COLLOQUIUM MATHEMATICUM

VOL. 134 2014 NO. 2

THE ALMOST DAUGAVET PROPERTY AND TRANSLATION-INVARIANT SUBSPACES

BY

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Abstract. Let G be a metrizable, compact abelian group and let Λ be a subset of its dual group $\widehat{G}$. We show that $C_\Lambda(G)$ has the almost Daugavet property if and only if Λ is an infinite set, and that $L^1_\Lambda(G)$ has the almost Daugavet property if and only if Λ is not a $\Lambda(1)$ set.

1. Introduction. I. K. Daugavet [3] proved in 1963 that all compact operators T on $C[0,1]$ fulfill the norm identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|,$$

which has become known as the *Daugavet equation.* C. Foiaş and I. Singer [5] extended this result to all weakly compact operators on $C(K)$ where K is a compact space without isolated points. Shortly afterwards G. Ya. Lozanovskiĭ [13] showed that the Daugavet equation holds for all compact operators on $L^1[0,1]$, and J. R. Holub [9] extended this result to all weakly compact operators on $L^1(\mu)$ where μ is a σ-finite non-atomic measure. V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner [12] proved that the validity of the Daugavet equation for weakly compact operators already follows from the corresponding statement for operators of rank one. This result led to the following definition: A Banach space X is said to have the *Daugavet property* if every operator $T : X \to X$ of rank one satisfies the Daugavet equation. During the studies of ultraproducts [2] and quotients [10] of Banach spaces with the Daugavet property a weaker notion was introduced. Let X be a Banach space and let Y be a subspace of X^*. We say that X has the *Daugavet property with respect to* Y if the Daugavet equation holds true for every rank-one operator $T : X \to X$ of the form $T = y^* \otimes x$ where $x \in X$ and $y^* \in Y$. A Banach space X is called an *almost Daugavet space*, or a space with the *almost Daugavet property*, if it has the Daugavet property with respect to some norming subspace $Y \subset X^*$.

2010 *Mathematics Subject Classification*: Primary 46B04; Secondary 43A46.

Key words and phrases: Daugavet property, thickness, translation-invariant subspace, L-embedded space.

DOI: 10.4064/cm134-2-1

Recall that a subspace $Y \subset X^*$ is said to be *norming* if, for every $x \in X$,

$$\sup_{y^* \in S_Y} |y^*(x)| = \|x\|.$$

The space ℓ^1 is an example of an almost Daugavet space that does not have the Daugavet property.

Separable almost Daugavet spaces can be characterized using a kind of inner measure of non-compactness of the unit sphere. We call a set F an *inner ε-net for S_X* if $F \subset S_X$ and for every $x \in S_X$ there exists $y \in F$ with $\|x - y\| \le \varepsilon$. Then the *thickness* $T(X)$ of a Banach space X is defined by

$$T(X) = \inf\{\varepsilon > 0 : \text{there exists a finite inner } \varepsilon\text{-net for } S_X\}.$$

R. Whitley [18] introduced this parameter and showed that $1 \le T(X) \le 2$ if X is infinite-dimensional, in particular that $T(\ell^p) = 2^{1/p}$ for $1 \le p < \infty$ and that $T(C(K)) = 2$ if K has no isolated points. It was shown by V. Kadets, V. Shepelska, and D. Werner that a separable Banach space X is an almost Daugavet space if and only if $T(X) = 2$ [11, Theorem 1.1].

Almost Daugavet spaces contain ℓ^1 [11, Corollary 3.3] and are considered "big". It is therefore an interesting question which subspaces of an almost Daugavet space inherit the almost Daugavet property. The most general result in this direction is that a closed subspace Z of a separable almost Daugavet space X has the almost Daugavet property as well if the quotient space X/Z contains no copy of ℓ^1 [14, Theorem 2.5].

Let us consider an infinite, compact abelian group G with its Haar measure m. Since G has no isolated points and since m has no atoms, the spaces $C(G)$ and $L^1(G)$ have the Daugavet property. Using the group structure of G, we can translate functions that are defined on G and look at closed, translation-invariant subspaces of $C(G)$ or $L^1(G)$. These subspaces can be described via subsets Λ of the dual group $\widehat{G}$ and are of the form

$$C_\Lambda(G) = \{f \in C(G) : \operatorname{spec} f \subset \Lambda\}, \qquad L^1_\Lambda(G) = \{f \in L^1(G) : \operatorname{spec} f \subset \Lambda\},$$

where

$$\operatorname{spec} f = \{\gamma \in \widehat{G} : \hat{f}(\gamma) \neq 0\}.$$

We are going to characterize the sets $\Lambda \subset \widehat{G}$ such that $C_\Lambda(G)$ and $L^1_\Lambda(G)$ are of thickness two. If G is metrizable, this leads to a characterization of the translation-invariant subspaces of $C(G)$ and $L^1(G)$ which have the almost Daugavet property.

2. Translation-invariant subspaces of $C(G)$. Let us start with translation-invariant subspaces of $C(G)$. We will show that $T(C_\Lambda(G)) = 2$ if and only if Λ is an infinite subset of $\widehat{G}$. We will split the proof into various cases that depend on the structure of G. For this reason we recall some definitions and results concerning abelian groups and compact abelian groups.

Let G be an abelian group with identity element e_G. A subset E of G is said to be *independent* if $x_1^{k_1} \cdots x_n^{k_n} = e_G$ implies $x_1^{k_1} = \cdots = x_n^{k_n} = e_G$ for every choice of distinct points $x_1, \ldots, x_n \in E$ and integers $k_1, \ldots, k_n$. The *order* $o(x)$ of an element $x \in G$ is the smallest positive integer m such that $x^m = e_G$. If no such m exists, x is said to have infinite order.

