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DETERMINATION OF THE DIFFUSION OPERATOR ON AN INTERVAL

BY

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Abstract. The inverse problem of spectral analysis for the diffusion operator with quasiperiodic boundary conditions is considered. A uniqueness theorem is proved, a solution algorithm is presented, and sufficient conditions for the solvability of the inverse problem are obtained.

1. Introduction. Inverse problems of spectral analysis consist in the reconstruction of the operators from their spectral data. One takes for the main spectral data, for instance, one, two, or more spectra, the spectral function, the spectrum and the normalizing constants, the Weyl function. Different statements of inverse problems are possible depending on the selected spectral data. The already existing literature on the theory of inverse problems of spectral analysis is abundant. Many aspects of the modern state of this theory and its applications are presented in [2], [6], [7], [11], [12], [15], [17]–[20], [27], [30], [31], [37].

The problem of describing the interactions between colliding particles is of fundamental interest in physics. For a radial static potential $V(x)$ the s-wave Schrödinger equation is written as

$$y'' + [E - V(E,x)]y = 0,$$

where $V(E,x)$ has the form:

$$V(E,x) = 2\sqrt{E}\,p(x) + q(x).$$

We note that with the additional condition $q(x) = -p^2(x)$, the above equation reduces to the Klein–Gordon s-wave equation for a particle of zero mass and energy $\sqrt{E}$ (see [13]).

In this paper we consider the boundary value problem L_t generated on the interval $[0,\pi]$ by the diffusion differential equation

$$l_\lambda y = y'' + [\lambda^2 - 2\lambda p(x) - q(x)]y = 0 \tag{1.1}$$

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with real coefficients $p \in W_2^1[0, \pi]$ and $q \in L_2[0, \pi]$ and boundary conditions

$$y(\pi) = e^{it}y(0), \quad y'(\pi) = e^{it}y'(0), \tag{1.2}$$

where t is a fixed real number. If $t = m\pi$, where m is an integer, then the boundary conditions (1.2) transform into periodic (if m is even) or antiperiodic (if m is odd) boundary conditions. For $t \neq m\pi$, the boundary conditions (1.2) are said to be *quasiperiodic* (see [3, p. 442]).

The first paper dedicated to the study of the periodic inverse problem for the Sturm–Liouville operator ($p(x) \equiv 0$) was [32]. In that paper, Lyapunov functions were applied. Subsequently, this problem was studied using different approaches in [28], [19] and [14]. In [28] a uniqueness theorem for the inverse problem is proved using certain mappings of solution spaces. In [19] a complete characterization of the spectrum of the Hill operator is obtained, based on a parametrization of a class of real entire functions using special conformal mappings of the upper half-plane onto the upper half-plane with vertical cuts. Using the direct method (in which the results of [19], the Gelfand–Levitan–Marchenko equation and the trace formulas are not used), in [14] the inverse problem with gap lengths for the Hill operator is solved. Here a real analytic isomorphism between Hilbert spaces and two-sided estimates of the potential in terms of spectral data are applied. Using the methods of [19], in [25] and [26] the problem of reconstruction of similar (including quasiperiodic) Sturm–Liouville problems is completely solved (recall that two boundary-value problems are called *similar* if their characteristic functions differ only by a constant). Such a problem is also studied in [35] using a different approach, where the characterization of the spectrum is obtained without using the asymptotic properties of special conformal mappings. In [5], [10] and [18] inverse problems for the Sturm–Liouville operators with regular nonseparated boundary conditions are solved.

Some versions of inverse problems for the equation (1.1), which is a natural generalization of the Sturm–Liouville equation, were thoroughly studied in [4], [8], [9], [11], [13], [16], [22], [24], [29], [36]. Thus, for example, inverse problems for l_λ on the half-line and on the whole line are considered in [13] and [20], where the scattering data and the Weyl function are used as spectral data. The problem of reconstructing the equation (1.1) from the spectra of two boundary-value problems with some separated boundary conditions is solved in [4]. Inverse spectral problems for (1.1) with other forms of separated boundary conditions, and also with periodic and antiperiodic boundary conditions are studied in [8] (see also [9]), where the corresponding results of the monograph [19] are generalized to the case $p(x) \not\equiv 0$. Problems of reconstructing nonsimilar boundary-value problems generated by (1.1) and nonseparated boundary conditions are studied in [11] and [22]. In [24] and [29] the uniqueness of the reconstruction of l_λ by three spectra is investigated. We also record the papers [1], [16], [33] and [38], in which inverse nodal and half-inverse problems for l_λ on the interval and on graphs are considered.

