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THE REAPING AND SPLITTING NUMBERS OF NICE IDEALS

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Abstract. We examine the splitting number $\mathfrak{s}(\mathbf{B})$ and the reaping number $\mathfrak{r}(\mathbf{B})$ of quotient Boolean algebras $\mathbf{B} = \mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is an F_σ ideal or an analytic P-ideal. For instance we prove that under Martin's Axiom $\mathfrak{s}(\mathcal{P}(\omega)/\mathcal{I}) = \mathfrak{c}$ for all F_σ ideals $\mathcal{I}$ and for all analytic P-ideals $\mathcal{I}$ with the BW property (and one cannot drop the BW assumption). On the other hand under Martin's Axiom $\mathfrak{r}(\mathcal{P}(\omega)/\mathcal{I}) = \mathfrak{c}$ for all F_σ ideals and all analytic P-ideals $\mathcal{I}$ (in this case we do not need the BW property). We also provide applications of these characteristics to the ideal convergence of sequences of real-valued functions defined on the reals.

1. Introduction. Let $\mathbf{B}$ be a Boolean algebra. A set S is a *splitting set* for $\mathbf{B}$ if for every nonzero $b \in \mathbf{B}$ there is an $s \in S$ such that $b\cdot s \neq 0 \neq b\cdot(-s)$. A set $D \subseteq \mathbf{B} \setminus \{0\}$ is *weakly dense* if for every $b \in \mathbf{B}$ there is $d \in D$ such that $d \leq b$ or $d \leq -b$. By the *splitting number* of $\mathbf{B}$ we mean the cardinal $\mathfrak{s}(\mathbf{B}) = \min\{|S| : S \text{ is a splitting set for } \mathbf{B}\}$, and by the *reaping number* of $\mathbf{B}$ we mean $\mathfrak{r}(\mathbf{B}) = \min\{|D| : D \text{ is weakly dense in } \mathbf{B}\}$. Many results on $\mathfrak{s}(\mathbf{B})$ and $\mathfrak{r}(\mathbf{B})$ for various Boolean algebras can be found in [23].

In the following we assume that if $\mathcal{I}$ is an ideal on ω then $[\omega]^{<\omega} \subseteq \mathcal{I}$ and $\omega \notin \mathcal{I}$.

For a set $A \subseteq \omega$ we put $A^0 = A$ and $A^1 = \omega \setminus A$.

Let $\mathcal{I}$ be an ideal on ω. We denote by $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ the *coideal* of $\mathcal{I}$. A set A *$\mathcal{I}$-splits* B if both $B \cap A^0, B \cap A^1$ are in $\mathcal{I}^+$. A family $\mathcal{R} \subseteq \mathcal{I}^+$ is *$\mathcal{I}$-unsplittable* if no single set $\mathcal{I}$-splits all members of $\mathcal{R}$. An *$\mathcal{I}$-splitting family* is a family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ such that each $A \in \mathcal{I}^+$ is $\mathcal{I}$-split by at least one $S \in \mathcal{S}$.

In this paper we are interested in the splitting and reaping numbers of quotient Boolean algebras of the form $\mathbf{B} = \mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is an F_σ ideal or an analytic P-ideal on ω (see Section 2 for definitions). We then write $\mathfrak{s}(\mathcal{I}) = \mathfrak{s}(\mathcal{P}(\omega)/\mathcal{I})$ and $\mathfrak{r}(\mathcal{I}) = \mathfrak{r}(\mathcal{P}(\omega)/\mathcal{I})$. In this case the definitions of $\mathfrak{s}(\mathcal{I})$

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and $\mathfrak{r}(\mathcal{I})$ can be rephrased in the following manner:

$$\mathfrak{s}(\mathcal{I}) = \min\{|\mathcal{S}| : \mathcal{S} \subseteq \mathcal{I}^+,\ \mathcal{S} \text{ is an } \mathcal{I}\text{-splitting family}\},$$
$$\mathfrak{r}(\mathcal{I}) = \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathcal{I}^+,\ \mathcal{R} \text{ is } \mathcal{I}\text{-unsplittable}\}.$$

In the case of the ideal $\mathcal{I} = \mathrm{Fin}$ of all finite subsets of ω, we obtain the classical cardinal characteristics of the continuum: $\mathfrak{s} = \mathfrak{s}(\mathrm{Fin})$ and $\mathfrak{r} = \mathfrak{r}(\mathrm{Fin})$ (see e.g. [2] and [27]). It is well known that $\mathfrak{s}$ and $\mathfrak{r}$ are uncountable, and if we assume Martin's Axiom (MA) then $\mathfrak{s} = \mathfrak{r} = \mathfrak{c}$.

In Section 3 we show that $\mathfrak{s}(\mathcal{I}), \mathfrak{r}(\mathcal{I})$ are uncountable for every F_σ ideal (Proposition 3.1), and that if we assume MA then $\mathfrak{s}(\mathcal{I}) = \mathfrak{r}(\mathcal{I}) = \mathfrak{c}$ for every F_σ ideal (Theorem 3.2).

In Section 4 we prove that $\mathfrak{r}(\mathcal{I})$ is uncountable for every analytic P-ideal (Proposition 4.1), and that if we assume MA then $\mathfrak{r}(\mathcal{I}) = \mathfrak{c}$ for every analytic P-ideal (Theorem 4.2).

In [9] it is proved that $\mathfrak{s}(\mathcal{I}) = \omega \Leftrightarrow$ the ideal $\mathcal{I}$ does not have the BW property (see Section 2 for the definition). We prove that if we assume MA then $\mathfrak{s}(\mathcal{I}) = \mathfrak{c}$ for all analytic P-ideals $\mathcal{I}$ with the BW property (Theorem 4.3).

The splitting, reaping and other cardinal characteristics (e.g. $\mathfrak{a}$, $\mathfrak{p}$ and $\mathfrak{t}$) of the quotient Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ were already considered in some papers; see e.g. [1], [3], [8], [13], [15], [16] and [26].

