda_k\}$ of positive numbers with $\lambda_{k+1}-\lambda_k\ge\delta>0$.

2000 *Mathematics Subject Classification*: 60F15, 42C15.

Key words and phrases: law of large numbers, limit theorem, gap theorem.

After all these and other studies, Marstrand [6] proved the existence of a bounded measurable function for which (1) does not hold for $\lambda_k = k$, and at last disproved Khintchin's conjecture.

Bourgain [1] proved (1) for $\lambda_k = \theta^k$ and $f \in L^2$ when $\theta > 1$ is an algebraic number.

When $f \in L^1$ and $\lambda_k = \theta^k$, Rio [10] proved that (1) is valid for almost all $\theta > 1$. By randomizing θ, the sequence $\{f(\theta^k x)\}$ becomes tame and easy to analyze. This is the idea which Rio used to derive the above result. Note that the exceptional set in this and the following results depends on f, and we cannot conclude, for almost every $\theta > 1$, that (1) is valid for every $f \in L^1$.

The purpose of this paper is to show that the idea applies to a wide variety of sequences, not only to the special sequence $\lambda_k = \theta^k$. We actually have the following result.

THEOREM. *Suppose that a sequence of real numbers* $0 < \mu_1 < \mu_2 < \ldots$ *satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^\eta} < \infty \quad \text{for some } \eta > 0.$$

Put $\lambda_k = \mu_k^t$. *If* $f \in L^1$, *then* (1) *is valid for almost every* $t > \eta$. *If* $f \in L^2$, *then* (1) *is valid for almost every* $t > 0$.

From this theorem, we can derive the following result of Koksma's type.

COROLLARY. *If* $f \in L_1$, $\mu_{n+1} - \mu_n > \delta > 0$, *and* $\lambda_k = \mu_k^t$, *then* (1) *is valid for almost every* $t > 1$. *If* $f \in L^2$, *then* (1) *is valid for almost every* $t > 0$.

2. Proof. We use the following inequality proved by Rio [10]:

$$\left| \int_0^1 dx \int_b^{b+1} f(a^k x) f(a^{k+m} x)\, da \right| \le (5 + k \log b) b^{1-k-m/2} \|f\|_2^2$$

for $f \in L^2$, $b > 1$, $k \ge 1$ and $m > 0$. The original proof by Rio deals with the case when m and k are integers, but we can see that it also works if we do not assume this. Note that the right hand side can be dominated by $C_{b,\beta} \|f\|_2^2 b^{-(k+k+m)\beta}$, where $0 < \beta < 1/2$ is arbitrary and $C_{b,\beta}$ is a constant depending only on b and β.

By putting $\gamma(k) = \log \mu_k$, we have

$$r_{i,j} := \left| \int_0^1 dx \int_b^{b+1} f(a^{\gamma(i)} x) f(a^{\gamma(j)} x)\, da \right| \le C_{b,\beta} \|f\|_2^2 (\mu_i \mu_j)^{-\beta \log b}$$

for $i \neq j$. If $M > 0$, then

$$\int_0^1 f^2(Mx)\,dx \le \int_0^{[M+1]/M} f^2(Mx)\,dx = \frac{[M+1]}{M}\|f\|_2^2 \le \Big(1 + \frac{1}{M}\Big)\|f\|_2^2,$$

and hence

$$r_{i,i} \le \int_b^{b+1} (1 + a^{-\gamma(i)})\,da\,\|f\|_2^2 \le (1 + b^{-\gamma(i)})\|f\|_2^2 \le C'_{b,\mu_1}\|f\|_2^2,$$

where $C'_{b,\mu_1} = 1 + \mu_1^{-\log b}$. By applying these estimates for the integral of the square of $S_{n,l}f = f(a^{\gamma(n)}x) + \ldots + f(a^{\gamma(n+l-1)}x)$, we have

$$\begin{aligned}\int_0^1 dx \int_b^{b+1} (S_{n,l}f)^2\,da &\le C'_{b,\mu_1} l\|f\|_2^2 + \sum_{i,j\in[n,n+l),\, i\neq j} r_{i,j} \\ &\le C'_{b,\mu_1} l\|f\|_2^2 + C_{b,\beta}\|f\|_2^2\Big(\sum_{i=n}^{n+l-1} \mu_i^{-\beta\log b}\Big)^2 \\ &\le C'_{b,\mu_1} l\|f\|_2^2 + C_{b,\beta}\|f\|_2^2 l^{2/q}\Big(\sum_{i=1}^{\infty} \mu_i^{-p\beta\log b}\Big)^{2/p}\end{aligned}$$

by Hölder's inequality, where $p, q > 1$ and $1/p + 1/q = 1$.