Note that the problem of reconstructing the diffusion equation (1.1) with quasiperiodic boundary conditions has special features and the conditions of solvability of the inverse problem in this case are essentially different from the results for other types of nonseparated boundary conditions. This paper is devoted to the inverse problem of reconstructing the boundary-value problem L_t. A uniqueness theorem is proved, an algorithm is constructed and sufficient conditions for the solvability of the inverse problem are obtained. Analogous results for the Dirac operator are obtained in [23].

2. Asymptotics of eigenvalues. We denote by $W_2^n[0,\pi]$ the Sobolev space of complex-valued functions on the interval $[0,\pi]$ which have $n-1$ absolutely continuous derivatives and square-summable nth derivative. In what follows, for brevity, we will say that *the condition* (T) *is satisfied* if the inequality

$$\int_0^\pi \{|y'(x)|^2 + q(x)|y(x)|^2\}\,dx > 0 \tag{2.1}$$

holds for all functions $y \in W_2^2[0,\pi]$, $y(x) \not\equiv 0$, satisfying (1.2). Note that the inequality (2.1) holds in particular if $q(x) > 0$.

Let $c(\lambda,\cdot)$ and $s(\lambda,\cdot)$ be the solutions of the equation (1.1) satisfying the initial conditions $c(\lambda,0) = s'(\lambda,0) = 1$, $c'(\lambda,0) = s(\lambda,0) = 0$. The Wronskian of these solutions is identically one:

$$c(\lambda,x)s'(\lambda,x) - c'(\lambda,x)s(\lambda,x) = 1. \tag{2.2}$$

It is easy to see that the function

$$\varDelta(\lambda) = c(\lambda,\pi) + s'(\lambda,\pi) - 2\cos t \tag{2.3}$$

is the characteristic function of the boundary-value problem L_t. The zeros of this function coincide with the eigenvalues of the problem L_t. When the condition (T) is satisfied, the eigenvalues are real and nonzero [21]. According to [21], a number λ_0 is a double eigenvalue of L_t if and only if $t = m\pi$ $(m = 0, \pm1, \pm2, \dots)$ and $c'(\lambda_0,\pi) = s(\lambda_0,\pi) = 1$. Since in this paper we assume $t \neq m\pi$, all eigenvalues of the problem L_t are simple.

In what follows, we will suppose that $p(0) = p(\pi)$, $0 < t < \pi/2$.

THEOREM 2.1. *The eigenvalues $a_k^\pm$ $(k = 0, \pm1, \pm2, \dots)$ of the boundary-value problem L_t $(p(0) = p(\pi)$, $0 < t < \pi/2)$ have the asymptotics*

$$a_k^\pm = 2k + a \pm \frac{t}{\pi} + \frac{A}{2k} + \frac{\tau_k^\pm}{k}, \tag{2.4}$$

where

$$a=\frac{1}{\pi}\int_0^\pi p(x)\,dx,\qquad A=\frac{1}{2\pi}\int_0^\pi [p^2(x)+q(x)]\,dx,\qquad \sum_{k=-\infty}^{\infty}|\tau_k^{\pm}|^2<\infty.$$

Proof. From the representations (see [9], [11])

$$c(\lambda,\pi)=\cos\pi(\lambda-a)+A\pi\frac{\sin\pi(\lambda-a)}{\lambda-a}+\frac{f_1(\lambda-a)}{\lambda-a},$$
$$s'(\lambda,\pi)=\cos\pi(\lambda-a)+A\pi\frac{\sin\pi(\lambda-a)}{\lambda-a}+\frac{f_2(\lambda-a)}{\lambda-a}$$

and the equality (2.3) we deduce that the eigenvalues of the problem L_t satisfy the equation

$$\cos\pi(\lambda-a)+A\pi\frac{\sin\pi(\lambda-a)}{\lambda-a}+\frac{f(\lambda-a)}{\lambda-a}-\cos t=0, \tag{2.5}$$