In Section 5 we apply the results on $\mathfrak{s}(\mathcal{I})$ and $\mathfrak{r}(\mathcal{I})$ to the ideal convergence of sequences of real-valued functions defined on the reals.

2. Preliminaries

2.1. Nice ideals. By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign topological complexity to ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is F_σ (resp. *analytic*) if it is an F_σ (resp. analytic) subset of the Cantor space.

An ideal $\mathcal{I}$ is a *P-ideal* if for every countable family $\{A_n : n \in \omega\} \subseteq \mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for every $n \in \omega$.

A map $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure on* ω if $\phi(\emptyset) = 0$ and $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$ for all $A, B \subseteq \omega$. In what follows we assume that $\phi(\{n\}) < \infty$ for every submeasure ϕ and $n \in \omega$. A submeasure ϕ is *lower semicontinuous* (we will write *lsc* for short) if for all $A \subseteq \omega$ we have $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0, 1, \ldots, n-1\})$. For a submeasure ϕ we write

$$\mathrm{Fin}(\phi) = \{A \subseteq \omega : \phi(A) < \infty\}, \qquad \mathrm{Exh}(\phi) = \{A \subseteq \omega : \|A\|_\phi = 0\},$$

where $\|A\|_\phi = \lim_{n\to\infty} \phi(A \setminus \{0, 1, \ldots, n-1\})$.

Theorem 2.1 ([21], [25]). *Let $\mathcal{I}$ be an ideal on ω (not necessarily proper).*

(1) *$\mathcal{I}$ is an F_σ ideal $\Leftrightarrow$ $\mathcal{I} = \mathrm{Fin}(\phi)$ for some lsc submeasure ϕ on ω.*
(2) *$\mathcal{I}$ is an analytic P-ideal $\Leftrightarrow$ $\mathcal{I} = \mathrm{Exh}(\phi)$ for some lsc submeasure ϕ on ω.*

2.1.1. *Examples.* For many examples of nice ideals see e.g. [16] or [7]. Below we list some of them.

(1) The ideal Fin is an F_σ P-ideal.
(2) Let $f : \omega \to [0, \infty)$ be such that $\sum_{n\in\omega} f(n) = \infty$. The *summable ideal generated by f*,
$$\mathcal{I}_f = \Big\{A \subseteq \omega : \sum_{n\in A} f(n) < \infty\Big\},$$
is an F_σ ideal [21].
(3) The *ideal of sets of asymptotic density* 0,
$$\mathcal{I}_d = \left\{A \subseteq \omega : \limsup_{n\to\infty} \frac{|A \cap \{0, 1, \ldots, n-1\}|}{n} = 0\right\},$$
is an analytic P-ideal (and it is not an F_σ ideal).
(4) Let $f : \omega \to [0, \infty)$ be such that
$$\sum_{i=0}^{\infty} f(i) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{f(n)}{\sum_{i\in n} f(i)} = 0.$$
The *Erdős–Ulam ideal generated by f*,
$$\mathcal{EU}_f = \left\{A \subseteq \omega : \lim_{n\to\infty} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i\in n} f(i)} = 0\right\},$$
is an analytic P-ideal [17]. Note that $\mathcal{I}_d$ is an Erdős–Ulam ideal.
(5) Assume that I_n are pairwise disjoint intervals on ω, and μ_n is a measure that concentrates on I_n. Then $\phi = \sup_n \mu_n$ is a lower semicontinuous submeasure and $\mathrm{Exh}(\phi)$ is called the *density ideal generated by $(\mu_n)_n$*. It is known that Erdős–Ulam ideals are density ideals.
(6) The *van der Waerden ideal*
$$\mathcal{W} = \{A \subseteq \omega : A \text{ does not contain arithmetic progressions of arbitrary length}\}$$
is an F_σ ideal (and it is not a P-ideal).
(7) The *eventually different ideal*
$$\mathcal{ED} = \{A \subseteq \omega \times \omega : \exists m, n \in \omega\ \forall k \geq n\ (|\{i \in \omega : (k, i) \in A\}| \leq m)\}$$
is an F_σ ideal (and it is not a P-ideal).

2.2. Ideal convergence. Let $\mathcal{I}$ be an ideal on ω and $A \subseteq \omega$. We say that a sequence $(x_n)_{n\in A}$ of reals is $\mathcal{I}$-*convergent* to $x \in \mathbb{R}$ if $\{n \in A : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. We say that an ideal $\mathcal{I}$ on ω has the *BW property* ($\mathcal{I} \in$ BW, for short) if for every bounded sequence $(x_n)_{n\in\omega}$ of reals there exists $A \in \mathcal{I}^+$ such that $(x_n)_{n\in A}$ is $\mathcal{I}$-convergent [9].

PROPOSITION 2.2 ([9]).

(1) *Every F_σ ideal has the BW property* (*hence* Fin, *summable ideals, $\mathcal{W}$ and $\mathcal{ED}$ have the BW property*).
(2) *Erdős–Ulam ideals* (*and $\mathcal{I}_d$*) *do not have the BW property.*
(3) *A density ideal does not have the BW property if and only if it is an Erdős–Ulam ideal.*

THEOREM 2.3 ([9]). *Let $\mathcal{I}$ be an ideal on ω. Then $\mathfrak{s}(\mathcal{I}) = \omega \Leftrightarrow \mathcal{I}$ does not have the BW property.*

2.3. Big intersections. Below we present some auxiliary results which we will need later (however they seem to be interesting on their own).