Let $\mathbb{T}$ be the *circle group*, i.e., the multiplicative group of all complex numbers with absolute value one. If G is a compact abelian group, we denote by $\mathbf{1}_G$ the identity element of $\widehat{G}$, which coincides with the function identically equal to one. Linear combinations of elements of $\widehat{G}$ are called *trigonometric polynomials*, and for every $\Lambda \subset \widehat{G}$, the space $T_\Lambda(G) = \operatorname{lin} \Lambda$ of trigonometric polynomials with spectrum contained in Λ is dense in $C_\Lambda(G)$.

Let H be a closed subgroup of G. The *annihilator* of H is defined by

$$H^\perp = \{\gamma \in \widehat{G} : \gamma(x) = 1 \text{ for all } x \in H\}$$

and is therefore a closed subgroup of $\widehat{G}$. We remark that $\widehat{H} = \widehat{G}/H^\perp$ and $\widehat{G/H} = H^\perp$ [17, Theorem 2.1.2].

If $(G_j)_{j\in J}$ is a family of abelian groups, we define their *direct product* (or their *complete direct sum*) by

$$\prod_{j\in J} G_j = \Big\{ f : J \to \bigcup_{j\in J} G_j : f(j) \in G_j \Big\}$$

and define the group operation coordinatewise. Their *direct sum* is the subgroup

$$\bigoplus_{j\in J} G_j = \Big\{ f \in \prod_{j\in J} G_j : f(j) = e_{G_j} \text{ for all but finitely many } j \in J \Big\}.$$

If all G_j coincide with G, we write G^J or $G^{(J)}$ for the direct product or the direct sum. We denote by p_{G_j} the projection from $\prod_{j\in J} G_j$ onto G_j. If we consider products of the form $\mathbb{Z}^{\mathbb{N}}$ or $\mathbb{Z}^n$, we denote by $p_1, p_2, \ldots$ the corresponding projections onto $\mathbb{Z}$. If all G_j are compact, then $\prod_{j\in J} G_j$ is a compact abelian group as well and its dual group is given by $\bigoplus_{j\in J} \widehat{G_j}$ [17, Theorem 2.2.3].

Proposition 2.1. *Let A be a compact abelian group, set $G = \mathbb{T} \oplus A$, and let Λ be a subset of $\widehat{G} = \mathbb{Z} \oplus \widehat{A}$. If $p_{\mathbb{Z}}[\Lambda]$ is infinite, then $T(C_\Lambda(G)) = 2$.*

Proof. Fix $f_1, \ldots, f_n \in S_{C_\Lambda(G)}$ and $\varepsilon > 0$. We have to find $g \in S_{C_\Lambda(G)}$ with $\|f_k + g\|_\infty \geq 2 - \varepsilon$ for $k = 1, \ldots, n$.

Every f_k is uniformly continuous and therefore there exists $\delta > 0$ such that, for $k = 1, \ldots, n$ and all $a \in A$,

$$|\varphi - \vartheta| \leq \delta \;\Rightarrow\; |f_k(\mathrm{e}^{\mathrm{i}\varphi}, a) - f_k(\mathrm{e}^{\mathrm{i}\vartheta}, a)| \leq \varepsilon.$$

Since $p_{\mathbb{Z}}[\Lambda]$ contains infinitely many elements, we can pick $s \in p_{\mathbb{Z}}[\Lambda]$ with $|s|2\delta \geq 2\pi$. By our choice of s, for all $\vartheta \in [0, 2\pi]$ we get

$$\{\mathrm{e}^{\mathrm{i}s\varphi} : |\varphi - \vartheta| \leq \delta\} = \{\mathrm{e}^{\mathrm{i}\varphi} : |\varphi - \vartheta| \leq |s|\delta\} = \mathbb{T}. \tag{2.1}$$

Choose $g \in \Lambda$ with $p_{\mathbb{Z}}(g) = s$ and fix $k \in \{1, \ldots, n\}$. Since $f_k \in S_{C_\Lambda(G)}$, there exists $(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)}) \in G$ with

$$|f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)})| = 1.$$

By (2.1), we can pick $\varphi^{(k)} \in \mathbb{R}$ with

$$|\varphi^{(k)} - \vartheta^{(k)}| \leq \delta \quad \text{and} \quad \mathrm{e}^{\mathrm{i}s\varphi^{(k)}} = \frac{f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)})}{g(1, a^{(k)})}.$$

Note that the right-hand side of the last equation has absolute value one because g is a character of G. Consequently,

$$g(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, a^{(k)}) = g(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, e_A)g(1, a^{(k)}) = e^{\mathrm{i}s\varphi^{(k)}}g(1, a^{(k)}) = f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)}).$$

Finally,

$$\begin{aligned}\|f_k + g\|_\infty &\geq |f_k(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, a^{(k)}) + g(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, a^{(k)})| \\ &\geq 2|f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)})| - |f_k(\mathrm{e}^{\mathrm{i}\varphi^{(k)}}, a^{(k)}) - f_k(\mathrm{e}^{\mathrm{i}\vartheta^{(k)}}, a^{(k)})| \geq 2 - \varepsilon. \ \blacksquare\end{aligned}$$

PROPOSITION 2.2. *Let A be a compact abelian group, set $G = \mathbb{T}^{\mathbb{N}} \oplus A$, and let Λ be a subset of $\widehat{G} = \mathbb{Z}^{(\mathbb{N})} \oplus \widehat{A}$. If we find arbitrarily large $l \in \mathbb{N}$ with $p_l[\Lambda] \neq \{0\}$, then $T(C_\Lambda(G)) = 2$.*

Proof. Fix $f_1, \ldots, f_n \in S_{C_\Lambda(G)}$. Since $T_\Lambda(G)$ is dense in $C_\Lambda(G)$, we may assume without loss of generality that $f_1, \ldots, f_n$ are trigonometric polynomials. We are going to find $g \in S_{C_\Lambda(G)}$ with $\|f_k + g\|_\infty = 2$ for $k = 1, \ldots, n$.

