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EXPANSIONS OF BINARY RECURRENCES IN THE ADDITIVE BASE FORMED BY THE NUMBER OF DIVISORS OF THE FACTORIAL

BY

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Abstract. We note that every positive integer N has a representation as a sum of distinct members of the sequence $\{d(n!)\}_{n\geq 1}$, where $d(m)$ is the number of divisors of m. When N is a member of a binary recurrence $\mathbf{u} = \{u_n\}_{n\geq 1}$ satisfying some mild technical conditions, we show that the number of such summands tends to infinity with n at a rate of at least $c_1 \log n/\log\log n$ for some positive constant c_1. We also compute all the Fibonacci numbers of the form $d(m!)$ and $d(m_1!) + d(m_2)!$ for some positive integers m, m_1, m_2.

1. Introduction. Let $d(m)$ be the number of divisors of the positive integer m. There are a few papers in the literature addressing the function $d(n!)$. For example, several results about this function can be found in [2]. In the more recent paper [6], it was shown that $d(n!)$ is a divisor of $n!$ for all $n \geq 6$.

For simplicity, put $a_n = d(n!)$. The sequence $\{a_n\}_{n\geq 1}$ starts as

$$1, 2, 4, 8, 16, 30, 60, 96, 160, 270, 540, 792, \ldots.$$

It would seem that $a_n \leq 2a_{n-1}$ with several cases of equality. This is our first result.

Proposition 1.1. *The inequality $a_n \leq 2a_{n-1}$ holds for all $n \geq 2$.*

Since $a_n \leq 2a_{n-1}$ for all $n \geq 1$ and $a_1 = 1$, it then follows, by the greedy algorithm, that for every positive integer N there exist $k \geq 1$ and $m_k > \cdots > m_1 \geq 1$ such that

$$N = a_{m_k} + \cdots + a_{m_1}.$$

There might be several such representations of a given N, but we shall pick the one with a minimal k. For example,

$$55 = 30 + 16 + 8 + 1 = a_6 + a_5 + a_4 + a_1.$$

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In this paper, we take a binary recurrent sequence $\mathbf{u} = \{u_n\}_{n\geq 0}$ where u_0, u_1 are integers

$$u_{n+2} = ru_{n+1} + su_n \quad \text{for all } n \geq 0,$$

and r and s are fixed nonzero integers. Assuming that $\Delta = r^2 + 4s \neq 0$ and writing α, β for the two roots of the quadratic equation $x^2 - rx - s = 0$, we see that there exist c and d in $\mathbb{Q}(\alpha)$ with

$$u_n = c\alpha^n + d\beta^n \quad \text{for all } n \geq 0. \tag{1.1}$$

The sequence $\mathbf{u}$ is called *nondegenerate* if $cd \neq 0$ and α/β is not a root of 1. Since α/β is not a root of 1, the sequence u_n has at most one zero. That is, $u_n \neq 0$ for all $n \geq 0$ with at most one exception.

We study the representation

$$|u_n| = a_{m_k} + \cdots + a_{m_1} \quad \text{with } m_k > \cdots > m_1 \geq 1. \tag{1.2}$$

Under mild assumptions on the sequence $\mathbf{u}$, we show that the number of terms k in the above representation grows with n.

THEOREM 1.2. *Assume that* $\mathbf{u}$ *is nondegenerate and has* $\gcd(r, s) = 1$. *Then there exist positive constants* n_0 *and* $c_0 > 0$ *depending on* $\mathbf{u}$ *which are computable such that in the representation* (1.2) *we have*

$$k > \frac{c_0 \log n}{\log\log n} \quad \textit{whenever } n > n_0.$$

Perhaps the condition $\gcd(r, s) = 1$ can be removed but we have not succeeded in doing so. Representation problems similar to the above one involving members of binary recurrences, factorials or both have been studied in the literature. For example, it was shown in [11] and [4] that if $b \geq 2$ is an integer and

$$|u_n| = d_k b^{m_k} + \cdots + d_1 b^{m_1} \quad \text{where } d_i \in \{1, \ldots, b-1\} \ (1 \leq i \leq k),$$

and $m_k > \cdots > m_1 \geq 1$, then $k > c_1 \log n/\log\log n$ whenever $n > n_0$, where $c_1 > 0$ is a computable constant depending on $\mathbf{u}$ and b. If we write

$$n! = d_k b^{m_k} + \cdots + d_1 b^{m_1} \quad \text{where } d_i \in \{1, \ldots, b-1\} \ (1 \leq i \leq k),$$

where again $m_k > \cdots > m_1 \geq 1$, then $k > c_2 \log n$, where $c_2 > 0$ depends only on b (see [5]). Finally, in [3], it was shown that if

$$|u_n| = m_1! + \cdots + m_k! \quad \text{with } m_k > \cdots > m_1 \geq 1, \tag{1.3}$$

and k is fixed, then n is bounded by some computable number depending on $\mathbf{u}$ and k. In particular, in any representation (1.3) with large n, the parameter k tends to infinity with n. Although this is not explicitly stated in [3], from the proof of the main result in that paper one can deduce $k > c_3 \log n/\log\log n$, whenever $n > n_0$, where $c_3 > 0$ is some computable constant depending on $\mathbf{u}$.