LEMMA 2.4. *Let $\mathcal{I}$ be an ideal on ω. There is a function $x : \mathcal{P}(\omega) \to \{0, 1\}$ such that*

$$\bigcap\{A^{x(A)} : A \in \mathcal{A}\} \notin \mathcal{I}$$

for every finite and nonempty family $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

Proof. Let $\mathcal{J}$ be a maximal ideal such that $\mathcal{I} \subseteq \mathcal{J}$. For $A \in \mathcal{P}(\omega)$ we define

$$x(A) = \begin{cases} 0 & \text{if } A \notin \mathcal{J}, \\ 1 & \text{if } A \in \mathcal{J}. \end{cases}$$

Since $A^{x(A)} \notin \mathcal{J}$ for every A, and $\mathcal{J}$ is a maximal ideal, we have $\bigcap\{A^{x(A)} : A \in \mathcal{A}\} \notin \mathcal{J}$. Thus $\bigcap\{A^{x(A)} : A \in \mathcal{A}\} \notin \mathcal{I}$. ∎

COROLLARY 2.5. *Let $\mathcal{I} = \mathrm{Fin}(\phi)$ be an F_σ ideal. There is $x : \mathcal{P}(\omega) \to \{0, 1\}$ such that*

$$\phi\Big(\bigcap\{A^{x(A)} : A \in \mathcal{A}\}\Big) = \infty$$

for every finite and nonempty family $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

Proof. Apply Lemma 2.4 and note that $A \notin \mathcal{I} \Leftrightarrow \phi(A) = \infty$. ∎

COROLLARY 2.6. *Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal. There is $x : \mathcal{P}(\omega) \to \{0, 1\}$ such that*

$$\Big\|\bigcap\{A^{x(A)} : A \in \mathcal{A}\}\Big\|_\phi > 0$$

for every finite and nonempty family $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

Proof. Apply Lemma 2.4 and note that $A \notin \mathcal{I} \Leftrightarrow \|A\|_\phi > 0$. ∎

Below we show that for ideals with the BW property we can obtain a strengthening of the above result.

LEMMA 2.7. *Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal. The ideal $\mathcal{I}$ has the BW property if and only if there are $\delta > 0$ and $x : \mathcal{P}(\omega) \to \{0,1\}$ such that*

$$\Big\| \bigcap \{A^{x(A)} : A \in \mathcal{A}\} \Big\|_\phi \geq \delta$$

for every finite and nonempty family $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

Proof. ($\Rightarrow$) By [9, Theorem 3.6] there exists $\delta > 0$ such that for every finite partition $A_1 \cup \cdots \cup A_n = \omega$ there exists $1 \leq i \leq n$ with $\|A_i\|_\phi \geq \delta$. We will show that this δ is as required.

For every finite and nonempty family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ we define

$$C_\mathcal{A} = \Big\{ x \in \{0,1\}^{\mathcal{P}(\omega)} : \Big\| \bigcap \{A^{x(A)} : A \in \mathcal{A}\} \Big\|_\phi \geq \delta \Big\}.$$

We will show that

(1) $C_\mathcal{A} \neq \emptyset$;
(2) $C_\mathcal{A}$ is a closed set in $\{0,1\}^{\mathcal{P}(\omega)}$;
(3) the family $\{C_\mathcal{A} : \mathcal{A}$ is finite and nonempty$\}$ is centered.

Then using compactness of the topological space $\{0,1\}^{\mathcal{P}(\omega)}$ we get

$$x \in \bigcap \{C_\mathcal{A} : \mathcal{A} \text{ is finite and nonempty}\}.$$

It is easy to see that this x is as required. Thus, the proof will be finished as soon as we derive (1)–(3).

(1) Take any finite and nonempty $\mathcal{A} \subseteq \mathcal{P}(\omega)$. Since the family

$$\Big\{ \bigcap \{A^{s(A)} : A \in \mathcal{A}\} : s \in \{0,1\}^\mathcal{A} \Big\}$$

is a finite partition of ω, there is $s \in \{0,1\}^\mathcal{A}$ with $\| \bigcap \{A^{s(A)} : A \in \mathcal{A}\}\|_\phi \geq \delta$. Then any $x \in \{0,1\}^{\mathcal{P}(\omega)}$ such that $s \subseteq x$ belongs to $C_\mathcal{A}$.

(2) Take any finite and nonempty $\mathcal{A} \subseteq \mathcal{P}(\omega)$. Since $S = \{x{\restriction}\mathcal{A} : x \in C_\mathcal{A}\} \subseteq \{0,1\}^\mathcal{A}$ is finite and $C_\mathcal{A} = \bigcup_{s \in S} \{x \in \{0,1\}^{\mathcal{P}(\omega)} : s \subseteq x\}$, $C_\mathcal{A}$ is a finite union of basic clopen sets, hence closed.

(3) Take any finite and nonempty $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq \mathcal{P}(\omega)$. Since $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ is finite, $C_\mathcal{A} \neq \emptyset$ by (1). On the other hand, it is not difficult to see that $C_\mathcal{A} \subseteq C_{\mathcal{A}_1} \cap \cdots \cap C_{\mathcal{A}_n}$.

($\Leftarrow$) Let $\delta > 0$ and $x : \mathcal{P}(\omega) \to \{0,1\}$ be such that

$$\Big\| \bigcap \{A^{x(A)} : A \in \mathcal{A}\} \Big\|_\phi \geq \delta$$

for every finite and nonempty family $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

By [9, Theorem 3.6], $\mathcal{I}$ has the BW property if and only if there is $\varepsilon > 0$ such that for every $N \in \omega$ and every partition $A_1, \ldots, A_N$ of ω there is $i \leq N$ with $\|A_i\|_\phi \geq \varepsilon$.

Let $\varepsilon = \delta$. Let $N \in \omega$ and $\mathcal{A} = \{A_1, \ldots, A_N\}$ be a partition of ω. Then there is $i \leq N$ with $x(A_i) = 0$ (otherwise $\bigcap\{A^{x(A)} : A \in \mathcal{A}\} = \emptyset$, hence $\|\bigcap\{A^{x(A)} : A \in \mathcal{A}\}\|_\phi = 0 < \delta$). Thus $A_i \supseteq \bigcap\{A^{x(A)} : A \in \mathcal{A}\}$, hence

$$\|A_i\|_\phi \geq \Big\|\bigcap\{A^{x(A)} : A \in \mathcal{A}\}\Big\|_\phi \geq \delta = \varepsilon. \blacksquare$$

3. F_σ ideals

PROPOSITION 3.1. *Let $\mathcal{I} = \mathrm{Fin}(\phi)$ be an F_σ ideal. Then $\mathfrak{s}(\mathcal{I}), \mathfrak{r}(\mathcal{I}) \geq \omega_1$.*

Proof. ($\mathfrak{s}(\mathcal{I}) \geq \omega_1$) Let $\mathcal{S} = \{S_n : n \in \omega\} \subseteq \mathcal{I}^+$. We will show that $\mathcal{S}$ is not an $\mathcal{I}$-splitting family, i.e. we will construct an $A \in \mathcal{I}^+$ such that for every $n \in \omega$, either $A \cap S_n^0 \in \mathcal{I}$ or $A \cap S_n^1 \in \mathcal{I}$.