Setting $\Delta = \bigcup_{k=1}^n \operatorname{spec} f_k$, we get a finite subset of Λ because every f_k is a trigonometric polynomial and therefore has a finite spectrum. Consequently, there exists $l_0 \in \mathbb{N}$ with $p_l[\Delta] = \{0\}$ for all $l > l_0$ and the evaluation of $f_1, \ldots, f_n$ at a point $(t_1, t_2, \ldots, a) \in G$ just depends on the coordinates $t_1, \ldots, t_{l_0}$ and a.

By assumption, we can find $l > l_0$ and $g \in \Lambda$ with $s = p_l(g) \neq 0$. Fix $k \in \{1, \ldots, n\}$. Since $f_k \in S_{C(G)}$, there exists $x^{(k)} = (t_1^{(k)}, t_2^{(k)}, \ldots, a^{(k)}) \in G$ with $|f_k(x^{(k)})| = 1$. Pick $u^{(k)} \in \mathbb{T}$ with

$$(u^{(k)})^s = \frac{f_k(x^{(k)})}{g(t_1^{(k)}, \ldots, t_{l-1}^{(k)}, 1, t_{l+1}^{(k)}, t_{l+2}^{(k)}, \ldots, a^{(k)})}.$$

Note that the right-hand side of the last equation has absolute value one because g is a character of G. With the same reasoning as at the end of the proof of Proposition 2.1 we get

$$g(t_1^{(k)},\dots,t_{l-1}^{(k)},u^{(k)},t_{l+1}^{(k)},t_{l+2}^{(k)},\dots,a^{(k)}) = f_k(x^{(k)}).$$

Finally,

$$\begin{aligned}\|f_k+g\|_\infty &\ge |(f_k+g)(t_1^{(k)},\dots,t_{l-1}^{(k)},u^{(k)},t_{l+1}^{(k)},t_{l+2}^{(k)},\dots,a^{(k)})|\\ &= 2|f_k(x^{(k)})| = 2. \ \blacksquare\end{aligned}$$

LEMMA 2.3. *Let $\varepsilon > 0$ and $z_1,\dots,z_n \in \{z\in\mathbb{C} : |z|\le 1\}$ with*

$$\Big|\sum_{k=1}^n z_k\Big| \ge n(1-\varepsilon).$$

Then

$$|z_k| \ge 1-n\varepsilon \quad \textit{and} \quad |z_k - z_l| \le 2n\sqrt{\varepsilon}$$

for $k,l = 1,\dots,n$.

Proof. The first assertion is an easy consequence of the triangle inequality. For fixed $k,l\in\{1,\dots,n\}$ we have

$$\begin{aligned}\operatorname{Re} z_k\overline{z_l} &= \operatorname{Re}\sum_{s,t=1}^n z_s\overline{z_t} - \operatorname{Re}\sum_{\substack{s,t=1\\(s,t)\ne(k,l)}}^n z_s\overline{z_t} = \Big|\sum_{k=1}^n z_k\Big|^2 - \operatorname{Re}\sum_{\substack{s,t=1\\(s,t)\ne(k,l)}}^n z_s\overline{z_t}\\ &\ge n^2(1-\varepsilon)^2 - (n^2-1) = 1 - 2n^2\varepsilon + n^2\varepsilon^2 \ge 1-2n^2\varepsilon.\end{aligned}$$

Using this inequality, we get

$$|z_k - z_l|^2 = |z_k|^2 + |z_l|^2 - 2\operatorname{Re}(z_k\overline{z_l}) \le 2 - 2(1-2n^2\varepsilon) = 4n^2\varepsilon. \ \blacksquare$$

The following lemma shows that if we are given n subsets of the unit circle that do not meet a circular sector with central angle larger than $2\pi/n$, then we can rotate these n subsets such that their intersection becomes empty.

LEMMA 2.4. *Let $W_1,\dots,W_n \subset \{z\in\mathbb{C} : |z|\le 1\}$. Suppose that for every $k\in\{1,\dots,n\}$ there exist $\varphi_k\in[0,2\pi]$ and $\vartheta_k\in[2\pi/n,2\pi]$ with*

$$W_k \cap \{r\mathrm{e}^{\mathrm{i}\alpha} : r\in[0,1],\ \alpha\in[\varphi_k,\varphi_k+\vartheta_k]\} = \emptyset.$$

Then there exist $t_1,\dots,t_n\in\mathbb{T}$ with

$$\bigcap_{k=1}^n t_kW_k = \emptyset.$$

Proof. Setting for $k=1,\dots,n$ (with $\vartheta_0 = 0$)

$$t_k = \mathrm{e}^{\mathrm{i}\sum_{l=0}^{k-1}\vartheta_l}\mathrm{e}^{-\mathrm{i}\varphi_k},$$

we get

$$t_kW_k \cap \Big\{r\mathrm{e}^{\mathrm{i}\alpha} : r\in[0,1],\ \alpha\in\Big[\sum_{l=0}^{k-1}\vartheta_l, \sum_{l=0}^{k}\vartheta_l\Big]\Big\} = \emptyset.$$

Fix $\alpha \in [0, 2\pi]$ and $r \in [0, 1]$. Since $\sum_{k=1}^{n} \vartheta_k \geq 2\pi$, there is $k \in \{1, \ldots, n\}$ with

$$\alpha \in \Big[\sum_{l=0}^{k-1} \vartheta_l, \sum_{l=0}^{k} \vartheta_l\Big].$$