When $r = s = 1$, $u_0 = 0$ and $u_1 = 1$ the sequence $\mathbf{u}$ becomes the famous sequence $\mathbf{F} = \{F_n\}_{n\geq 0}$ of Fibonacci numbers. We prove the following result.

THEOREM 1.3. *If* $F_n = a_m$, *then* $n \in \{1, 2, 3, 6\}$. *If* $F_n = a_{m_2} + a_{m_1}$ *with* $m_2 > m_1 \geq 1$, *then* $n \in \{4, 5, 9\}$.

2. Proof of Proposition 1.1. The fact that the stated inequality is an equality when n is prime is a consequence of the multiplicativity of the function "number of divisors". The stated inequality also holds for $n = 4, 8$. Assume now that $n \notin \{4, 8\}$ is composite.

Inequality (5) in [2] shows that

$$\frac{a_n}{a_{n-1}} \leq \exp \frac{S(n)}{n}, \tag{2.1}$$

where

$$S(n) = \sum_{p^{\alpha_p} \| n} \alpha_p p$$

is the sum of the prime factors of n with multiplicity. Note that

$$\frac{\alpha_p p}{p^{\alpha_p}} \leq 1 \quad (\alpha_p \geq 1,\, p \geq 2) \quad \text{and} \quad \frac{\alpha_p}{p^{\alpha_p - 1}} \leq \frac{2}{3} \quad (\alpha_p \geq 2,\, p \geq 3).$$

Put $\omega(n) = k$. If $k = 1$, then $n = p^{\alpha_p}$ for some prime p and $\alpha_p \geq 2$. If $p = 2$, then $\alpha_p \geq 4$, therefore $S(n)/n = \alpha_p/2^{\alpha_p - 1} \leq 1/2$, while if $p \geq 3$, then $S(n)/n \leq 2/3$ by the above inequality. If $\omega(n) = k \geq 3$ and $p_1 < \cdots < p_k$ are all the distinct prime factors of n, then

$$\frac{S(n)}{n} \leq \sum_{j=1}^{k} \frac{1}{\prod_{i \neq j} p_i} \leq \frac{k}{p_1 \cdots p_{k-1}} \leq \frac{k}{k!} = \frac{1}{(k-1)!} \leq \frac{1}{2}. \tag{2.2}$$

If $k = 2$ and n is divisible by some prime $p \geq 7$, then

$$\frac{S(n)}{n} \leq \frac{1}{2} + \frac{1}{7} = \frac{9}{14}. \tag{2.3}$$

If $n = 3^a \cdot 5^b$, then

$$\frac{S(n)}{n} \leq \frac{1}{3} + \frac{1}{5} = \frac{8}{15}. \tag{2.4}$$

Finally, if $n = 2^a \cdot 3^b$ or $n = 2^a \cdot 5^b$, then

$$\frac{S(n)}{n} \leq \frac{2a + 3b}{2^a 3^b} \quad \text{and} \quad \frac{S(n)}{n} \leq \frac{2a + 5b}{2^a 5^b},$$

respectively. Unless $n = 6, 10$, we have either $a \geq 2$ or $b \geq 2$, so

$$\frac{S(n)}{n} \leq \max\left\{\frac{7}{12}, \frac{4}{9}, \frac{9}{20}, \frac{6}{25}\right\} = \frac{7}{12}. \tag{2.5}$$

In conclusion, collecting all the above inequalities (2.2)–(2.5), we find that if n is composite, then, unless $n \in \{6, 10\}$, we have

$$\frac{S(n)}{n} \leq \max\left\{\frac{1}{2}, \frac{2}{3}, \frac{9}{14}, \frac{8}{15}, \frac{7}{12}\right\} = \frac{2}{3},$$

so

$$\frac{a_n}{a_{n-1}} < \exp\frac{2}{3} < 2$$

by (2.1). One can check that the inequality $a_n \leq 2a_{n-1}$ holds also for the values $n = 6, 10$.

3. Preliminary results for the proofs of Theorems 1.2 and 1.3. We write $\pi(x)$ for the counting function of the primes $p \leq x$.

LEMMA 3.1. *The following inequalities hold:*

(i)

$$\begin{aligned} \pi(x) &> \frac{x}{\log x} && \textit{for all } x \geq 17,\\ \pi(x) &< \frac{5x}{4\log x} && \textit{for all } x \geq 114,\\ \pi(x) &> \frac{x}{\log x - 0.5} && \textit{for all } x \geq 67,\\ \pi(x) &< \frac{x}{\log x - 1.5} && \textit{for all } x \geq 5. \end{aligned}$$

(ii)

$$\pi(x) - \pi\left(\frac{x}{2}\right) > \frac{3x}{10\log x} \qquad \textit{for all } x \geq 41.$$

