COLLOQUIUM MATHEMATICUM

VOL. 90 2001 NO. 2

ON MINIMAL GENERIC SUBMANIFOLDS IMMERSED IN S^{2m+1}

BY

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Abstract. We give a pinching theorem for a compact minimal generic submanifold with flat normal connection immersed in an odd-dimensional sphere with standard Sasakian structure.

1. Introduction. Let M be an $(n+1)$-dimensional submanifold of the unit sphere S^{2m+1} with standard Sasakian structure (ϕ,ξ,η,g). *We assume that M is tangent to the structure vector field ξ.* If the normal space of M is mapped by ϕ into the tangent space of M at each point, that is, $\phi T_x(M)^\perp \subset T_x(M)$ for any point x of M, then M is called a *generic submanifold* of S^{2m+1}. Any hypersurface of S^{2m+1} is a generic submanifold. We denote by K_{ts} the sectional curvature of M spanned by e_t and e_s orthogonal to the structure vector ξ. The sectional curvature of a generic submanifold M spanned by ξ and e_t is always zero. So, we consider the sectional curvatures K_{ts} only.

In [1] we proved that a compact minimal hypersurface of S^{2m+1} with $K_{ts}+3g(Pe_t,e_s)^2 \geq 1/n$ is congruent to $S^{2m-1}(r_1)\times S^1(r_2)$, where P is defined by $\phi X = PX + FX$, PX and FX being the tangential and normal parts of ϕX. The purpose of the present paper is to prove that if the normal connection of a compact minimal generic submanifold M is flat and if $K_{ts}+3g(Pe_t,e_s)^2 \geq 1/n$, then M is a hypersurface and, consequently, it is congruent to $S^{2m-1}(r_1)\times S^1(r_2)$.

2. Preliminaries. Let S^{2m+1} be the $(2m+1)$-dimensional unit sphere and let (ϕ,ξ,η,g) be the standard Sasakian structure on S^{2m+1}. Then we have

$$\phi^2X = -X+\eta(X)\xi,\quad \phi\xi=0,\quad \eta(\xi)=1,\quad \eta(\phi X)=0,$$
$$g(\phi X,\phi Y) = g(X,Y)-\eta(X)\eta(Y),\quad \eta(X)=g(X,\xi),$$
$$\overline{\nabla}_X\xi = \phi X,\quad (\overline{\nabla}_X\phi)Y = -g(X,Y)\xi+\eta(Y)X$$

for any vector fields X and Y on S^{2m+1}, where $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Levi-Civita connection.

2000 *Mathematics Subject Classification*: Primary 53B25, 53C25, 53C40.

Let M be an $(n+1)$-dimensional submanifold of S^{2m+1}. *Throughout this paper, we assume that the submanifold M is tangent to the structure vector field ξ of S^{2m+1}.* We denote by the same g the Riemannian metric tensor field induced on M from that of S^{2m+1}. The operator of covariant differentiation with respect to the induced connection on M will be denoted by ∇. Then the Gauss and Weingarten formulas are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \quad \text{and} \quad \overline{\nabla}_X V = -A_V X + D_X V$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where D denotes the operator of covariant differentiation with respect to the linear connection induced in the normal bundle $T(M)^{\perp}$ of M. Both A and B are called the second fundamental forms of M, and are related by $g(B(X,Y),V) = g(A_V X, Y)$. For the second fundamental form A we define its covariant derivative $\nabla_X A$ by

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

If $\mathrm{Tr}\, A_V = 0$ for any vector field V normal to M, then M is said to be *minimal*, where Tr denotes the trace of an operator. If the second fundamental form of M vanishes, then M is said to be *totally geodesic*.

For any vector field X tangent to M, we put

$$\phi X = PX + FX,$$

where PX is the tangential part of ϕX and FX the normal part of ϕX. Then P is an endomorphism of the tangent bundle $T(M)$, and F is a normal bundle valued 1-form on the tangent bundle $T(M)$. Let R be the Riemannian curvature tensor of M. Then the Gauss equation is given by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(PY,Z)PX - g(PX,Z)PY \\ + 2g(X,PY)PZ + A_{B(Y,Z)}X - A_{B(X,Z)}Y. \end{aligned}$$

The Codazzi equation of M is given by

$$g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z) = 0$$

for any vector fields X, Y tangent to M and any vector field V normal to M. We now define the curvature tensor $R^{\perp}$ of the normal bundle of M by

$$R^{\perp}(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

Then we have the equation of Ricci

$$g(R^{\perp}(X,Y)U,V) = g([A_U, A_V]X, Y),$$

where $[A_U, A_V] = A_U A_V - A_V A_U$. If $R^{\perp} = 0$, the normal connection of M is said to be *flat*. The normal connection of M is flat if and only if $A_U A_V = A_V A_U$.