Consequently, $re^{i\alpha}$ does not belong to $t_k W_k$ and $\bigcap_{k=1}^{n} t_k W_k = \emptyset$. ∎

LEMMA 2.5. *Let $\varepsilon, \delta > 0$, let $W \subset \{z \in \mathbb{C} : 1 - \delta \leq |z| \leq 1\}$, and set $W_\varepsilon = \{z \in \mathbb{C} :$ there exists $w \in W$ with $|w - z| \leq \varepsilon\}$. Suppose that there exists $\vartheta \in [0, 2\pi]$ such that for every $\varphi \in [0, 2\pi]$,*

$$W_\varepsilon \cap \{re^{i\alpha} : r \in [0, 1],\, \alpha \in [\varphi, \varphi + \vartheta]\} \neq \emptyset.$$

Then W is a $(2\varepsilon + \delta + \vartheta)$-net for $\mathbb{T}$.

Proof. Fix $e^{i\varphi} \in \mathbb{T}$. We have to find $w \in W$ with $|w - e^{i\varphi}| \leq 2\varepsilon + \delta + \vartheta$.

By assumption, there exist $se^{i\beta} \in W_\varepsilon \cap \{re^{i\alpha} : r \in [0, 1], \alpha \in [\varphi, \varphi + \vartheta]\}$ and $w \in W$ with $|w - se^{i\beta}| \leq \varepsilon$. It is easy to see that $s \geq 1 - \delta - \varepsilon$. Finally,

$$\begin{aligned} |w - e^{i\varphi}| &\leq |w - se^{i\beta}| + |se^{i\beta} - se^{i\varphi}| + |se^{i\varphi} - e^{i\varphi}| \\ &\leq \varepsilon + \vartheta + (\delta + \varepsilon) = 2\varepsilon + \delta + \vartheta. \; ∎ \end{aligned}$$

PROPOSITION 2.6. *Let A be a compact abelian group, let $(G_l)_{l\in\mathbb{N}}$ be a sequence of finite abelian groups, set $G = \prod_{l=1}^{\infty} G_l \oplus A$, and let Λ be an infinite subset of $\widehat{G} = \bigoplus_{l=1}^{\infty} \widehat{G_l} \oplus \widehat{A}$. If $p_{\widehat{A}}[\Lambda]$ is a finite set, then $T(C_\Lambda(G)) = 2$.*

Proof. The beginning is almost like in the proof of Proposition 2.2.

Fix $f_1, \ldots, f_n \in S_{C_\Lambda(G)}$ and $\varepsilon > 0$. Since $T_\Lambda(G)$ is dense in $C_\Lambda(G)$, we may assume without loss of generality that $f_1, \ldots, f_n$ are trigonometric polynomials. We have to find $g \in S_{C_\Lambda(G)}$ with $\|f_k + g\|_\infty \geq 2 - \varepsilon$ for $k = 1, \ldots, n$.

Setting $\Delta = \bigcup_{k=1}^{n} \operatorname{spec} f_k$, we get a finite subset of Λ because every f_k is a trigonometric polynomial and therefore has a finite spectrum. Consequently, there exists $l_0 \in \mathbb{N}$ with $p_{\widehat{G_l}}[\Delta] = \{\mathbf{1}_{G_l}\}$ for all $l > l_0$ and the evaluation of $f_1, \ldots, f_n$ at a point $(x_1, x_2, \ldots, a) \in G$ just depends on the coordinates $x_1, \ldots, x_{l_0}$ and a.

Since $\widehat{G_1}, \ldots, \widehat{G_{l_0}}$ and $p_{\widehat{A}}[\Lambda]$ are finite sets and Λ is an infinite set, there exist an infinite subset Λ_0 of Λ and elements $\gamma_1 \in \widehat{G_1}, \ldots, \gamma_{l_0} \in \widehat{G_{l_0}}, \gamma_A \in \widehat{A}$ with $p_{\widehat{G_l}}[\Lambda_0] = \{\gamma_l\}$ for $l = 1, \ldots, l_0$ and $p_{\widehat{A}}[\Lambda_0] = \{\gamma_A\}$. In other words, all elements of Λ_0 coincide in the first l_0 coordinates of $\bigoplus_{l=1}^{\infty} \widehat{G_l}$ and in the coordinate that corresponds to $\widehat{A}$. We can also assume that Λ_0 is a Sidon set because every infinite subset of $\widehat{G}$ contains an infinite Sidon set [17, Example 5.7.6(a)]. (Recall that Λ_0 is said to be a *Sidon set* if there exists a constant $C > 0$ such that $\sum_{\gamma\in\Lambda_0} |\hat{f}(\gamma)| \leq C\|f\|_\infty$ for all $f \in T_{\Lambda_0}(G)$.)

So if $\{\lambda_1, \lambda_2, \dots\}$ is an enumeration of Λ_0, then $(\lambda_s)_{s\in\mathbb{N}}$ is equivalent to the canonical basis of ℓ^1.

Set $\gamma = (\overline{\gamma_1}, \dots, \overline{\gamma_{l_0}}, \mathbf{1}_{G_{l_0+1}}, \mathbf{1}_{G_{l_0+2}}, \dots, \overline{\gamma_A}) \in \widehat{G}$. The sequence $(\gamma\lambda_s)_{s\in\mathbb{N}}$ is still equivalent to the canonical basis of ℓ^1 and for every character $\gamma\lambda_s$ we have $p_{\widehat{A}}(\gamma\lambda_s) = \mathbf{1}_A$ and $p_{\widehat{G}_l}(\gamma\lambda_s) = \mathbf{1}_{G_l}$ for $l = 1, \dots, l_0$. Thus the evaluation of $\gamma\lambda_1, \gamma\lambda_2, \dots$ at a point $(x_1, x_2, \dots, a) \in G$ does not depend on the coordinates $x_1, \dots, x_{l_0}$ and a.