(iii) *For each prime number q, there exists x_0 depending on q such that*

$$\pi\left(\frac{x}{q-1}\right) - \pi\left(\frac{x}{q}\right) > \frac{x}{2q^2\log x} \qquad \textit{for all } x > x_0.$$

(iv)

$$\begin{aligned} \log a_n &> \frac{n}{\log n} && \textit{for all } n \geq 17,\\ \log a_n &< \frac{3n}{\log n} && \textit{for all } n \geq 286. \end{aligned}$$

Proof. Parts (i) and (ii) follow from inequalities (3.3), (3.4), (3.5), (3.7) and (3.8) in [7]. Part (iii) follows from the last two inequalities in (i). For part (iv), we start with the lower bound and assume that $n \geq 17$. Recall that

$$a_n = d(n!) = \prod_{p\leq n}(\alpha_p(n) + 1), \quad \text{where} \quad \alpha_p(n) = \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \cdots.$$

Thus,

$$\alpha_p(n) + 1 \geq \left\lfloor \frac{n}{p} \right\rfloor + 1 > \frac{n}{p}$$

leading to

$$\begin{aligned}
\log a_n &\geq \log \prod_{p \leq n} \frac{n}{p} = \sum_{p \leq n} (\log n - \log p) = \pi(n)\log n - \sum_{p \leq n} \log p \\
&= \pi(n)\log n - \left(\pi(n)\log n - \int_2^n \frac{\pi(t)}{t}\,dt \right) \\
&= \int_2^n \frac{\pi(t)}{t}\,dt = \int_{17}^n \frac{\pi(t)}{t}\,dt + \int_2^{17} \frac{\pi(t)}{t}\,dt \\
&\geq \int_{17}^n \frac{dt}{\log t} + c_1 = \frac{t}{\log t}\bigg|_{t=17}^{n} + \int_{17}^n \frac{dt}{(\log t)^2} + c_1 \\
&> \frac{n}{\log n} + (c_1 - c_2) > \frac{n}{\log n},
\end{aligned}$$

where

$$c_1 = \int_2^{17} \frac{\pi(t)}{t}\,dt = 6.68933\ldots > c_2 = \frac{17}{\log 17} = 6.00025\ldots.$$

and where in the above chain of inequalities we also used integration by parts, the Abel summation formula, as well as (i).

For the upper bound, we assume that $n \geq 286$ and use the fact that

$$\alpha_p(n) \leq \frac{n}{p} + \frac{n}{p^2} + \cdots = \frac{n}{p-1},$$

to get

$$\begin{aligned}
\log a_n &\leq \log \prod_{p \leq n} \left(\frac{n}{p-1} + 1 \right) = \log \prod_{p \leq n} \frac{n}{p}\,\frac{p}{p-1} \left(1 + \frac{p-1}{n} \right) \\
&= \sum_{p \leq n} (\log n - \log p) + \sum_{p \leq n} \log \frac{p}{p-1} + \frac{1}{n} \sum_{p \leq n} p = S_1 + S_2 + S_3.
\end{aligned}$$

We have already seen that

$$S_1 = \int_2^n \frac{\pi(t)}{t}\,dt,$$

but we need an upper bound for it. Using (i), we have

$$S_1 \leq \frac{5}{4} \int_{114}^n \frac{dt}{\log t} + \int_2^{114} \frac{\pi(t)}{t}\,dt = \frac{5}{4} \int_{114}^n \frac{dt}{\log t} + c_3,$$

where $c_3 = \int_2^{114}(\pi(t)/t)\,dt$. Integrating by parts, we get

$$\int_{114}^{n} \frac{dt}{\log t} = \frac{t}{\log t}\bigg|_{t=114}^{n} + \int_{114}^{n} \frac{dt}{(\log t)^2} \le \frac{n}{\log n} - c_4 + \frac{1}{\log 114}\int_{114}^{n} \frac{dt}{\log t},$$

where $c_4 = 114/\log 114$. Thus,

$$\int_{114}^{n} \frac{dt}{\log t} \le \frac{1}{1-1/\log 114}\left(\frac{n}{\log n} - c_4\right).$$

Hence,

$$S_1 \le \left(\frac{5}{4(1-1/\log 114)}\right)\frac{n}{\log n} + \left(c_3 - \frac{5c_4}{4(1-1/\log 114)}\right) < \frac{8n}{5\log n},$$

where we used the facts that

$$c_3 = 35.0161\ldots < 38.1404\ldots = \frac{5c_4}{4(1-1/\log 114)}$$

and

$$\frac{5}{4(1-1/\log 114)} = 1.58456\ldots < \frac{8}{5}.$$

By (3.29) in [7], which holds under the assumption that $n \ge 286$, we have

$$S_2 < c_5 + \log\log n + \log\left(1 + \frac{1}{2(\log n)^2}\right) < \log\log n + \left(c_5 + \frac{1}{2(\log 286)^2}\right)$$
$$< \log\log n + 1,$$

where $c_5 = 0.57721\ldots$ is the Euler constant. For S_3, we have

$$S_3 = \frac{1}{n}\sum_{p\le n} p \le n\pi(n) \le \frac{5n}{4\log n}.$$

Thus,

$$\log a_n < \left(\frac{8}{5} + \frac{5}{4}\right)\frac{n}{\log n} + \log\log n + 1 < \frac{3n}{\log n},$$