where $f_1(\cdot), f_2(\cdot), f(\cdot)$ are entire functions of exponential type not exceeding π, square-summable on the real line. By Rouché's theorem the roots $a_k^{\pm}$ $(k=0,\pm1,\pm2,\dots)$ of this equation have the asymptotics

$$a_k^{\pm}=2k+a\pm\frac{t}{\pi}+\varepsilon_k^{\pm}, \tag{2.6}$$

where $\sum_{k=-\infty}^{\infty}(\varepsilon_k^{\pm})^2<\infty$. It is obvious that

$$\cos\pi(a_k^{\pm}-a)=\cos t\mp\varepsilon_k^{\pm}\pi\sin t+O\left(\frac{1}{k^2}\right), \tag{2.7}$$

$$\frac{\sin\pi(a_k^{\pm}-a)}{a_k^{\pm}-a}=\pm\frac{\pi\sin t}{2k\pi\pm t}+O\left(\frac{1}{k^2}\right), \tag{2.8}$$

$$\frac{f(a_k^{\pm}-a)}{a_k^{\pm}-a}=\frac{\pi f(2k\pm t/\pi)}{2k\pi\pm t}+O\left(\frac{1}{k^2}\right) \tag{2.9}$$

(here we used [19, Lemma 1.4.3]). Substituting (2.6)–(2.9) into (2.5) we obtain the asymptotics

$$\varepsilon_k^{\pm}=\frac{A}{2k}+\frac{\tau_k^{\pm}}{k},$$

the substitution of which in (2.6) leads to (2.4). ∎

3. Uniqueness theorem and algorithm for solving the inverse problem. Let λ_k $(k=\pm1,\pm2,\dots)$ be the zeros of the function $s(\cdot,\pi)$, i.e. the eigenvalues of the boundary-value problem generated by the equation (1.1) and the Dirichlet boundary conditions

$$y(0)=y(\pi)=0. \tag{3.1}$$

We denote $\sigma_k=\operatorname{sign}[1-|s'(\lambda_k,\pi)|]$, $k=\pm1,\pm2,\dots$.

THEOREM 3.1. *The boundary-value problem* L_t ($p(0) = p(\pi)$, $0 < t < \pi/2$) *is uniquely determined by the three sequences* $\{a_k^\pm\}$, $\{\lambda_k\}$ *and* $\{\sigma_k\}$.

Proof. Given the spectrum $\{a_k^\pm\}$ of the problem L_t the parameter t of the boundary conditions is uniquely determined from

$$t = \frac{\pi}{2} \lim_{k\to\infty} (a_k^+ - a_k^-), \tag{3.2}$$

since $\{a_k^\pm\}$ has the asymptotics (2.4). The characteristic function $\Delta(\cdot)$ of the boundary-value problem L_t, which is an entire function of exponential type, is uniquely determined by the sequence $\{a_k^\pm\}$ in the form of an infinite product. We denote

$$u_+(\lambda) = c(\lambda, \pi) + s'(\lambda, \pi). \tag{3.3}$$

Knowing $\Delta(\cdot)$ and t, the function $u_+(\cdot)$ can be determined from (2.2):

$$u_+(\lambda) = \Delta(\lambda) + 2\cos t. \tag{3.4}$$

Given the sequence $\{\lambda_k\}$ we construct the function

$$s(\lambda, \pi) = \pi \prod_{k=-\infty}^{\infty}{}' \frac{\lambda_k - \lambda}{k}, \tag{3.5}$$

where the prime on the product (or summation) symbol (here and afterwards) indicates that the term corresponding to $k = 0$ is excluded. If we also knew the function

$$u_-(\lambda) = c(\lambda, \pi) - s'(\lambda, \pi), \tag{3.6}$$

we would be able to determine the function $s'(\cdot, \pi)$ from the identity

$$s'(\lambda, \pi) = \tfrac{1}{2}[u_+(\lambda) - u_-(\lambda)], \tag{3.7}$$

and the knowledge of the functions $s(\cdot, \pi)$, $s'(\cdot, \pi)$, according to [8], is sufficient to uniquely determine $p(\cdot)$ and $q(\cdot)$. From (3.3), (3.6) using (2.2) we have

$$u_+^2(\lambda) - u_-^2(\lambda) = 4 + 4c'(\lambda, \pi)s(\lambda, \pi).$$

Putting $\lambda = \lambda_k$ in this equality, we get $u_+^2(\lambda_k) - u_-^2(\lambda_k) = 4$. Hence

$$u_-(\lambda_k) = (\operatorname{sign} u_-(\lambda_k))\sqrt{u_+^2(\lambda_k) - 4}.$$