Choose $n_0 \in \mathbb{N}$ with $2\pi/n_0 \le \varepsilon/3$ and $\delta \in (0,1)$ with $4n_0\sqrt{\delta} \le \varepsilon/3$. By James's ℓ^1 distortion theorem [1, Theorem 10.3.1], there is a normalized block basis sequence $(g_s)_{s\in\mathbb{N}}$ of $(\gamma\lambda_s)_{s\in\mathbb{N}}$ with

$$(1-\delta)\sum_{s=1}^{\infty}|z_s| \le \Big\|\sum_{s=1}^{\infty} z_s g_s\Big\|_\infty \le \sum_{s=1}^{\infty}|z_s|$$

for any sequence of complex numbers $(z_s)_{s\in\mathbb{N}}$ with finite support. It follows that for every n_0-tuple $(z_1, \dots, z_{n_0}) \in \mathbb{T}^{n_0}$ there is $x \in G$ with

$$\Big|\sum_{s=1}^{n_0} z_s g_s(x)\Big| \ge n_0(1-\delta).$$

Using Lemma 2.3, for $s, t = 1, \dots, n_0$ we have

$$|g_s(x)| \ge 1 - n_0\delta \quad \text{and} \quad |z_s g_s(x) - z_t g_t(x)| \le 2n_0\sqrt{\delta}.$$

If for $s = 1, \dots, n_0$, we set

$$W_s = g_s[G] \cap \{z \in \mathbb{C} : |z| \ge 1 - n_0\delta\},$$
$$\widetilde{W_s} = \{z \in \mathbb{C} : \text{there exists } w \in W_s \text{ with } |w - z| \le 2n_0\sqrt{\delta}\},$$

we conclude that for every tuple $(z_1, \dots, z_{n_0}) \in \mathbb{T}^{n_0}$,

$$\bigcap_{s=1}^{n_0} z_s\widetilde{W_s} \ne \emptyset.$$

By Lemma 2.4, there is $s_0 \in \{1, \dots, n_0\}$ such that for any $\varphi \in [0, 2\pi]$,

$$\widetilde{W_{s_0}} \cap \left\{r\mathrm{e}^{\mathrm{i}\alpha} : r \in [0,1], \alpha \in \left[\varphi, \varphi + \frac{2\pi}{n_0}\right]\right\} \ne \emptyset.$$

It follows from Lemma 2.5 and our choice of n_0 and δ that W_{s_0} is an ε-net for $\mathbb{T}$.

The function $g = \overline{\gamma} g_{s_0}$ is by construction a normalized trigonometric polynomial with spectrum contained in Λ. Fix $k \in \{1, \dots, n\}$. There exists $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, a^{(k)}) \in G$ with $|f_k(x^{(k)})| = 1$. By our choice of g_{s_0}, we can find $y^{(k)} = (y_1^{(k)}, y_2^{(k)}, \dots, b^{(k)}) \in G$ with

$$|\gamma(x^{(k)}) f_k(x^{(k)}) - g_{s_0}(y^{(k)})| \le \varepsilon.$$

Note that $\gamma(x^{(k)})f_k(x^{(k)}) \in \mathbb{T}$ since γ is a character. We therefore get

$$\begin{aligned}\|f_k+g\|_\infty &= \|\gamma f_k + g_{s_0}\|_\infty \\ &\geq |(\gamma f_k + g_{s_0})(x_1^{(k)},\ldots,x_{l_0}^{(k)},y_{l_0+1}^{(k)},y_{l_0+2}^{(k)},\ldots,a^{(k)})| \\ &= |\gamma(x^{(k)})f_k(x^{(k)}) + g_{s_0}(y^{(k)})| \\ &\geq 2|\gamma(x^{(k)})f_k(x^{(k)})| - |\gamma(x^{(k)})f_k(x^{(k)}) - g_{s_0}(y^{(k)})| \geq 2-\varepsilon. \quad\blacksquare\end{aligned}$$

LEMMA 2.7. *Let G be a compact abelian group and let $\gamma \in \widehat{G}$.*

(a) *If $o(\gamma) = m$, then $\gamma[G] = \{e^{2\pi i k/m} : k = 0,\ldots,m-1\}$, i.e., the image of G under γ is the set of mth roots of unity.*
(b) *If $o(\gamma) = \infty$, then $\gamma[G] = \mathbb{T}$.*

Proof. If $o(\gamma) = m$, we have $\gamma(x)^m = 1$ for every $x \in G$. Thus every element of $\gamma[G]$ is an mth root of unity. Setting $n = |\gamma[G]|$, it follows from Lagrange's theorem that $\gamma(x)^n = 1$ for every $x \in G$. Therefore $n \geq m$ and $\gamma[G]$ has to coincide with $\{e^{2\pi i k/m} : k = 0,\ldots,m-1\}$.

The set $\gamma[G]$ is a compact and therefore closed subgroup of $\mathbb{T}$. Since all proper closed subgroups of $\mathbb{T}$ are finite [15, Corollary 2.3], we have $\gamma[G] = \mathbb{T}$ if $o(\gamma) = \infty$. $\blacksquare$

THEOREM 2.8. *Let G be a compact abelian group and let Λ be an infinite subset of $\widehat{G}$. Then $T(C_\Lambda(G)) = 2$.*

Proof. We start like in the proofs of Propositions 2.2 and 2.6.