where for the last inequality we used the fact that

$$\frac{0.15n}{\log n} > \log\log n + 1 \quad \text{for all } n > 70.$$

This completes the proof of Lemma 3.1. ∎

4. Proof of Theorem 1.2. Throughout this section, we label positive constants $c_1, c_2, \ldots$ in the order of appearance in our arguments. We need a result from the theory of linear forms in p-adic logarithms. For an algebraic number η with minimal polynomial

$$f(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d \in \mathbb{Z}[X] \quad \text{with } \gcd(a_0, a_1, \ldots, a_d) = 1,$$

we put $H(\eta) = \max\{|a_i| : i = 0, 1, \ldots, d\}$. For example, if $\eta = m/n$ is a rational number written in reduced form, i.e., with coprime integers m and $n > 0$, then $H(\eta) = \max\{|m|, n\}$. Given a number field $\mathbb{K}$, a prime ideal π of its ring of algebraic integers $\mathcal{O}_{\mathbb{K}}$ and $\eta \in \mathbb{K}^*$, we write $\operatorname{ord}_\pi(\eta)$ for the exponent with which π appears in the prime ideal factorization of the principal fractional ideal $\eta\mathcal{O}_{\mathbb{K}}$ generated by η inside $\mathbb{K}$. The following result is due to Kunrui Yu [12].

Lemma 4.1. *Let $\mathbb{K}$ be an algebraic number field, π a prime ideal of $\mathcal{O}_{\mathbb{K}}$ and let $\eta_1, \eta_2 \in \mathbb{K}^*$. Let H_1, H_2 be real numbers, $H_i \geq \max\{H(\eta_i), 3\}$, $i = 1, 2$. Let m_1, m_2 be integers and put $M = \max\{|m_1|, |m_2|, 3\}$. Assume $\eta_1^{m_1}\eta_2^{m_2} - 1 \neq 0$. Then*

$$\operatorname{ord}_\pi(\eta_1^{m_1}\eta_2^{m_2} - 1) < c_1 \log H_1 \log H_2 \log M,$$

where $c_1 = c_1(\mathbb{K}, \pi)$ is a positive constant depending on $\mathbb{K}$ and π.

We work with the nondegenerate binary recurrent sequence $\mathbf{u} = \{u_n\}_{n\geq 0}$ whose Binet formula for its general term is given by (1.1). We label the roots α, β in such a way that $|\alpha| \geq |\beta|$. Clearly, $|\alpha| > 1$. The following result is due to Stewart [10] (see also [9, Theorem 3.1]).

Lemma 4.2. *There exist computable constants n_0 and c_2 such that*

$$|u_n| > |\alpha|^{n - c_2 \log n} \qquad \textit{for all } n \geq n_0.$$

Proof of Theorem 1.2. Throughout this proof, $n_0 \geq 286$ and m_0 are large numbers not necessarily the same at each occurrence. Consider the equation (1.2). Assume that $n > n_0$ so $u_n \neq 0$. We only consider the case of positive u_n since the case of negative u_n can be treated in a similar way. Clearly,

$$c_3 n + c_4 > \log u_n \geq \log a_{m_k} > \frac{m_k}{\log m_k}, \tag{4.1}$$

with $c_3 = \log|\alpha|$ and $c_4 = \log\max\{|c| + |d|, 1\}$, while by Lemma 4.2, we have

$$\begin{aligned} c_3 n - c_5 \log n < \log u_n \leq \log(k a_{m_k}) &\leq \frac{3m_k}{\log m_k} + \log k \\ &\leq \frac{3m_k}{\log m_k} + \log m_k, \end{aligned} \tag{4.2}$$

where $c_5 = c_2 c_3$. The combination of (4.1) and (4.2) shows that

$$c_6 n \log n < m_k < c_7 n \log n \qquad \text{whenever } n > n_0, \tag{4.3}$$

where we can take $c_6 = c_3/2$ and $c_7 = 2c_3$. Let q be the smallest prime exceeding s and let π be a prime ideal dividing q in $\mathbb{K} = \mathbb{Q}(\alpha)$.

Note that since $q \nmid s$, it follows that π divides neither α nor β. Then

$$\begin{aligned} \operatorname{ord}_q(u_n) \leq \operatorname{ord}_\pi(c\alpha^n + d\beta^n) &= \operatorname{ord}_\pi(d\beta^n) + \operatorname{ord}_\pi((-c/d)(\alpha/\beta)^n - 1) \\ &\leq c_8 \log n + c_9; \end{aligned}$$

here $c_9 = \max\{\mathrm{ord}_\pi(d), 1\}$ and for the right-most inequality above we used Lemma 4.1 with $\eta_1 = -c/d$ and $\eta_2 = \alpha/\beta$. Observe that if $m > q^2$ and p is a prime in $(m/q, m/(q-1)]$, then $p^2 > (m/q)^2 > m$, therefore

$$\alpha_p(m) = \mathrm{ord}_q(m!) = \left\lfloor \frac{m}{p} \right\rfloor = q - 1.$$

Hence, $\alpha_p(m) + 1 = q$ for all primes $p \in (m/q, m/(q-1)]$. Lemma 3.1(iii) now shows that if $m > m_0$, then the interval $(m/q, m/(q-1)]$ contains at least $c_{10}m/\log m$ primes p, where $c_{10} = 1/(2q^2)$.