If $\phi T_x(M)^\perp \subset T_x(M)$ for any point x of M, then M is called a *generic submanifold* of S^{2m+1}. Let M be a generic submanifold of S^{2m+1}. For any vector field V normal to M we put

$$\phi V = tV,$$

where tV is a tangent vector and t is a tangent bundle valued 1-form on the normal bundle. Since ξ is tangent to M, for any vector field X tangent to M, we have

$$\overline{\nabla}_X \xi = \phi X = \nabla_X \xi + B(X, \xi),$$

from which it follows that

$$PX = \nabla_X \xi, \quad B(X, \xi) = FX, \quad A_V \xi = -tV.$$

Moreover, we obtain

$$\nabla_X tV = -PA_V X + tD_X V.$$

We also have

$$A_V tU = A_U tV.$$

We define the covariant derivatives of P, F and t by

$$(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y, \quad (\nabla_X F)Y = D_X(FY) - F\nabla_X Y,$$
$$(\nabla_X t)V = \nabla_X(tV) - tD_X V.$$

We then have

$$(\nabla_X P)Y = A_{FY} X + tB(X, Y), \quad (\nabla_X F)Y = -B(X, PY),$$
$$(\nabla_X t)V = -PA_V X.$$

3. Pinching theorem. Let M be an $(n+1)$-dimensional minimal generic submanifold of S^{2m+1}. We take an orthonormal basis $e_0 = \xi, e_1, \dots, e_n$ of the tangent bundle of M. We use the convention that the ranges of indices are

$$i, j, k = 0, 1, \dots, n; \quad r, s, t = 1, \dots, n.$$

In what follows we assume that the normal connection of M is flat. Then we can choose an orthonormal basis $v_1, \dots, v_p$ in the normal bundle of M such that $Dv_a = 0$, $a = 1, \dots, p$, $p = 2m - n$. Hence

$$\nabla_X tv_a = -PA_a X$$

for all a, where we have put $A_a = A_{v_a}$ to simplify the notation. We also have

$$\sum_i (\nabla_i P)e_i = -n\xi - \sum_b A_b t v_b, \qquad (\nabla_i t)v_a = -PA_a e_i,$$

where we have put $\nabla_i = \nabla_{e_i}$. From these equations we get

$$\begin{aligned}
-\operatorname{div}(PA_a t v_a) &= -\sum_i g(\nabla_{ei}(PA_a t v_a), e_i) \\
&= -\sum_i [g((\nabla_i P)A_a t v_a, e_i) \\
&\quad + g(P(\nabla_i A)_a t v_a, e_i) + g(PA_a(\nabla_i t)v_a, e_i)] \\
&= \sum_i [g(A_a t v_a, (\nabla_i P)e_i) + g((\nabla_i t)v_a, A_a P e_i)] \\
&= -ng(A_a t v_a, \xi) \\
&\quad - \sum_b g(A_a t v_a, A_b t v_a) - \sum_i g(PA_a e_i, A_a P e_i) \\
&= (n+1) - \operatorname{Tr} A_a^2 + \frac{1}{2}|[P, A_a]|^2.
\end{aligned}$$

Consequently, we obtain

$$\operatorname{div}(\nabla_{tv_a} t v_a) = (n+1) - \operatorname{Tr} A_a^2 + \frac{1}{2}|[P, A_a]|^2.$$

Generally we have (cf. [3])

$$\begin{aligned}
g(\nabla^2 A, A) &= \sum_{a,i,j} g((R(e_i, e_j)A_a)e_i, A_a e_j) \\
&= \sum_{a,i,j} g(R(e_i, e_j)A_a e_i, A_a e_j) - \sum_{a,i,j} g(A_a R(e_i, e_j)e_i, A_a e_j).
\end{aligned}$$

For a fixed a, we choose an orthonormal basis $e_0 = \xi, e_1, \dots, e_n$ such that

$$A_a e_t = \lambda_t e_t + u(e_t)\xi, \qquad t = 1, \dots, n,$$

where $u(e_t) = g(A_a e_t, \xi) = -g(e_t, t v_a)$. Then we have

$$\begin{aligned}
&\sum_{i,j} g(R(e_i, e_j)A_a e_i, A_a e_j) \\
&\qquad = 2\sum_t g(R(\xi, e_t)A_a \xi, A_a e_t) + \sum_{t,s} g(R(e_t, e_s)A_a e_t, A_a e_s) \\
&\qquad = -\sum_{t,s} \lambda_t \lambda_s K_{t,s} + 4\sum_{a,b} [g(A_a t v_b, A_a t v_b) - g(A_a e_t, A_a e_t)],
\end{aligned}$$