It is known [9] that $\operatorname{sign} s'(\lambda_k, \pi) = (-1)^k$. Then

$$\operatorname{sign} u_-(\lambda_k) = \operatorname{sign}[c(\lambda_k, \pi) - s'(\lambda_k, \pi)] = \operatorname{sign} \frac{1 - [s'(\lambda_k, \pi)]^2}{s'(\lambda_k, \pi)} = (-1)^k \sigma_k.$$

Taking into account (3.6) and the representations of the functions $c(\cdot, \pi)$ and $s'(\cdot, \pi)$, we infer that $u_-(\lambda) = o(e^{\pi|\operatorname{Im}\lambda|}/|\lambda|)$ as $|\lambda| \to \infty$. Then according to

[34, p. 178] the interpolation formula

$$u_-(\lambda) = s(\lambda,\pi) \sum_{k=-\infty}^{\infty}{}' \frac{u_-(\lambda_k)}{(\lambda-\lambda_k)\dot{s}(\lambda_k,\pi)} \tag{3.8}$$

holds, where $u_-(\lambda_k) = (-1)^k \sigma_k \sqrt{u_+^2(\lambda_k) - 4}$. Therefore, knowing the function $u_+(\cdot)$ and the sequences $\{\lambda_k\}$ and $\{\sigma_k\}$ we can determine the function $u_-(\cdot)$ using (3.8). Uniqueness of $u_-(\cdot)$ follows from the fact that this interpolation formula sets up a bijection between l_2 and the space of entire functions of exponential type not exceeding π, square-summable on the real line. Hence, with the help of $u_+(\cdot)$ and $u_-(\cdot)$ we can determine the function $s'(\cdot,\pi)$ by (3.7). The zeros ν_k, $k=\pm1, \pm2, \dots$, of this function are the eigenvalues of the boundary value problem generated by the equation (1.1) and the boundary conditions

$$y(0) = y'(\pi) = 0. \tag{3.9}$$

As noted above, the sequences $\{\lambda_k\}$ and $\{\nu_k\}$ uniquely determine the functions $p(\cdot)$ and $q(\cdot)$.

Therefore the coefficient functions $p(\cdot)$ and $q(\cdot)$, and the parameter t, are uniquely determined by the spectrum $\{a_k^\pm\}$ of the problem L_t and the sequences $\{\lambda_k\}$, $\{\sigma_k\}$. ■

Bearing in mind the proof of Theorem 3.1 we arrive at the following solution algorithm for the inverse problem of recovering the boundary-value problem L_t.

Algorithm. Let the sequences $\{a_k^\pm\}$, $\{\lambda_k\}$ and $\{\sigma_k\}$ be given.

(1) Determine the parameter t from (3.2).
(2) With the help of the sequence $\{a_k^\pm\}$, construct the function $\Delta(\cdot)$ in the form of an infinite product.
(3) Find the function $u_+(\cdot)$ from (3.4).
(4) Construct the function (3.5).
(5) Reconstruct the function $u_-(\cdot)$ using the interpolation formula (3.8).
(6) Define the characteristic function $s'(\cdot,\pi)$ of the boundary-value problem (1.1), (3.9) by (3.7).
(7) Determine the coefficients $p(\cdot)$ and $q(\cdot)$ from the sequences of zeros of the functions $s(\cdot,\pi)$ and $s'(\cdot,\pi)$ by a well-known procedure (see [8], [9]).

4. Sufficient solvability conditions for the problem of recovering the boundary-value problem L_t. First we prove the following auxiliary proposition.

LEMMA 4.1. *The function*

$$\varDelta(z) = 4\sin^2\frac{t}{2}\prod_{k=-\infty}^{\infty}\frac{\pi^2(a_k^- - z)(a_k^+ - z)}{4\pi^2k^2 - t^2} \tag{4.1}$$

has the representation

$$\varDelta(z) = 2\cos\pi(z-a) + 2A\pi\frac{\sin\pi(z-a)}{z-a} + \frac{g(z-a)}{z-a} - 2\cos t, \tag{4.2}$$

where $g(z) = M(\cos\pi z - \cos t) + \int_{-\pi}^{\pi}\tilde{g}(x)e^{ixz}\,dx$, *M is some constant,* $\tilde{g} \in L_2[-\pi;\pi]$, $a_k^\pm$ *has the form* (2.4), *in which* a, A *are real numbers and* $0 < t < \pi/2$.