In particular, for such m we have $\mathrm{ord}_q(a_m) \geq c_{10}m/\log m$. Thus, assuming $m_1 \geq m_0$, we see that

$$\begin{aligned} c_8 \log n + c_9 \geq \mathrm{ord}_q(u_n) = \mathrm{ord}_q(a_{m_k} + \cdots + a_{m_1}) &\geq \min_{1\leq i\leq k}\{\mathrm{ord}_q(a_{m_i})\} \\ &\geq c_{10}m_1/\log m_1, \end{aligned}$$

giving

$$m_1 \leq c_{11}(\log n)^2 \quad \text{for } n \geq n_0. \tag{4.4}$$

We further assume $c_{11} \geq m_0$ so that the above inequality includes also the case when $m_1 \leq m_0$. Comparing (4.3) with (4.4) tells us that $k \geq 2$ once $n \geq n_0$, for if not we would get $c_6 n \log n < m_1 < c_{11}(\log n)^2$, so $n < n_0$. We also assume that $c_{11}(\log n)^2 < n$ for $n > n_0$.

We will show, recursively, that the following holds:

Claim. *There exists a constant $c_{12} > 1$ such that if $j < k$ and*

$$m_j < (c_{12} \log n)^{2j} \quad \textit{and} \quad (c_{12}\log n)^{2j+2} < n, \tag{4.5}$$

then

$$m_{j+1} < (c_{12}\log n)^{2j+2}. \tag{4.6}$$

Let us see that once we have proved the above implication, we are through. Indeed, let $j \leq k$ be maximal such that inequalities (4.5) hold. If $j = k$, then

$$c_6 n \log n < m_k \leq (c_{12}\log n)^{2k}. \tag{4.7}$$

If $j < k$, then (4.6) holds. By the maximality of j, we must have

$$(c_{12}\log n)^{2k} \geq (c_{12}\log n)^{2j+2} \geq n. \tag{4.8}$$

Inequalities (4.7) and (4.8) together show $k \geq c_0 \log n/\log\log n$ for $n \geq n_0$, which is the desired conclusion.

To prove the Claim, notice that we have already proved it with $c_{12} = c_{11}^{1/2}$ for the case when $j = 1$. Assume now that (4.5) holds for some $j < k$ with some c_{12} to be determined later. We distinguish two cases.

Case 1: $s \neq \pm 1$. In this case, there exists a prime ideal $\mathbf{p} \in \mathcal{O}_{\mathbb{K}}$ dividing α. Indeed, if this were not true, then α would be a unit. Since $|\alpha| > 1$, this unit cannot be rational because the only rational units are ± 1. Hence, α is a quadratic unit, β is its conjugate and $s = -\alpha\beta = \pm 1$, which is not the case we are treating. Further, since $\mathbf{p}$ divides α, and r and s are coprime, it follows that $\mathbf{p}$ does not divide β. Moreover, it is clear that $\mathbf{p}$ sits above a rational prime p dividing s. We write

$$N_j = a_{m_j} + \cdots + a_{m_1}. \tag{4.9}$$

Rewrite (1.2) as

$$c\alpha^n + (d\beta^n - N_j) = a_{m_k} + \cdots + a_{m_{j+1}}. \tag{4.10}$$

Suppose that

$$d\beta^n - N_j \neq 0. \tag{4.11}$$

We then compute an upper bound for the $\mathbf{p}$-adic order of the above nonzero number using Lemma 4.1. For this, note first that if $m_j < 286$, then

$$\log N_j \leq c_{13}. \tag{4.12}$$

Assume next that $m_j \geq 286$. Then

$$\log N_j \leq \log(j a_{m_j}) \leq \frac{3m_j}{\log m_j} + \log m_j \leq 2m_j \leq 2(c_{12}\log n)^{2j} \tag{4.13}$$

because $\log m_j > 3$ when $m_j \geq 286$. We assume that n_0 is such that $c_{12}\log n > c_{13}$, so (4.13) incorporates the case of (4.12) when $m_j < 286$ as well. Thus,

$$\begin{aligned} \mathrm{ord}_{\mathbf{p}}(d\beta^n - N_j) &= \mathrm{ord}_{\mathbf{p}}(d\beta^n) + \mathrm{ord}_{\mathbf{p}}(\beta^{-n}(N_j d^{-1}) - 1) \\ &\leq c_{14} + c_{15}(c_{12}\log n)^{2j+1} \leq 2c_{15}(c_{12}\log n)^{2j+1} \end{aligned} \tag{4.14}$$

for $n \geq n_0$, where $c_{14} = \max\{\mathrm{ord}_{\mathbf{p}}(d), 1\}$ and we applied Lemma 4.1 with $\eta_1 = \beta$, $\eta_2 = N_j d^{-1}$, $m_1 = -n$, $m_2 = 1$, and we also used the fact that

$$\log\max\{H(\eta_2), 3\} \leq c_{16}\log\max\{H(d), 3\}\log\max\{N_j, 3\} \leq c_{17}(c_{12}\log n)^{2j}$$