Fix $f_1,\ldots,f_n \in S_{C_\Lambda(G)}$ and $\varepsilon > 0$. Since $T_\Lambda(G)$ is dense in $C_\Lambda(G)$, we may assume without loss of generality that $f_1,\ldots,f_n$ are trigonometric polynomials. We have to find $g \in S_{C_\Lambda(G)}$ with $\|f_k + g\|_\infty \geq 2-\varepsilon$ for $k = 1,\ldots,n$.

Setting $\Delta = \bigcup_{k=1}^n \operatorname{spec} f_k$, we get a finite subset of Λ because every f_k is a trigonometric polynomial and therefore has a finite spectrum.

We can assume, by passing to a countably infinite subset if necessary, that Λ is countable. Hence $\Gamma = \langle\Lambda\rangle$, the group generated by Λ, is a countable subgroup of $\widehat{G}$.

Let M be a maximal independent subset of Γ and let $\Gamma_1 = \langle M\rangle$ be the subgroup of Γ that is generated by M. Defining inductively

$$\Gamma_l = \{\gamma \in \Gamma : \gamma^l \in \Gamma_{l-1}\}$$

for $l = 2,3,\ldots$, we get an increasing sequence $(\Gamma_l)_{l\in\mathbb{N}}$ of subgroups of Γ. Since M is a maximal independent subset of Γ, we see that $\bigcup_{l=1}^\infty \Gamma_l = \Gamma$. Furthermore, every Γ_l is a direct sum of cyclic groups [6, Corollary 18.4]. We distinguish two cases depending on whether or not there exists Γ_l that contains Δ and infinitely many elements of Λ.

First case: Suppose that there exists $l_0 \in \mathbb{N}$ such that $\Delta \subset \Gamma_{l_0}$ and $\Lambda_0 = \Lambda \cap \Gamma_{l_0}$ is an infinite set.

By our choice of Γ_{l_0}, the functions $f_1, \ldots, f_n$ and all characters $\gamma \in \Lambda_0$ are constant on the cosets of $G/(\Gamma_{l_0})^\perp$ and can therefore be considered as functions and characters on $G_0 = G/(\Gamma_{l_0})^\perp$. (To simplify notation, we continue to write $f_1, \ldots, f_n$.) Note that Γ_{l_0} is the dual group of G_0. Since Γ_{l_0} is a direct sum of cyclic groups, there exists a sequence $(\widehat{G_s})_{s\in\mathbb{N}}$ of finite abelian groups such that $\Gamma_{l_0} = \mathbb{Z}^{(\mathbb{N})} \oplus \bigoplus_{s=1}^\infty \widehat{G_s}$ or $\Gamma_{l_0} = \mathbb{Z}^{n_0} \oplus \bigoplus_{s=1}^\infty \widehat{G_s}$ for adequate $n_0 \in \mathbb{N}$. Hence $G_0 = \mathbb{T}^{\mathbb{N}} \oplus \prod_{s=1}^\infty G_s$ or $G_0 = \mathbb{T}^{n_0} \oplus \prod_{s=1}^\infty G_s$. Let $p_1, p_2, \ldots$ be the projections from Γ_{l_0} onto $\mathbb{Z}$.

If there exists $s_0 \in \mathbb{N}$ such that $p_{s_0}[\Lambda_0]$ contains infinitely many elements or there exist arbitrarily large $s \in \mathbb{N}$ with $p_s[\Lambda_0] \neq \{0\}$, then $T(C_{\Lambda_0}(G_0)) = 2$ by Proposition 2.1 or 2.2. Otherwise $p_{\mathbb{Z}^{(\mathbb{N})}}[\Lambda_0]$ (or $p_{\mathbb{Z}^{n_0}}[\Lambda_0]$) is a finite set and $T(C_{\Lambda_0}(G_0)) = 2$ by Proposition 2.6. So we can find $\tilde{g} \in S_{C_{\Lambda_0}(G_0)}$ with $\|f_k + \tilde{g}\|_\infty \geq 2 - \varepsilon$ for $k = 1, \ldots, n$. Setting $g = \tilde{g} \circ \pi$ where π is the canonical map from G onto $G_0 = G/(\Gamma_0)^\perp$, we get $\|f_k + g\|_\infty \geq 2 - \varepsilon$ for $k = 1, \ldots, n$.

Second case: Suppose that there exist arbitrarily large $l \in \mathbb{N}$ with $\Lambda \cap (\Gamma_l \setminus \Gamma_{l-1}) \neq \emptyset$.

Fix $l_0 \in \mathbb{N}$ with $\Delta \subset \Gamma_{l_0}$ and choose $l_1 \in \mathbb{N}$ with $l_1 > l_0^2$, $2\pi/l_1 \leq \varepsilon$ and $(\Gamma_{l_1} \setminus \Gamma_{l_1 - 1}) \cap \Lambda \neq \emptyset$. By our choice of Γ_{l_0}, the functions $f_1, \ldots, f_n$ are constant on the cosets of $G/(\Gamma_{l_0})^\perp$ and therefore

$$f_k(xy) = f_k(x) \quad (x \in G, y \in (\Gamma_{l_0})^\perp) \tag{2.2}$$

for $k = 1, \ldots, n$. Pick $g \in (\Gamma_l \setminus \Gamma_{l-1}) \cap \Lambda$ and denote by $\tilde{g}$ the restriction of g to $(\Gamma_{l_0})^\perp$. What can we say about the order of $\tilde{g}$? Since $(\Gamma_{l_0})^{\perp\perp} = \Gamma_{l_0}$, we see that, for every $m \in \mathbb{N}$, $\tilde{g}^m = \mathbf{1}_{(\Gamma_{l_0})^\perp}$ if and only if $g^m \in \Gamma_{l_0}$.