Let $\varepsilon \in \{0,1\}^\omega$ be a sequence such that $\bigcap_{i \leq n} S_i^{\varepsilon_i} \in \mathcal{I}^+$ for every $n \in \omega$. By lsc of ϕ, we can find finite sets F_n ($n \in \omega$) such that $F_n \subseteq \bigcap_{i \leq n} S_i^{\varepsilon_i}$ and $\phi(F_n) \geq n$.

Let $A = \bigcup_n F_n$. Then $A \in \mathcal{I}^+$ and $A \cap S_n^{1-\varepsilon_n} \subseteq \bigcup_{i<n} F_i \in \mathcal{I}$ for every $n \in \omega$.

($\mathfrak{r}(\mathcal{I}) \geq \omega_1$) Let $\mathcal{R} = \{R_n : n \in \omega\} \subseteq \mathcal{I}^+$. We will show that $\mathcal{R}$ is not an $\mathcal{I}$-unsplittable family, i.e. we will construct a set $A \subseteq \omega$ such that $R_n \cap A^0 \in \mathcal{I}^+$ and $R_n \cap A^1 \in \mathcal{I}^+$ for every $n \in \omega$.

By lsc of ϕ, we can find pairwise disjoint finite sets $F_{i,n}^k$ ($i, n \in \omega$ and $k \in \{0,1\}$) such that $F_{i,n}^k \subseteq R_n$ and $\phi(F_{i,n}^k) \geq i$ for every $i, n \in \omega$ and $k \in \{0,1\}$.

Let $A = \bigcup_{i,n \in \omega} F_{i,n}^0$. If $n \in \omega$ and $k \in \{0,1\}$, then $R_n \cap A^k \supseteq \bigcup_{i \in \omega} F_{i,n}^k$ and hence $R_n \cap A^k \in \mathcal{I}^+$. $\blacksquare$

THEOREM 3.2. *Assume MA. Let $\mathcal{I} = \mathrm{Fin}(\phi)$ be an F_σ ideal. Then $\mathfrak{s}(\mathcal{I}) = \mathfrak{r}(\mathcal{I}) = \mathfrak{c}$.*

Proof. ($\mathfrak{s}(\mathcal{I}) = \mathfrak{c}$) Let $\mathcal{S} \subseteq \mathcal{P}(\omega)$ be such that $|\mathcal{S}| = \kappa < \mathfrak{c}$. We will show that $\mathcal{S}$ is not an $\mathcal{I}$-splitting family.

Let $x : \mathcal{P}(\omega) \to \{0,1\}$ be as in Corollary 2.5. Let $\mathcal{F} = \{S^{x(S)} : S \in \mathcal{S}\}$ and $\mathbb{P} = [\omega]^{<\omega} \times [\mathcal{F}]^{<\omega}$. For $(s, A), (t, B) \in \mathbb{P}$ we define $(s, A) \leq (t, B)$ if

(1) $s \supseteq t$,
(2) $A \supseteq B$, and
(3) $s \setminus t \subseteq \bigcap B$.

Then it is not difficult to show that $\langle \mathbb{P}, \leq \rangle$ is a ccc poset.

Define

$$D_F = \{(s,A) \in \mathbb{P} : F \in A\} \qquad \text{for every } F \in \mathcal{F},$$
$$D_n = \{(s,A) \in \mathbb{P} : \phi(s) > n\} \qquad \text{for every } n \in \omega.$$

It is easy to see that D_F is dense for every F. We show that D_n is also dense for every n.

Let $(s,A) \in \mathbb{P}$ and $A = \{F_0, \ldots, F_{m-1}\}$. Let $F_i = S_i^{x(S_i)}$ with $S_i \in \mathcal{S}$ for $i < m$. Since $\bigcap A = \bigcap_{i<m} F_i = \bigcap_{i<m} S_i^{x(S_i)}$, we have $\phi(\bigcap A) = \infty$. By lsc of ϕ there is a finite set $t \subseteq \bigcap A$ such that $\phi(t) > n$. Then $(s \cup t, A) \in D_n$ and $(s \cup t, A) \le (s, A)$.

By Martin's Axiom, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_n \neq \emptyset$ and $G \cap D_F \neq \emptyset$ for every $n \in \omega$ and $F \in \mathcal{F}$. Let

$$X = \bigcup \{s : (s,A) \in G\}.$$

Clearly $X \in \mathcal{I}^+$, and X is not $\mathcal{I}$-split by any member of $\mathcal{S}$ because if $F = S^{x(S)} \in \mathcal{F}$ and $(s,A) \in G \cap D_F$, then $X \cap S^{1-x(S)} \subseteq s$ and hence $X \cap S^{1-x(S)} \in \mathcal{I}$.

($\mathfrak{r}(\mathcal{I}) = \mathfrak{c}$) Let $\kappa < \mathfrak{c}$ and $\mathcal{F} = \{F_\alpha : \alpha < \kappa\} \subseteq \mathcal{I}^+$. We will show that there is a set which $\mathcal{I}$-splits all members of $\mathcal{F}$.

Let $\mathbb{P} = 2^{<\omega}$. Then $\langle \mathbb{P}, \supseteq \rangle$ is a ccc poset. Define

$$D_{\alpha,n} = \{s \in \mathbb{P} : \phi(s^{-1}(0) \cap F_\alpha) > n \wedge \phi(s^{-1}(1) \cap F_\alpha) > n\}$$

for every $n \in \omega$ and $\alpha < \kappa$. Using lsc of ϕ it is not difficult to show that the sets $D_{\alpha,n}$ are dense in $\mathbb{P}$.