(see (4.13)); here c_{16} is absolute (see [9, Lemma A.3]), c_{15} depends also on $\mathbf{p}$ and β, and $c_{17} = 2c_{16}\log\max\{H(d), 3\}$. Assume that n_0 is such that the inequality $\log n > 4c_{15}/c_{12}$ holds for $n > n_0$. Then (4.14) implies

$$\mathrm{ord}_{\mathbf{p}}(d\beta^n - N_j) < (c_{12}\log n)^{2j+2}/2 < n/2 \leq \mathrm{ord}_{\mathbf{p}}(c\alpha^n)$$

for $n > n_0$, which shows that

$$\begin{aligned} \mathrm{ord}_{\mathbf{p}}(u_n - N_j) &= \mathrm{ord}_{\mathbf{p}}(c\alpha^n + (d\beta^n - N_j)) = \mathrm{ord}_{\mathbf{p}}(d\beta^n - N_j) \\ &< 2c_{15}(c_{12}\log n)^{2j+1}. \end{aligned} \tag{4.15}$$

However, if $m_{j+1} \geq m_0$, we have $\mathrm{ord}_{\mathbf{p}}(a_{m_{j+1}}) \geq c_{18} m_{j+1}/\log m_{j+1}$, where $c_{18} = 1/(2p^2)$ plays the same role for the rational prime p sitting above $\mathbf{p}$ as $c_{10} = 1/(2q^2)$ played for q. So, assuming that $m_{j+1} \geq m_0$, we have

$$\mathrm{ord}_{\mathbf{p}}(a_{m_k} + \cdots + a_{m_{j+1}}) \geq \min_{j+1 \leq \ell \leq k}\{\mathrm{ord}_{\mathbf{p}}(a_{m_\ell})\} \geq c_{18}\frac{m_{j+1}}{\log m_{j+1}}. \tag{4.16}$$

Comparing the bounds (4.15) and (4.16), we get

$$c_{18}\frac{m_{j+1}}{\log m_{j+1}} \le 2c_{15}(c_{12}\log n)^{2j+1},$$

so

$$\frac{m_{j+1}}{\log m_{j+1}} \le c_{19}(c_{12}\log n)^{2j+1},$$

where $c_{19} = 2c_{15}/c_{18}$.

If $A > 10$, then the inequality $x/\log x < A$ implies $x < 2A\log A$. Taking $A = c_{19}(c_{12}\log n)^{2j+1}$ and assuming that n_0 is such that $A > 10$ for $n > n_0$, we get

$$\begin{aligned} m_{j+1} &< 2c_{19}(\log n)^{2j+1}\big(\log c_{19} + (2j+1)\log(c_{12}\log n)\big) \\ &< 4c_{19}(c_{12}\log n)^{2j+2} \end{aligned} \tag{4.17}$$

for $n > n_0$, because $(2j+1)\log(c_{12}\log n) < \log((c_{12}\log n)^{2j+2}) < \log n$, and for $n > n_0$ we have $\log c_{19} < \log n$. We enlarge c_{19} by replacing it with $\max\{c_{19}, m_0\}$ so that the last inequality above incorporates the case when $m_{j+1} < m_0$ as well. Comparing (4.17) with (4.6), we see that it is sufficient to choose c_{12} such that $c_{12}^2 > 4c_{19}$ and then indeed inequality (4.6) is a consequence of inequalities (4.5).

This was however under the assumption (4.11). Let us see what happens if on the contrary $N_j = d\beta^n$. Then $d \in \mathbb{Q}^*$ and $\beta \in \mathbb{Z}$. If $|\beta| \ge 2$, we then have

$$n\log 2 + \log|d| \le \log|d\beta^n| = \log N_j < 2(c_{12}\log n)^{2j} < 2c_{12}^{-2}\frac{n}{(\log n)^2},$$

which is not possible for $n \ge n_0$. Thus, $\beta = \pm 1$. Treating separately the cases when n is even and when n is odd, i.e., replacing the sequence $\{u_n\}_{n\ge 0}$ by the two sequences $\{u_{2n}\}_{n\ge 0}$ and $\{u_{2n+1}\}_{n\ge 0}$, which results in replacing the pair of roots (α, β) by the pair of roots (α^2, β^2), we may assume that $\beta = 1$. Then $d = N_j$, and so $j = j_0 \le c_{20}$. We fix j_0 and work with the relation

$$a\alpha^n = a_{m_k} + \cdots + a_{m_{j_0+1}}.$$

Relabeling the indices on the right-hand side above, we may assume that they are $m_k > \cdots > m_1 \ge 1$. We let q be the smallest prime not dividing a. Then $m_1 < c_{21}$. We rewrite our equation as

$$a\alpha^n - a_{m_1} = a_{m_k} + \cdots + a_{m_2}.$$

We apply Lemma 4.1 to bound the q-adic valuation of the the left-hand side above getting $\operatorname{ord}_q(a\alpha^n - a_{m_1}) < c_{22}\log n$. This implies, by an argument similar to the one used before, that $m_2 < c_{23}\log n$. From now on, the proof proceeds in the same way as before and shows that (4.5) implies (4.6) with some appropriate constant $c_{12} > c_{23}$.