Suppose that $\tilde{g}^m = \mathbf{1}_{(\Gamma_{l_0})^\perp}$ for some $2 \leq m \leq l_0$. Then $\tilde{g}^{ml_0} = \mathbf{1}_{(\Gamma_{l_0})^\perp}$ as well and $g^{ml_0} \in \Gamma_{l_0}$. Consequently, $g \in \Gamma_{ml_0}$ because $g^{ml_0} \in \Gamma_{l_0} \subset \Gamma_{ml_0 - 1}$. But this contradicts our choice of g and l_1 because $l_1 > ml_0$. Assuming that $\tilde{g}^m = \mathbf{1}_{(\Gamma_{l_0})^\perp}$ for some $l_0 < m < l_1$ leads to the same contradiction. The order of $\tilde{g}$ is therefore at least l_1. By our choice of l_1 and by Lemma 2.7, we find that $\tilde{g}[(\Gamma_{l_0})^\perp]$ is an ε-net for $\mathbb{T}$.

Fix now $k \in \{1, \ldots, n\}$ and choose $x^{(k)} \in G$ with $|f_k(x^{(k)})| = 1$ and $y^{(k)} \in (\Gamma_{l_0})^\perp$ with

$$|f_k(x^{(k)}) - g(x^{(k)})\tilde{g}(y^{(k)})| \leq \varepsilon. \tag{2.3}$$

Note that g is a character and hence $g(x^{(k)}) \in \mathbb{T}$. Using (2.2) and (2.3), we get

$$\begin{aligned}\|f_k + g\|_\infty &\geq |f_k(x^{(k)}y^{(k)}) + g(x^{(k)}y^{(k)})| = |f_k(x^{(k)}) + g(x^{(k)})\tilde{g}(y^{(k)})| \\ &\geq 2|f_k(x^{(k)})| - |f_k(x^{(k)}) - g(x^{(k)})\tilde{g}(y^{(k)})| \geq 2 - \varepsilon. \ \blacksquare\end{aligned}$$

COROLLARY 2.9. *Let G be a metrizable, compact abelian group and let Λ be a subset of $\widehat{G}$. The space $C_\Lambda(G)$ has the almost Daugavet property if and only if Λ contains infinitely many elements.*

Proof. Every almost Daugavet space is infinite-dimensional and so the condition is necessary.

If G is a metrizable, compact abelian group, then $\widehat{G}$ is countable [17, Theorem 2.2.6] and $C(G)$ is separable. Since for separable Banach spaces the almost Daugavet property can be characterized via thickness [11, Theorem 1.1], it is sufficient to prove that $T(C_\Lambda(G)) = 2$. But this is given by Theorem 2.8. ∎

3. Subspaces of L-embedded spaces. To deal with translation-invariant subspaces of $L^1(G)$ we will consider a more general class of Banach spaces. A linear projection P on a Banach space X is called an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\| \quad (x \in X).$$

A closed subspace of X is called an *L-summand* if it is the range of an L-projection, and X is called *L-embedded* if X is an L-summand in X^{**}. Classical examples of L-embedded spaces are $L^1(\mu)$-spaces, preduals of von Neumann algebras, and the Hardy space H^1 [8, Example IV.1.1].

Using the principle of local reflexivity, it is easy to see that a non-reflexive, L-embedded space has thickness two. We will strengthen this and will show that every non-reflexive subspace of an L-embedded space has thickness two. Let us recall the following result from the theory of L-embedded spaces [8, claim in the proof of Theorem IV.2.7].

PROPOSITION 3.1. *Let X be an L-embedded space with $X^{**} = X \oplus_1 X_s$, let ε be a number with $0 < \varepsilon < 1/4$, and let $(y_l)_{l\in\mathbb{N}}$ be a sequence in X with*

$$(1-\varepsilon)\sum_{l=1}^{\infty}|a_l| \le \Big\|\sum_{l=1}^{\infty}a_l y_l\Big\| \le \sum_{l=1}^{\infty}|a_l|$$

for any sequence of scalars $(a_l)_{l\in\mathbb{N}}$ with finite support. Then there exists $x_s \in X_s$ such that

$$1 - 4\sqrt{\varepsilon} \le \|x_s\| \le 1$$

and for all $\delta > 0$, all $x_1^*, \dots, x_n^* \in X^*$, and all $l_0 \in \mathbb{N}$ there is $l \ge l_0$ with

$$|x_s(x_k^*) - x_k^*(y_l)| \le 3\sqrt{\varepsilon}\|x_k^*\| + \delta \quad (k = 1, \dots, n).$$

In other words, there is $x_s \in X_s$ which is "close" to a weak accumulation point of $(y_l)_{l\in\mathbb{N}}$.*

THEOREM 3.2. *Let X be an L-embedded space with $X^{**} = X \oplus_1 X_s$ and let Y be a closed subspace of X which is not reflexive. Then $T(Y) = 2$.*

Proof. Fix $x_1, \ldots, x_n \in S_Y$ and $\varepsilon > 0$. We have to find $y \in S_Y$ with $\|x_k + y\| \geq 2 - \varepsilon$ for $k = 1, \ldots, n$.