Proof. We denote

$$p^\pm(z) = \pm\frac{a_0^\pm - z}{t}\sin\frac{t}{2}\prod_{k=-\infty}^{\infty}{}'\frac{a_k^\pm - z}{2k}\prod_{k=-\infty}^{\infty}{}'\frac{1}{1 \pm t/(2\pi k)}. \tag{4.3}$$

According to [9], [11] the function $l(z) = \pi(z-a)\prod_{k=-\infty}^{\prime\infty}(u_k - z)/k$ can be represented as

$$l(z) = \sin\pi(z-a) + A_0\pi\frac{4(z-a)}{4(z-a)^2 - 1}\cos\pi(z-a) + \frac{f_0(z-a)}{z-a},$$

where

$$u_k = k + a - \frac{A_0}{k} + \frac{\delta_k}{k}, \qquad f_0(z) = a_0\sin\pi z + \int_{-\pi}^{\pi}\tilde{f}_0(t)e^{itz}\,dt, \qquad \tilde{f}_0 \in L_2[-\pi,\pi],$$

$f_0(0) = f_0'(0) = 0$, a, a_0, A_0 are some numbers, $\sum_{k=-\infty}^{\prime\infty}|\delta_k|^2 < \infty$. Using this fact and the formula $\sin\pi z = \pi z\prod_{k=-\infty}^{\prime\infty}(1 - z/k)$, from (4.3) we have

$$p^\pm(z) = \pm\frac{a_0^\pm - z}{2\pi(z-a)}\left[\sin\pi z^\pm - \frac{A\pi z^\pm}{4(z^\pm)^2 - 1}\cos\pi z^\pm + \frac{g^\pm(z^\pm)}{z^\pm}\right], \tag{4.4}$$

where $z^\pm = \frac{1}{2}(z - a \mp t/\pi)$, $g^\pm(z) = a^\pm\sin\pi z + \int_{-\pi}^{\pi}\tilde{g}^\pm(t)e^{itz}\,dt$, $\tilde{g}^\pm \in L_2[-\pi,\pi]$, $g^\pm(0) = dg^\pm(0)/dz = 0$ and $a^\pm$ are some numbers. The formulas (4.1) and (4.3) imply $\varDelta(z) = 4p^-(z)p^+(z)$. From this, taking into account the representation (4.4) and the Paley–Wiener theorem [34, p. 101], after elementary transformations we obtain the formula (4.2). ∎

Now we prove our main result, which gives a characterization of the spectrum of the problem L_t.

THEOREM 4.2. *For a sequence* $\{a_k^\pm\}$ *to consist of the eigenvalues of a quasiperiodic problem* L_t ($p(0) = p(\pi)$, $0 < t < \pi/2$), *it is sufficient that the following conditions hold:*

(1) *the* $a_k^\pm$ *satisfy the asymptotic condition* (2.4), *in which a and A can be arbitrary real numbers;*

(2) $\lim_{k\to\infty} k[\Delta(2k+a)-4\sin^2(t/2)] = 0$, *where* $\Delta(z)$ *is defined by* (4.1)*;*
(3) *the zeros* $\mu^{\pm}_{2k}$ *and* $\mu^{\pm}_{2k+1}$ $(k = 0, \pm 1, \pm 2, \dots)$ *of the functions* $\Delta_p(z) = \Delta(z) - 4\sin^2\frac{t}{2}$ *and* $\Delta_a(z) = \Delta(z) + 4\cos^2\frac{t}{2}$*, respectively, satisfy*

$$0 > \mu_0^- > \mu_{-1}^+ \ge \mu_{-1}^- > \mu_{-2}^+ \ge \mu_{-2}^- > \cdots, \tag{4.5}$$
$$0 < \mu_0^+ < \mu_1^- \le \mu_1^+ < \mu_2^- \le \mu_2^+ < \cdots$$

and the asymptotic formula (*as* $|m| \to \infty$)

$$\mu_m^{\pm} = m + a + \frac{A}{m} + \frac{\gamma_m^{\pm}}{m}, \qquad \sum_{k=-\infty}^{\infty}{}' (\gamma_m^{\pm})^2 < \infty. \tag{4.6}$$