CASE 2: $s = \pm 1$. Here, α and β are quadratic units. As we already mentioned before, treating separately the cases n even and n odd amounts to replacing $\{u_n\}_{n\geq 0}$ by the two sequences $\{u_{2n}\}_{n\geq 0}$ and $\{u_{2n+1}\}_{n\geq 0}$. Thus, we replace (α, β) by (α^2, β^2), and therefore we may assume that $s = -1$ and $\beta = \alpha^{-1}$. We write

$$\begin{aligned} u_n - N_k = c\alpha^n + d\beta^n - N_j &= c\beta^n(\alpha^{2n} - c^{-1}N_j\alpha^n + dc^{-1}) \\ &= c\beta^n(\alpha^n - z_{1,j})(\alpha^n - z_{2,j}), \end{aligned}$$

where

$$z_{i,j} = \frac{c^{-1}N_j \pm \sqrt{c^{-2}N_j^2 - 4dc^{-1}}}{2} \quad \text{for } i = 1, 2$$

are the roots of the quadratic equation $x^2 - c^{-1}N_jx + dc^{-1} = 0$. We let $\mathbb{L}_j = \mathbb{K}(z_{1,j})$ and let $\mathbf{p}$ be some prime ideal of $\mathbb{L}$ sitting above the prime 2 (which is the smallest prime that does not divide $s = 1$). Then, by Lemma 4.2, we have

$$\mathrm{ord}_2(u_n - N_j) \leq \mathrm{ord}_{\mathbf{p}}(u_n - N_j) = \mathrm{ord}_{\mathbf{p}}(c\alpha^n) + \mathrm{ord}_{\mathbf{p}}(\alpha^n - z_{1,j}) + \mathrm{ord}_{\mathbf{p}}(\alpha^n - z_{2,j}).$$

Since $\mathbb{L}_j$ is of degree at most 4 and α is a unit, $\mathrm{ord}_{\mathbf{p}}(c\alpha^n) = \mathrm{ord}_{\mathbf{p}}(c) \leq c_{24}$, where we can take $c_{24} = 4\max\{\mathrm{ord}_2(c), 1\}$. The other two quantities can be bounded as

$$\max_{i=1,2}\{\mathrm{ord}_{\mathbf{p}}(\alpha^n - z_{i,j})\} \leq c_{25}\log H(z_{i,j})\log n \leq c_{26}(c_{12}\log n)^{2j+1},$$

where we used (4.13), as well as the fact that

$$\log H(z_{i,j}) \leq c_{27}\log N_j \quad \text{for } i = 1, 2$$

(see again [9, Lemma A.3]). Hence,

$$\begin{aligned} \mathrm{ord}_2(u_n - N_j) &< c_{24} + 2c_{26}(c_{12}\log n)^{2j+1} \\ &< 3c_{26}(\log n)^{2j+1} \quad \text{for } n \geq n_0. \end{aligned} \tag{4.18}$$

From now on the argument continues as in the preceding case, by writing a lower bound for $\mathrm{ord}_2(u_n - N_j) = \mathrm{ord}_2(a_{m_k} + \cdots + a_{m_{j+1}})$ of the form $c_{28}m_{j+1}/\log m_{j+1}$, where, by Lemma 3.1(iii), we can take $c_{28} = 3/10$ provided that $m_{j+1} \geq m_0 = 41$, and by comparing this lower bound with the upper bound (4.18) to get

$$\frac{m_{j+1}}{\log m_{j+1}} < 10c_{26}(c_{12}\log n)^{2j+1}.$$

This leads to

$$\begin{aligned} m_{j+1} &< 20c_{26}(c_{12}\log n)^{2j+1}\big(\log(10c_{26}) + (2j+1)\log(c_{12}\log n)\big) \\ &< 40c_{26}(c_{12}\log n)^{2j+2} \end{aligned} \tag{4.19}$$

for $n \geq n_0$. We assume that n_0 is sufficiently large such that the inequality $40c_{26}(c_{12}\log n)^2 \geq 41$ holds, so that the case $m_{j+1} \leq 41$ is also incorporated in (4.19). Thus, we deduce that also in this case inequality (4.6) is a consequence of inequalities (4.5) provided that we choose c_{12} such that $c_{12}^2 > 40c_{26}$. Theorem 1.2 is therefore proved. ∎

5. Proof of Theorem 1.3. Here, we need the following result from [1].

LEMMA 5.1. *Let N be a positive integer not of the form F_m for some positive integer m. Then for all positive integers $n \geq 3$ one has*

$$\operatorname{ord}_2(F_n - N) < 1730 \log(6N^2) \max\{10, \log n\}^2. \tag{5.1}$$

Let $\gamma = (1+\sqrt{5})/2$ be the golden section. We also need the following well-known facts about the size and 2-adic valuation of the Fibonacci numbers.