Choose $\delta > 0$ with $7\sqrt{\delta} + 2\delta \leq \varepsilon$. Every non-reflexive subspace of X contains a copy of ℓ^1 [8, Corollary IV.2.3] and by James's ℓ^1 distortion theorem [1, Theorem 10.3.1] there is a sequence $(y_l)_{l\in\mathbb{N}}$ in Y with

$$(1-\delta)\sum_{l=1}^{\infty}|a_l| \leq \Big\|\sum_{l=1}^{\infty} a_l y_l\Big\| \leq \sum_{l=1}^{\infty}|a_l|$$

for any sequence of scalars $(a_l)_{l\in\mathbb{N}}$ with finite support. Let $x_s \in X_s$ be "close" to a weak* accumulation point of $(y_l)_{l\in\mathbb{N}}$ as in Proposition 3.1. Since $X^{**} = X \oplus_1 X_s$, for $k = 1, \ldots, n$ we have

$$\|x_k + x_s\| = \|x_k\| + \|x_s\| \geq 2 - 4\sqrt{\delta}.$$

Thus there exist functionals $x_1^*, \ldots, x_n^* \in S_{X^*}$ with

$$|x_k^*(x_k) + x_s(x_k^*)| \geq 2 - 4\sqrt{\delta} - \delta$$

and $l \in \mathbb{N}$ with

$$|x_s(x_k^*) - x_k^*(y_l)| \leq 3\sqrt{\delta} + \delta$$

for $k = 1, \ldots, n$.

Fix $k \in \{1, \ldots, n\}$. The last two inequalities lead to

$$\begin{aligned}\|x_k + y_l\| &\geq |x_k^*(x_k) + x_k^*(y_l)| \geq |x_k^*(x_k) + x_s(x_k^*)| - |x_s(x_k^*) - x_k^*(y_l)| \\ &\geq (2 - 4\sqrt{\delta} - \delta) - (3\sqrt{\delta} + \delta) \geq 2 - \varepsilon. \ \blacksquare\end{aligned}$$

Corollary 3.3. *Let X be an L-embedded space and let Y be a separable, closed subspace of X. If Y is not reflexive, then Y has the almost Daugavet property.*

Proof. The space Y has thickness two by Theorem 3.2, and for separable spaces this is equivalent to the almost Daugavet property [11, Theorem 1.1]. ∎

Let us use this result in the setting of translation-invariant subspaces of $L^1(G)$. Suppose that G is a compact abelian group, Λ is a subset of its dual group $\widehat{G}$, and $0 < r < p < \infty$. The set Λ is said to *be of type* (r, p) if there is a constant $C > 0$ such that

$$\|f\|_p \leq C\|f\|_r$$

for every $f \in T_\Lambda(G)$, or in other words, if $\|\cdot\|_r$ and $\|\cdot\|_p$ are equivalent on $T_\Lambda(G)$. Furthermore we say that Λ is a $\Lambda(p)$ *set* if Λ is of type (r, p) for some $r < p$.

Corollary 3.4. *Let G be a metrizable, compact abelian group and let Λ be a subset of $\widehat{G}$. The space $L^1_\Lambda(G)$ has the almost Daugavet property if and only if Λ is not a $\Lambda(1)$ set.*

Proof. Every almost Daugavet space contains a copy of ℓ^1 [11, Corollary 3.3] and is therefore not reflexive. So the condition is necessary because $L^1_\Lambda(G)$ is reflexive if and only if Λ is a $\Lambda(1)$ set [7, Corollary].

If G is a metrizable, compact abelian group, then $\widehat{G}$ is countable [17, Theorem 2.2.6] and $L^1(G)$ is separable. If Λ is not a $\Lambda(1)$ set, then $L^1_\Lambda(G)$ is not reflexive and $T(L^1_\Lambda(G)) = 2$ by Theorem 3.2. But for separable spaces this is equivalent to the almost Daugavet property [11, Theorem 1.1]. ∎

4. Remarks. The almost Daugavet property is strictly weaker than the Daugavet property for translation-invariant subspaces of $C(G)$ or $L^1(G)$. If we set $\Lambda = \{3^n : n \in \mathbb{N}\}$, then Λ is a Sidon set. So $C_\Lambda(\mathbb{T})$ is isomorphic to ℓ^1, has the Radon–Nikodým property and therefore not the Daugavet property. But Λ is an infinite set and $C_\Lambda(\mathbb{T})$ has the almost Daugavet property. Analogously, $L^1_{\mathbb{N}}(\mathbb{T})$ is isomorphic to the Hardy space H^1_0, has therefore the Radon–Nikodým property and fails the Daugavet property. But $\mathbb{N}$ is not a $\Lambda(1)$ set and $L^1_{\mathbb{N}}(\mathbb{T})$ has the almost Daugavet property.

We say that a Banach space X has the *fixed point property* if, given any non-empty, closed, bounded and convex subset C of X, every non-expansive mapping $T : C \to C$ has a fixed point. Here T is non-expansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. By considering

$$C = \{(x_n)_{n\in\mathbb{N}} \in S_{\ell^1} : x_n \ge 0\}$$

and the right shift operator, it can be shown that ℓ^1 does not have the fixed point property [4, Theorem 1.2]. This counterexample can be transferred to all Banach spaces that contain an *asymptotically isometric copy of* ℓ^1. A Banach space X is said to contain an asymptotically isometric copy of ℓ^1 if there is a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \le \Big\|\sum_{n=1}^{\infty} a_n x_n\Big\| \le \sum_{n=1}^{\infty}|a_n|$$

for any sequence of scalars $(a_n)_{n\in\mathbb{N}}$ with finite support. Every Banach space X with $T(X) = 2$ contains an asymptotically isometric copy of ℓ^1 [11, implicitly in the proof of Propositions 3.2 and 3.4]. So Theorem 3.2 gives another proof of the fact that every non-reflexive subspace of $L^1[0,1]$, or more generally every non-reflexive subspace of an L-embedded space, fails the fixed point property (cf. [4, Theorem 1.4] and [16, Corollary 4]).

Acknowledgements. This is part of the author's Ph.D. thesis, written under the supervision of D. Werner at the Freie Universität Berlin.

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Received 2 August 2013 (5998)