By Martin's Axiom, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_{\alpha,n} \neq \emptyset$ for every $n \in \omega$ and $\alpha < \kappa$. Let

$$f = \bigcup G \quad \text{and} \quad X = f^{-1}(0).$$

It is easy to see that $X \in \mathcal{I}^+$. We will show that X $\mathcal{I}$-splits all sets in $\mathcal{F}$.

Let $\alpha < \kappa$. For any $n \in \omega$ there is $s_n \in G \cap D_{\alpha,n}$. Since $F_\alpha \cap X^i \supseteq F_\alpha \cap s_n^{-1}(i)$ for $i = 0,1$ and every n, we have $\phi(F_\alpha \cap X^i) > n$ for $i = 0,1$ and every n, and so $F_\alpha \cap X^i \in \mathcal{I}^+$ $(i = 0,1)$. ∎

4. Analytic P-ideals

PROPOSITION 4.1. *Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal. Then $\mathfrak{r}(\mathcal{I}) \geq \omega_1$.*

Proof. Let $\mathcal{F} = \{F_n \in \mathcal{I}^+ : n \in \omega\}$. We will show that there is a set which $\mathcal{I}$-splits all members of $\mathcal{F}$.

Let $\delta_n > 0$ be such that $\|F_n\|_\phi > \delta_n$ for every $n \in \omega$. Let $\langle G_n : n \in \omega \rangle$ be a sequence such that $\{G_n : n \in \omega\} = \{F_n : n \in \omega\}$ and $\{k \in \omega : G_k = F_n\}$ is infinite for each $n \in \omega$. Let $f : \omega \to \omega$ be such that $G_n = F_{f(n)}$ for every

$n \in \omega$. We will construct sequences $\langle s_n : n \in \omega \rangle$ and $\langle t_n : n \in \omega \rangle$ such that

(1) s_n, t_n are finite,
(2) $s_n, t_n \subseteq G_n \setminus \{0, 1, \ldots, n-1\}$ for every $n \in \omega$,
(3) $s_n \cap s_k = \emptyset$, $t_n \cap t_k = \emptyset$ and $s_n \cap t_k = \emptyset$ for every $n, k \in \omega$,
(4) $\phi(s_n) > \delta_{f(n)}$, $\phi(t_n) > \delta_{f(n)}$.

Suppose that we have already constructed s_i, t_i for $i \le n$. Let $s = s_0 \cup \cdots \cup s_n$ and $t = t_0 \cup \cdots \cup t_n$. Let $G = G_{n+1} \setminus (s \cup t)$. Since $s \cup t$ is finite we have $\|G\|_\phi > \delta_{f(n+1)}$. By the definition of $\|\cdot\|_\phi$ and lsc of ϕ there is a finite set $s_{n+1} \subseteq G \setminus \{0, 1, \ldots, n\}$ with $\phi(s_{n+1}) > \delta_{f(n+1)}$. Applying the definition of $\|\cdot\|_\phi$ and lsc of ϕ again, there is a finite set $t_{n+1} \subseteq G \setminus s_{n+1}$ with $\phi(t_{n+1}) > \delta_{f(n+1)}$.

Let $X = \bigcup_{n \in \omega} s_n$. Then $s_n \subseteq G_n \setminus \{0, 1, \ldots, n-1\} = F_0 \setminus \{0, 1, \ldots, n-1\}$ for every $n \in f^{-1}(0)$. Thus $\phi(X \setminus \{0, 1, \ldots, n-1\}) \ge \phi(s_n) > \delta_0 > 0$ for every $n \in f^{-1}(0)$, hence $\|X\|_\phi \ge \delta_0 > 0$. We will show that X $\mathcal{I}$-splits all sets in the family $\mathcal{F}$.

First of all, we will show that $F_k \cap X \in \mathcal{I}^+$. Let $i \in \omega$. Then there is $n \in f^{-1}(k)$ with $n > i$. Then

$$\begin{aligned}\phi((F_k \cap X) \setminus \{0, 1, \ldots, i-1\}) &= \phi((G_n \cap X) \setminus \{0, 1, \ldots, i-1\}) \\ &\ge \phi((G_n \cap X) \setminus \{0, 1, \ldots, n-1\}) \ge \phi(s_n) > \delta_k.\end{aligned}$$

Thus $\|F_k \cap X\|_\phi \ge \delta_k > 0$.

Using the same argument as above one can show that $F_k \setminus X \in \mathcal{I}^+$. ∎

THEOREM 4.2. *Assume MA. Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal. Then $\mathfrak{r}(\mathcal{I}) = \mathfrak{c}$.*

Proof. Let $\kappa < \mathfrak{c}$ and $\mathcal{F} = \{F_\alpha : \alpha < \kappa\} \subseteq \mathcal{I}^+$. Let $\delta_\alpha > 0$ be such that $\|F_\alpha\|_\phi > \delta_\alpha$ for every $\alpha < \kappa$.

Let $\mathbb{P} = 2^{<\omega}$. Then $\langle \mathbb{P}, \supseteq \rangle$ is a ccc poset. Define

$$D_{\alpha,n} = \big\{ s \in \mathbb{P} : \phi\big((F_\alpha \cap s^{-1}(i)) \setminus \{0, 1, \ldots, n-1\}\big) > \delta_\alpha \text{ for } i = 0, 1 \big\}$$

for every $n \in \omega$ and $\alpha < \kappa$. It is not difficult to show that $D_{\alpha,n}$ is dense in $\mathbb{P}$.

By Martin's Axiom, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_{\alpha,n} \neq \emptyset$ for every $n \in \omega$ and $\alpha < \kappa$. Let $f = \bigcup G$ and $X = f^{-1}(0)$. Then $X \in \mathcal{I}^+$ and X $\mathcal{I}$-splits all sets in $\mathcal{F}$. ∎

THEOREM 4.3. *Assume MA. Let $\mathcal{I} = \mathrm{Exh}(\phi)$ be an analytic P-ideal with the BW property. Then $\mathfrak{s}(\mathcal{I}) = \mathfrak{c}$.*

Proof. Let $\mathcal{S} \subseteq \mathcal{P}(\omega)$ be such that $|\mathcal{S}| = \kappa < \mathfrak{c}$. We will show that $\mathcal{S}$ is not an $\mathcal{I}$-splitting family.