Proof. According to the above lemma the function $\Delta(z)$ has the representation (4.2). It is obvious that

$$\Delta(2k+a) = M(1-\cos t) + \int_{-\pi}^{\pi} \tilde{g}(x)e^{2ikx}\,dx + 4\sin^2\frac{t}{2}.$$

Hence, by (2) and the Riemann–Lebesgue lemma we have $M = 0$. The representation (4.2) and the asymptotic formula (4.6) imply

$$\Delta_p(z) = \pi^2(z-\mu_0^-)(\mu_0^+ - z)\prod_{k=-\infty}^{\infty}{}' \frac{(\mu_k^- - z)(\mu_k^+ - z)}{(2k)^2}, \tag{4.7}$$
$$\Delta_a(z) = 4\prod_{k=-\infty}^{\infty} \frac{(\mu_{2k+1}^- - z)(\mu_{2k+1}^+ - z)}{(2k+1)^2}.$$

We put $u_1(z) = \Delta(z) + 2\cos t$. It is easy to see that

$$\Delta_p(z) = u_1(z) - 2, \qquad \Delta_a(z) = u_1(z) + 2, \tag{4.8}$$

where

$$u_1(z) = 2\cos\pi(z-a) + 2A\pi\frac{\sin\pi(z-a)}{z-a} + \frac{g(z-a)}{z-a}. \tag{4.9}$$

Take any point $\lambda_k \in [\mu_k^-, \mu_k^+]$, $k = \pm 1, \pm 2, \dots$. Then from (4.6) we have

$$\lambda_k = k + a + \frac{A}{k} + \frac{\delta_k}{k}, \qquad \sum_{k=-\infty}^{\infty}{}' \delta_k^2 < \infty. \tag{4.10}$$

As noted above (see the proof of Lemma 4.1) the function

$$(z-a)s(z) = \pi(z-a)\prod_{k=-\infty}^{\infty}{}' \frac{\lambda_k - z}{k} \tag{4.11}$$

can be represented in the form

$$(z-a)s(z) = \sin\pi(z-a) - A\pi\frac{4(z-a)}{4(z-a)^2-1}\cos\pi(z-a) + \frac{\psi(z-a)}{z-a},$$

where

$$\psi(z) = b\sin\pi z + \int_{-\pi}^{\pi} \tilde\psi(t)e^{itz}\,dt, \qquad \tilde\psi \in L_2[-\pi,\pi], \qquad \psi(0)=\psi'(0)=0.$$

According to (4.7) and (4.8) we have

$$u_1^2(z) - 4 = \varDelta_p(z)\varDelta_a(z) = -4\pi^2(\mu_0^- - z)(\mu_0^+ - z)\prod_{k=-\infty}^{\infty}{}' \frac{(\mu_k^- - z)(\mu_k^+ - z)}{k^2}.$$

Hence using the inequalities $\mu_k^- \le \lambda_k \le \mu_k^+$ we obtain $u_1^2(\lambda_k) - 4 \ge 0$ and $\operatorname{sign} u_1(\lambda_k) = (-1)^k$. Then it is obvious that

$$u_1(\lambda_{2m-1}) \le -2, \qquad u_1(\lambda_{2m}) \ge 2 \qquad (m = 0, \pm1, \pm2, \ldots).$$

Therefore, there exist numbers h_k such that

$$u_1(\lambda_k) = 2(-1)^k \operatorname{ch} h_k. \tag{4.12}$$

Since at both endpoints of $[\mu_k^-, \mu_k^+]$ the function $u_1(\cdot)$ takes the same value $2(-1)^k$, its derivative has at least one zero β_k in that interval. From (4.9) we have, as $|z| \to \infty$,

$$u_1'(z) = -2\pi \sin\pi(z-a) + O(e^{\pi|\operatorname{Im} z|}/|z|).$$

Using this estimate and Rouché's theorem it is easy to show that, the function $u_1'(\cdot)$ has exactly $2n+1$ zeros in each strip $|\operatorname{Re} z - a| \le n + 1/2$ for sufficiently large n, and from (4.6) it is seen that

$$-(n+1/2) + a < \beta_{-n} < \beta_{-n+1} < \cdots < \beta_0 < \beta_1 < \cdots < \beta_n < n + 1/2 + a.$$