Assume $f \in L^2$ and take $b > 1$ arbitrary. Take $p > 1$ and $\beta \in (0, 1/2)$ such that $p\beta\log b > \eta$. Then $\mu_i^{-p\beta\log b}$ is summable in i, and hence we have $\int_0^1 dx \int_b^{b+1} (S_{n,l}f)^2\,da \le C_1 l + C_2 l^{2/q} < C l^{2-\varepsilon}$ for some C_1, C_2, C, and $\varepsilon > 0$. By using the following law of large numbers for a quasi-orthogonal sequence (cf. e.g. Stout [11, Th. 3.7.3]), we have the law of large numbers on $[0,1] \times [b, b+1]$.

THEOREM A. *If* $E(X_n + \ldots + X_{n+l-1})^2 \le K l^{2-\varepsilon}$ *for some* $\varepsilon > 0$ *and* $K > 0$, *then* $(X_1 + \ldots + X_n)/n \to 0$ *almost surely.*

Noting $a^{\gamma(i)} = \mu_k^{\log a}$ and applying Fubini's theorem, we have the second conclusion of the Theorem.

Moreover, if we put $p = q = 2$, take $b > e^\eta$ arbitrary, and choose $\beta \in (0, 1/2)$ greater than $\eta/(2\log b)$, then we see that $\mu_i^{-p\beta\log b}$ is summable in i, and hence we have $\int_0^1 dx \int_b^{b+1} (S_{n,l}f)^2\,da < C\|f\|_2^2 l$. Following the argument of Rio [10], which applies the result from [9], we see that, if $f \in L^1$, the law of large numbers holds on $(b, b+1) \times [0,1]$. Therefore for a.e. $a > e^\eta$, the law holds. Thus for a.e. $t > \eta$, (1) holds for λ_k^t.

If $\mu_{n+1} - \mu_n \ge \delta > 1$, then $\sum 1/\mu_n^c < \infty$ for any $c > 1$. By applying the above result, if $f \in L^1$, the law of large numbers holds for almost every $t > c$. Since $c > 1$ is arbitrary, it holds for almost every $t > 1$.

REFERENCES

[1] J. Bourgain, *The Riesz–Raikov theorem for algebraic numbers*, in: Israel Math. Conf. Proc. 3, Weizmann Sci. Press, Jerusalem, 1990, 1–45.

[2] P. Erdős, *On the strong law of large numbers*, Trans. Amer. Math. Soc. 67 (1949), 51–56.

[3] M. Kac, R. Salem and A. Zygmund, *A gap theorem*, ibid. 63 (1948), 235–243.

[4] A. Khintchin, *Ein Satz über Kettenbruche mit arithmetischen Anwendungen*, Math. Z. 18 (1923), 289–306.

[5] J. F. Koksma, *An arithmetical property of some summable functions*, Indag. Math. 13 (1951), 354–367.

[6] J. M. Marstrand, *On Khinchin's conjecture about strong uniform distribution*, Proc. London Math. Soc. 21 (1970), 540–556.

[7] D. Raikov, *On some arithmetical properties of summable functions*, Rec. Math. Moscou 1 (1936), 377–384.

[8] F. Riesz, *Sur la théorie ergodique*, Comment. Math. Helv. 17 (1945), 221–239.

[9] E. Rio, *Vitesses de convergence dans la loi forte pour des suites dépendantes*, C. R. Acad. Sci. Paris Sér. I 320 (1995), 469–474.

[10] —, *Lois fortes des grands nombres presque sûres pour les sommes de Riesz–Raikov*, Probab. Theory Related Fields 118 (2000), 342–348.

[11] W. Stout, *Almost Sure Convergence*, Academic Press, New York, 1974.

[12] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. 77 (1916), 313–352.

Department of Mathematics
Kobe University
Rokko, Kobe, 657-8501 Japan
E-mail: fukuyama@math.kobe-u.ac.jp

Received 3 January 2001;
revised 25 May 2001 (4015)