Let $\delta > 0$ and $x : \mathcal{P}(\omega) \to \{0, 1\}$ be as in Lemma 2.7.

Let $\mathcal{F} = \{S^{x(S)} : S \in \mathcal{S}\}$ and $\mathbb{P} = [\omega]^{<\omega} \times [\mathcal{F}]^{<\omega}$. For $(s, A), (t, B) \in \mathbb{P}$ we define $(s, A) \le (t, B)$ if

$$s \supseteq t, \quad A \supseteq B, \quad s \setminus t \subseteq \bigcap B.$$

Then it is not difficult to show that $\langle \mathbb{P}, \le \rangle$ is a ccc poset.

Define

$$\begin{aligned} D_n &= \{(s, A) \in \mathbb{P} : \phi(s \setminus \{0, 1, \ldots, n-1\}) > \delta/2\} && \text{for every } n \in \omega, \\ D_F &= \{(s, A) \in \mathbb{P} : F \in A\} && \text{for every } F \in \mathcal{F}. \end{aligned}$$

Clearly D_F is dense for every $F \in \mathcal{F}$. We will show that the sets D_n are dense.

Let $(s, A) \in \mathbb{P}$ and $A = \{F_0, \ldots, F_{m-1}\}$. Let $F_i = S_i^{x(S_i)}$ with $S_i \in \mathcal{S}$ for $i < m$. Since $\bigcap A = \bigcap_{i<m} F_i = \bigcap_{i<m} S_i^{x(S_i)}$, we have $\|\bigcap A\|_\phi \ge \delta$. Since

$$\Big\|\bigcap A\Big\|_\phi = \lim_{k\to\infty} \phi\Big(\bigcap A \setminus \{0, 1, \ldots, k-1\}\Big)$$

it follows that $\phi(\bigcap A \setminus \{0, 1, \ldots, n-1\}) > \delta/2$. By lsc of ϕ there is a finite set $t \subseteq \bigcap A \setminus \{0, 1, \ldots, n-1\}$ such that $\phi(t) > \delta/2$. Then $(s \cup t, A) \in D_n$ and $(s \cup t, A) \le (s, A)$.

By Martin's Axiom, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_n \neq \emptyset$ and $G \cap D_F \neq \emptyset$ for every $n \in \omega$ and $F \in \mathcal{F}$. Let

$$X = \bigcup \{s : (s, A) \in G\}.$$

Clearly, $\|X\|_\phi \ge \delta/2$ so $X \in \mathcal{I}^+$, and X is not $\mathcal{I}$-split by any member of $\mathcal{S}$ because if $S \in \mathcal{S}$, $F = S^{x(S)}$, and $(s, A) \in G \cap D_F$, then $X \cap S^{1-x(S)} \subseteq s$. ∎

5. Applications. It is not difficult to prove that the Bolzano–Weierstrass theorem (that every bounded sequence of reals has a convergent subsequence) fails if we consider sequences of functions instead of reals (i.e. there exists a uniformly bounded sequence $(f_n)_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ such that no subsequence of $(f_n)_{n\in\omega}$ is pointwise convergent). The ideal version of this result is presented below (in this case we have to consider two cases: either $\mathcal{I}$ is a "somewhere" maximal ideal or not).

Let $\mathcal{I}$ be an ideal on ω and $A \subseteq \omega$. We say that a sequence $(f_n)_{n\in A}$ of real-valued functions defined on a set X is *pointwise $\mathcal{I}$-convergent* to $f : X \to \mathbb{R}$ if for every $x \in X$ the sequence of reals $(f_n(x))_{n\in A}$ is $\mathcal{I}$-convergent to $f(x)$. (See [18], [20] and [6] for description of pointwise $\mathcal{I}$-limits of continuous functions; in [12], [5] and [11] the authors also consider an ideal version of discrete convergence and equal convergence of sequences of functions.)

For an ideal $\mathcal{I}$ on ω and $A \subseteq \omega$ we define the ideal $\mathcal{I}\restriction A = \{B \subseteq \omega : B \cap A \in \mathcal{I}\}$.

PROPOSITION 5.1. *Let $\mathcal{I}$ be an ideal on ω. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \omega)$ be a uniformly bounded sequence of functions.*

(1) *If $\mathcal{I}$ is a maximal ideal then $(f_n)_{n\in\omega}$ is pointwise $\mathcal{I}$-convergent.*
(2) *If there is $A \in \mathcal{I}^+$ such that $\mathcal{I}\restriction A$ is a maximal ideal then the subsequence $(f_n)_{n\in A}$ is pointwise $\mathcal{I}$-convergent.*

Proof. (1) Follows from the fact that every bounded sequence of reals is $\mathcal{I}$-convergent, for each maximal ideal $\mathcal{I}$.

(2) Follows from (1). ∎

PROPOSITION 5.2. *Let $\mathcal{I}$ be an ideal on ω such that $\mathcal{I}\restriction A$ is not maximal for any $A \in \mathcal{I}^+$. There exists a uniformly bounded sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \omega)$ such that $(f_n)_{n\in A}$ is not pointwise $\mathcal{I}$-convergent for any $A \in \mathcal{I}^+$.*

Proof. Let $\{0,1\}^\omega = \{s_\alpha : \alpha < \mathfrak{c}\}$ and $\mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\}$. We define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x_\alpha) = s_\alpha(n)$ $(n \in \omega,\ \alpha < \mathfrak{c})$.

Let $A \in \mathcal{I}^+$. Then there are $B, C \subseteq \omega$ such that $A = B \cup C$, $B \cap C = \emptyset$ and $B, C \in \mathcal{I}^+$.