Therefore all zeros β_k are simple and the derivative $u'(\cdot)$ has no other zeros. So, on each interval (β_k, β_{k+1}) the function $u_1'(\cdot)$ has constant sign. Since $\beta_k \in [\mu_k^-, \mu_k^+]$, we have $u_1^2(\beta_k) \ge 4$ and $\operatorname{sign} u_1(\beta_k) = (-1)^k$. Thus there exists a number $h_k' \ge 0$ such that

$$u_1(\beta_k) = 2(-1)^k \operatorname{ch} h_k'. \tag{4.13}$$

From the equalities $u_1(\mu_k^\pm) = 2(-1)^k$, $u_1'(\beta_k) = 0$ and the Taylor formula we have

$$2(-1)^k = u_1(\mu_k^\pm) = u_1(\beta_k) + \tfrac12(\mu_k^\pm - \beta_k)^2 u_1''(\beta_k^\pm),$$

and using (4.13) we get

$$\operatorname{ch} h_k' = 1 + \frac{(-1)^{k+1}}{4}(\mu_k^\pm - \beta_k)^2 u_1''(\beta_k^\pm),$$

where the points $\beta_k^\pm$ are located between $\mu_k^\pm$ and β_k. Then

$$1 + \tfrac12 h_k'^2 \le 1 + \tfrac14(\mu_k^\pm - \beta_k)^2 M_k, \qquad M_k = \max_{z\in[\mu_k^-,\mu_k^+]} |u_1''(z)|,$$

and hence

$$h'_k \le \frac{\sqrt{2M_k}}{4}(\mu_k^+ - \mu_k^-),$$

since one of the numbers $|\mu_k^- - \beta_k|$, $|\mu_k^+ - \beta_k|$ does not exceed $\frac{1}{2}(\mu_k^+ - \mu_k^-)$. According to the Bernstein inequality [34, p. 104] for the derivatives of bounded entire functions of exponential type,

$$\sup_{z\in(-\infty,\infty)} |u_1''(z)| \le \pi^2 M, \qquad M = \sup_{z\in(-\infty,\infty)} |u_1(z)| < \infty,$$

so $h'_k \le \frac{\pi\sqrt{2M}}{4}(\mu_k^+ - \mu_k^-)$. From (4.6) we have

$$\mu_k^+ - \mu_k^- = \frac{\gamma_k^+ - \gamma_k^-}{k}.$$

Therefore $\sum_{k=-\infty}^{\infty}(kh'_k)^2 < \infty$. It is easy to see that $|h_k| \le h'_k$ and hence

$$\sum_{k=-\infty}^{\infty} (kh_k)^2 < \infty. \tag{4.14}$$

Let $\{\sigma_k\}$ $(k = \pm 1, \pm 2, \ldots)$ be any sequence (of signs) consisting of the numbers -1, 0 and 1. Since (4.14) holds, as in [9] (see also [19]) it can be shown that the function

$$u_2(z) = 2s(z) \sum_{k=-\infty}^{\infty}{}' \frac{\sigma_k |\operatorname{sh} h_k|}{(z-\lambda_k)s'(\lambda_k)} \tag{4.15}$$

can be represented as

$$u_2(z) = \frac{m(z-a)}{z-a}, \tag{4.16}$$

where

$$m(z) = \int_{-\pi}^{\pi} \tilde{m}(t)e^{itz}\,dt, \qquad \tilde{m} \in L_2[-\pi,\pi], \qquad m(0) = 0.$$

Consider the function

$$s_1(z) = \tfrac{1}{2}[u_1(z) - u_2(z)]. \tag{4.17}$$

Taking into account (4.9) and (4.17), we have

$$s_1(z) = \cos\pi(z-a) + A\pi\frac{\sin\pi(z-a)}{z-a} + \frac{\varphi_1(z-a)}{z-a},$$

where $\varphi_1(z) = \frac{1}{2}[g(z) - m(z)]$, $\varphi_1(0) = 0$. Therefore (see [9], [11]) the zeros $\nu_k (k = \pm 1, \pm 2, \ldots)$ of the function $s_1(\cdot)$ satisfy the asymptotic formula

$$\nu_k = k - \frac{1}{2}\operatorname{sign} k + a + \frac{a_1}{k} + \frac{\delta'_k}{k}, \qquad \sum_{k=-\infty}^{\infty} |\delta'_k|^2 < \infty. \tag{4.18}$$