where $K_{t,s}$ denotes the sectional curvature spanned by e_t and e_s, and

$$
\begin{aligned}
-\sum_{i,j} & g(A_a R(e_i, e_j)e_i, A_a e_j) \\
&= -\sum_t g(R(\xi, e_t)\xi, A_a^2 e_t) \\
&\quad -\sum_{t,s} g(A_a R(e_t, e_s)e_t, A_a e_s) - \sum_t g(R(e_t, \xi)e_t, A_a^2 \xi) \\
&= \sum_{t,s} \lambda_t^2 K_{ts} + (2n+p) + \sum_{a,t} g(A_a e_t, A_a e_t) - 4\sum_{a,b} g(A_a t v_b, A_a t v_b),
\end{aligned}
$$

where we have used the fact that $A_a A_b = A_b A_a$ by the assumption $R^\perp = 0$. Consequently, we obtain

$$
\begin{aligned}
\sum_{i,j} g((R(e_i, e_j)A_a)e_i, A_a e_j) = & -\sum_{t,s} \lambda_t \lambda_s K_{t,s} + \sum_{t,s} \lambda_t^2 K_{ts} \\
& + (2n+p) - 3\sum_t g(A_a e_t, A_a e_t),
\end{aligned}
$$

from which

$$
\begin{aligned}
-\sum_{i,j} g((R(e_i, e_j)A_a)e_i, A_a e_j) = & -\frac{1}{2}\sum_{t,s} (\lambda_t - \lambda_s)^2 (K_{ts} + 3g(Pe_t, e_s)^2) \\
& + \frac{3}{2}|[P, A_a]|^2 + 3(\operatorname{Tr} A_a^2 - 1) - (2n+p).
\end{aligned}
$$

Suppose that

$$
K_{ts} + 3g(Pe_t, e_s)^2 \geq \frac{1}{n}.
$$

Then

$$
\begin{aligned}
-\sum_{i,j} & g((R(e_i, e_j)A_a)e_i, A_a e_j) - 3|[P, A_a]|^2 \\
&\leq -\operatorname{Tr} A_a^2 + 2 - \frac{3}{2}|[P, A_a]|^2 + 3\operatorname{Tr} A_a^2 - 3(n+1) + (n-p) \\
&= -2\operatorname{div}(\nabla_{tv_a} t v_a) - (p-1) - \frac{1}{2}|[P, A_a]|^2.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
-\frac{1}{2}\triangle\Big(\sum_a \operatorname{Tr} A_a^2\Big) + g(\nabla A, \nabla A) &= -g(\nabla^2 A, A) \\
&= -\sum_{a,i,j} g((R(e_i, e_j)A_a)e_i, A_a e_j)
\end{aligned}
$$

and

$$
g(\nabla A, \nabla A) = \sum_{a,t,s} g((\nabla_t A_a)e_s, e_r)^2 + 3\sum_a |[P, A_a]|^2.
$$

Hence

$$-\frac{1}{2}\triangle\Big(\sum_a \operatorname{Tr} A_a^2\Big) + \sum_{a,t,s} g((\nabla_t A_a)e_s, e_r)^2$$
$$= -\sum_{a,i,j} g((R(e_i, e_j)A_a)e_i, A_a e_j) - 3\sum_a |[P, A_a]|^2$$
$$\le -2\operatorname{div}\sum_a (\nabla_{tv_a} tv_a) - p(p-1) - \frac{1}{2}\sum_a |[P, A_a]|^2.$$

If M is compact, we have

$$\int_M \sum_{a,t,s} g((\nabla_t A_a)e_s, e_r)^2 * 1 \le -\int_M \Big[p(p-1) + \frac{1}{2}\sum_a |[P, A_a]|^2\Big] * 1.$$

This implies that $g((\nabla_t A_a)e_s, e_r) = 0$ for all t, s, r and a, and $PA_a = A_a P$ for all a. Thus we also have $p = 1$. Consequently, M is a hypersurface of S^{2m+1}. Combining this fact with a theorem of [1], we obtain the following

THEOREM. *Let M be an $(n+1)$-dimensional compact minimal generic submanifold of S^{2m+1} with flat normal connection. If the sectional curvature K of M satisfies*

$$K_{ts} + 3g(Pe_t, e_s)^2 \ge 1/n,$$

then M is a hypersurface of S^{m+1} and M is congruent to $S^{2m-1}(r_1) \times S^1(r_2)$, where

$$r_1 = \left(\frac{2m-1}{2m}\right)^{1/2}, \quad r_2 = \left(\frac{1}{2m}\right)^{1/2}.$$

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Received 24 October 2000;
revised 30 May 2001 (3985)