Let α be such that $s_\alpha(n) = 0$ for $n \in B$ and $s_\alpha(n) = 1$ for $n \in C$.

Since $\mathcal{I}^+ \ni C \subseteq \{n : f_n(x_\alpha) \neq 0\}$ and $\mathcal{I}^+ \ni B \subseteq \{n : f_n(x_\alpha) \neq 1\}$, the sequence $(f_n)_{n\in A}$ is not $\mathcal{I}$-convergent. ∎

Saks asked (see [24]) whether for every uniformly bounded sequence $(f_n)_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ there exists an infinite set $A \subseteq \omega$ such that the subsequence $(f_n(x))_{n\in A}$ is convergent for uncountably many $x \in \mathbb{R}$. This question was answered in the negative by Sierpiński [24] under the assumption of the Continuum Hypothesis (CH). Later, Fuchino and Plewik [14] proved that if $\mathfrak{s} > \omega_1$ then the answer to the question is positive. In fact, they proved that for every uniformly bounded sequence $f_n : \mathbb{R} \to \mathbb{R}$ and every $X \subseteq \mathbb{R}$ with $|X| < \mathfrak{s}$ there exists an infinite $A \subseteq \omega$ such that $(f_n\restriction X)_{n\in A}$ is pointwise convergent. The ideal versions of these results are presented below.

First, if $\mathcal{I}$ is a "somewhere" maximal ideal then the answer to the ideal version of Saks' question is positive (by Proposition 5.1).

Second, if $\mathcal{I} \notin \mathrm{BW}$ then there exists (in ZFC) a uniformly bounded sequence $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \omega)$ such that for every $A \in \mathcal{I}^+$ the subsequence $(f_n(x))_{n\in A}$ is not pointwise $\mathcal{I}$-convergent for any $x \in \mathbb{R}$. (Indeed, let $(x_n)_{n\in\omega}$ be a bounded sequence such that $(x_n)_{n\in A}$ is not $\mathcal{I}$-convergent for any $A \in \mathcal{I}^+$. Then the functions $f_n(x) = x_n$ $(n \in \omega,\ x \in \mathbb{R})$ are as required.) Thus, the answer to the ideal version of Saks' question is negative.

Below (Corollaries 5.4 and 5.7) we prove that in the third case (i.e. $\mathcal{I} \in \mathrm{BW}$ and $\mathcal{I}\restriction A$ is not a maximal ideal) the answer to the ideal version of Saks' question is independent of ZFC for F_σ ideals and analytic P-ideals.

Proposition 5.3. *Let $\mathcal{I}$ be an ideal on ω. If $\mathfrak{r}(\mathcal{I}) = \mathfrak{c}$ then there exists a uniformly bounded sequence $(f_n)_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ such that for every $A \in \mathcal{I}^+$ the subsequence $(f_n(x))_{n\in A}$ is $\mathcal{I}$-convergent for less than $\mathfrak{c}$ many $x \in \mathbb{R}$.*

Proof. Let $\mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{I}^+ = \{A_\alpha : \alpha < \mathfrak{c}\}$. We define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x_\alpha) = \begin{cases} 0 & \text{for } n \in S_\alpha, \\ 1 & \text{for } n \in \omega \setminus S_\alpha, \end{cases}$$

where $S_\alpha \in \mathcal{I}^+$ is a set that $\mathcal{I}$-splits the family $\{A_\beta : \beta < \alpha\}$ (there is one since $|\alpha| < \mathfrak{r}(\mathcal{I})$).

Let $A = A_\beta \in \mathcal{I}^+$. We will show that the subsequence $(f_n(x_\alpha))_{n\in A}$ is not $\mathcal{I}$-convergent for every $\alpha > \beta$, and that will finish the proof.

Let $\alpha > \beta$. Then $\{n \in A : f_n(x_\alpha) = 0\} = A_\beta \cap S_\alpha \in \mathcal{I}^+$ and $\{n \in A : f_n(x_\alpha) = 1\} = A_\beta \setminus S_\alpha \in \mathcal{I}^+$. Thus $(f_n(x_\alpha))_{n\in A}$ is not $\mathcal{I}$-convergent. ■

Corollary 5.4. *Assume CH. Let $\mathcal{I}$ be an F_σ ideal or an analytic P-ideal on ω. There exists a uniformly bounded sequence $(f_n)_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ such that $\{x : (f_n(x))_{n\in A}$ is $\mathcal{I}$-convergent$\}$ is countable for every $A \in \mathcal{I}^+$.*

Proof. Apply Proposition 5.3 and Proposition 3.1 or 4.1 respectively. ■

Proposition 5.5. *Let $\mathcal{I}$ be an ideal on ω with the BW property. Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n \in \omega)$ be a uniformly bounded sequence of functions. Let $X \subseteq \mathbb{R}$ be such that $|X| < \mathfrak{s}(\mathcal{I})$. There exists $A \in \mathcal{I}^+$ such that $(f_n{\restriction}X)_{n\in A}$ is pointwise $\mathcal{I}$-convergent.*

Proof. The proof is a slight modification of the proof of [14, Lemma 4]. We provide it for completeness.

Let $|X| = \kappa < \mathfrak{s}(\mathcal{I})$. For every $x, y \in \mathbb{R}$ let $C_x^y = \{n \in \omega : f_n(x) < y\}$. Let $\mathcal{C} = \{C_x^q : q \in \mathbb{Q}, x \in X\}$. Since $|\mathcal{C}| < \mathfrak{s}(\mathcal{I})$, there exists $A \in \mathcal{I}^+$ such that $A \cap C \in \mathcal{I}$ or $A \setminus C \in \mathcal{I}$ for every $C \in \mathcal{C}$.

We claim that $(f_n{\restriction}X)_{n\in A}$ is $\mathcal{I}$-convergent to the function $f : X \to \mathbb{R}$ given by $f(x) = \inf\{y \in \mathbb{R} : \{n \in A : f_n(x) < y\} \in \mathcal{I}^+\} = \inf\{y \in \mathbb{R} : A \cap C_x^y \in \mathcal{I}^+\}$.