Putting $z = \lambda_k$ in (4.17) and taking into account (4.12) and the equality

$$u_2(\lambda_k) = 2\sigma_k|\operatorname{sh} h_k|,$$

which can be obtained from (4.15), we have

$$s_1(\lambda_k) = (-1)^k \operatorname{ch} h_k - \sigma_k|\operatorname{sh} h_k| = (-1)^k \operatorname{ch} h_k[1 - (-1)^k|\operatorname{th} h_k|]$$

and since $|\operatorname{th} h_k| < 1$, it follows that

$$\operatorname{sign} s_1(\lambda_k) = (-1)^k \quad (k = \pm 1, \pm 2, \ldots). \tag{4.19}$$

Now we will show that the sequence $\{\sigma_k\}$ can be chosen so that

$$s_1(0) > 0. \tag{4.20}$$

Indeed, according to (4.17) the inequality (4.20) is equivalent to

$$u_2(0) > u_1(0). \tag{4.21}$$

From (4.11) and (4.15) we obtain

$$s(0) = \pi \prod_{k=-\infty}^{\infty}{}' \frac{\lambda_k}{k} > 0, \qquad u_2(0) = -2s(0) \sum_{k=-\infty}^{\infty}{}' \frac{\sigma_k|\operatorname{sh} h_k|}{\lambda_k s'(\lambda_k)}.$$

Therefore (4.21) is equivalent to

$$\sum_{k=-\infty}^{\infty}{}' \frac{2\sigma_k|\operatorname{sh} h_k|}{-\lambda_k s'(\lambda_k)} < \frac{u_1(0)}{s(0)}. \tag{4.22}$$

Since $u_1(0) = \mathit{\Delta}_1(0) + 2$ and $\mathit{\Delta}_1(0) > 0$, we have $u_1(0) > 0$ and hence the right-hand side of (4.22) is positive. Thus if we take, for example, $\sigma_k = \operatorname{sign}(\lambda_k s'(\lambda_k))$, then the left-hand side of (4.22), will be negative. So, for such σ_k the inequality (4.22), and therefore (4.20), will be satisfied.

The relations (4.19) and (4.20) show that, in each interval

$$\ldots, (\lambda_{-2}, \lambda_{-1}), (\lambda_{-1}, 0), (0, \lambda_1), (\lambda_1, \lambda_2), \ldots$$

the function $s_1(\cdot)$ has one, and according to (4.18), only one zero. Therefore, the zeros of the functions $s_1(\cdot)$ and $s_2(\cdot)$ satisfy the inequalities

$$\cdots < \nu_{-3} < \lambda_{-2} < \nu_{-2} < \lambda_{-1} < \nu_{-1} < 0 < \nu_1 < \lambda_1 < \nu_2 < \lambda_2 < \nu_3 < \cdots.$$

In addition, $\{\lambda_k\}$ and $\{\nu_k\}$ have the asymptotics (4.10) and (4.18). So the sequences $\{\lambda_k\}$ and $\{\nu_k\}$ satisfy all the conditions of [8, Theorem 3]. Hence there exist unique real-valued functions $q \in L_2[0, \pi]$ and $p \in W_2^1[0, \pi]$ such that $\{\lambda_k\}$ and $\{\nu_k\}$ are the spectra of the boundary-value problems generated by one and the same equation

$$y'' + [\lambda^2 - 2\lambda p(x) - q(x)]y = 0$$

and the boundary conditions (3.1) and (3.9), and $s(\lambda) = s(\lambda, \pi)$, $s_1(\lambda) = s'(\lambda, \pi)$, where $s(\lambda, \cdot)$ is a solution of this equation with initial conditions $s(\lambda, 0) = 0$, $s'(\lambda, 0) = 1$.

It is easy to prove that $\{a_k^{\pm}\}$ is indeed the spectrum of the reconstructed problem L_t. ■

REMARK 4.3. Note that the conditions of Theorem 4.2 are also necessary when the condition (T) is satisfied. But the real function $q(\cdot)$ constructed in the proof of Theorem 4.2 may not satisfy the inequality (2.1).

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