Let $x \in X$ and $\varepsilon > 0$. Let $B_1 = \{n \in A : f_n(x) < f(x) - \varepsilon\}$ and $B_2 = \{n \in A : f_n(x) > f(x) + \varepsilon\}$.

Since $\{n \in A : |f_n(x) - f(x)| > \varepsilon\} = B_1 \cup B_2$, it is enough to show that $B_1, B_2 \in \mathcal{I}$.

Suppose that $B_1 \in \mathcal{I}^+$. Since $A \cap C_x^{f(x)-\varepsilon} = B_1 \in \mathcal{I}^+$, it follows that $f(x) = \inf\{y \in \mathbb{R} : A \cap C_x^y \in \mathcal{I}^+\} \le f(x) - \varepsilon$, a contradiction.

Suppose that $B_2 \in \mathcal{I}^+$. Let $q \in \mathbb{Q}$ be such that $f(x) < q < f(x) + \varepsilon$. Since $B_2 \subseteq A \setminus C_x^q$, we have $A \setminus C_x^q \notin \mathcal{I}$. But $C_x^q \in \mathcal{C}$ and $\mathcal{C}$ does not

$\mathcal{I}$-split A, so $A \cap C_x^q \in \mathcal{I}$. Thus $f(x) = \inf\{y \in \mathbb{R} : A \cap C_x^y \in \mathcal{I}^+\} \geq q$, a contradiction. ∎

REMARK 5.6. The assumption that $\mathcal{I}$ has the BW property is necessary in Proposition 5.5. Indeed, let $\mathcal{I}$ be an ideal without the BW. By Theorem 2.3, $\mathfrak{s}(\mathcal{I}) = \omega$. If $(f_n)_{n\in\omega}$ is the sequence defined before Proposition 5.3, and $X = \{0\}$, then $|X| < \mathfrak{s}(\mathcal{I})$, but $(f_n{\restriction}X)_{n\in A} = (x_n)_{n\in A}$ is not $\mathcal{I}$-convergent for any $A \in \mathcal{I}^+$.

COROLLARY 5.7. *Assume MA and $\neg$CH. Let $\mathcal{I}$ be an F_σ ideal, or an analytic P-ideal with the BW property, on ω. For every uniformly bounded sequence $(f_n)_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ there exists $A \in \mathcal{I}^+$ such that the subsequence $(f_n(x))_{n\in A}$ is $\mathcal{I}$-convergent for uncountably many $x \in \mathbb{R}$.*

Proof. Apply Proposition 5.5 and Theorems 3.2 and 4.3 respectively. ∎

Mazurkiewicz [22] proved that if one takes a uniformly bounded sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ ($n \in \omega$) then there always exist a perfect set $P \subseteq \mathbb{R}$ and an infinite set $A \subseteq \omega$ such that $(f_n(x))_{n\in A}$ is convergent for every $x \in P$. (Since perfect sets are uncountable, his result yields a positive answer to Saks' question in the realm of continuous functions.) In [10] it is proved that the ideal version of Mazurkiewicz's result holds for F_σ ideals and for analytic P-ideals with the BW property.

Mazurkiewicz's result shows (taking into account that perfect sets are of cardinality $\mathfrak{c}$) that for a uniformly bounded sequence $(f_n)_{n\in\omega}$ of continuous functions one always finds an infinite $A \subseteq \omega$ such that the subsequence $(f_n(x))_{n\in A}$ is convergent for $\mathfrak{c}$ many $x \in \mathbb{R}$. Of course, Sierpiński's result shows that under CH there is a uniformly bounded sequence $(f_n)_{n\in\omega}$ such that there is no infinite $A \subseteq \omega$ such that $(f_n(x))_{n\in A}$ is convergent for $\mathfrak{c}$ many $x \in \mathbb{R}$. Ciesielski and Pawlikowski [4] proved that it is consistent with the axioms of ZFC that for every uniformly bounded sequence $(f_n)_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ there exists an infinite $A \subseteq \omega$ such that $(f_n(x))_{n\in A}$ is convergent for $\mathfrak{c}$ many $x \in \mathbb{R}$. We do not know if the result of Ciesielski and Pawlikowski can be generalized to ideal convergence.

It is known (see e.g. [4] or [19]) that under MA for every uniformly bounded sequence $(f_n)_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ and every $|X| < \mathfrak{c}$ there exists an infinite $A \subseteq \omega$ such that the subsequence $(f_n{\restriction}X)_{n\in A}$ is pointwise convergent, and on the other hand, there exists a uniformly bounded sequence $(f_n)_{n\in\omega}$ of real-valued functions defined on $\mathbb{R}$ such that for every infinite $A \subseteq \omega$ the subsequence $(f_n(x))_{n\in A}$ is convergent for less than $\mathfrak{c}$ many $x \in \mathbb{R}$.

COROLLARY 5.8. *Assume MA. Let $\mathcal{I}$ be an F_σ ideal, or an analytic P-ideal with the BW property, on ω. For every uniformly bounded sequence*

$(f_n)_{n\in\omega}$ *of real-valued functions defined on* $\mathbb{R}$ *and every* $|X| < \mathfrak{c}$ *there exists* $A \in \mathcal{I}^+$ *such that the subsequence* $(f_n \restriction X)_{n\in A}$ *is pointwise* $\mathcal{I}$*-convergent.*

Proof. Apply Proposition 5.5 and Theorems 3.2 or 4.3 respectively. ∎

COROLLARY 5.9. *Assume MA. Let* $\mathcal{I}$ *be an* F_σ *ideal or an analytic P-ideal on* ω*. There exists a uniformly bounded sequence* $(f_n)_{n\in\omega}$ *of real-valued functions defined on* $\mathbb{R}$ *such that for every* $A \in \mathcal{I}^+$ *the subsequence* $(f_n(x))_{n\in A}$ *is* $\mathcal{I}$*-convergent for less than* $\mathfrak{c}$ *many* $x \in \mathbb{R}$*.*

Proof. Apply Proposition 5.3 and Theorems 3.2 or 4.2 respectively. ∎

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