LEMMA 5.2.

(i) *We have*

$$\gamma^{n-2} \leq F_n \leq \gamma^{n-1} \quad \textit{for all } n \geq 1.$$

(ii) *F_n is even if and only if $3 \mid n$. If $n = 3m$ and m is odd, then $2 \parallel F_n$, while if m is even, then* $\operatorname{ord}_2(F_n) = \operatorname{ord}_2(n) + 2$. *In particular,*

$$\operatorname{ord}_2(F_n) \leq \frac{\log(n/3)}{\log 2} + 2.$$

We use $\{L_n\}_{n\geq 0}$ for the Lucas companion of the Fibonacci sequence which is given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Furthermore

$$F_a - F_b = F_{(a\pm b)/2} L_{(a\mp b)/2} \tag{5.2}$$

whenever $a \equiv b \pmod 2$. For a positive integer m we let $z(m)$ be the minimal positive integer k such that $m \mid F_k$. This exists for all m. Moreover, $m \mid F_n$ if and only if $z(m) \mid n$. With this notation, Lemma 5.2(ii) can be formulated as $z(2^s) = 3 \cdot 2^{s-2}$ for all $s \geq 3$.

Proof of Theorem 1.3. (I) Assume that $F_n = a_m$. By Lemma 5.2(i), we have

$$(n-2)\log\gamma \leq \log F_n \leq (n-1)\log\gamma. \tag{5.3}$$

Assume that $m \geq 286$. Then, by Lemma 3.1(iv),

$$\log a_m < \frac{3m}{\log m}. \tag{5.4}$$

Further, by Lemma 3.1(ii), we see that

$$\operatorname{ord}_2(a_m) \geq \pi(m) - \pi(m/2) > \frac{3m}{10\log m}.$$

The left-hand side is $15.1698\ldots$ at $m = 286$, so that $\mathrm{ord}_2(F_n) \ge 16$. By Lemma 5.2(ii), we have

$$\mathrm{ord}_2(F_n) \le \frac{\log(n/3)}{\log 2} + 2.$$

So, we get

$$\frac{3m}{10\log m} < \mathrm{ord}_2(a_m) = \mathrm{ord}_2(F_n) \le \frac{\log(n/3)}{\log 2} + 2. \tag{5.5}$$

From (5.3)–(5.5), we obtain

$$(n-2)\log\gamma < \frac{3m}{\log m} < \frac{10\log(n/3)}{\log 2} + 20,$$

which gives $n \le 163$. Thus, we have showed that either $m \le 286$ or $n \le 163$. Since $F_{188} > a_{286}$, we deduce that $n \le 187$ and a quick computer calculation shows that $(n, m) = (1, 1), (2, 1), (3, 2)$ and $(6, 4)$ are the only solutions of this equation.

(II) Assume that $F_n = a_{m_2} + a_{m_1}$ with $m_2 > m_1 \ge 1$. Assume first that $n \le 1000$. By Lemmas 5.2(i) and 3.1(iv) we find, assuming $m_2 \ge 17$, that

$$(n-1)\log\gamma \ge \log F_n = \log(a_{m_2} + a_{m_1}) > \log a_{m_2} > \frac{m_2}{\log m_2},$$

giving $m_2 \le 3985$. If $m_1 \ge 41$, then by Lemmas 5.2(ii) and 3.1(ii), we have

$$\begin{aligned} \frac{\log(n/3)}{\log 2} + 2 &\ge \mathrm{ord}_2(F_n) = \mathrm{ord}_2(a_{m_1} + a_{m_2}) \\ &\ge \min\{\mathrm{ord}_2(a_{m_1}), \mathrm{ord}_2(a_{m_2})\} \ge \frac{3m_1}{10\log m_1}, \end{aligned} \tag{5.6}$$

giving $m_1 \le 179$. We wrote a Mathematica code that searched among all $n \le 1000$, $1 \le m_1 \le 179$ and $m_1 < m_2 \le 3985$, and gave only the solutions indicated in the statement of the theorem. We now prove that there are no others.

We may assume that $n > 1000$. One checks that $m_2 > 286$ just because $2a_{286} < F_{1000}$. In particular, by Lemmas 3.1(iv) and 5.2(i), we have

$$(n-2)\log\gamma \le \log F_n = \log(a_{m_1}+a_{m_2}) \le \log(2a_{m_2}) < \log 2 + \frac{3m_2}{\log m_2}, \tag{5.7}$$

and since $n > 1000$, we get $m_2 \ge 1123$. Further,

$$\mathrm{ord}_2(a_{m_2}) \ge \frac{3m_2}{10\log m_2} \ge 47.9658\ldots \qquad \text{for } m_2 \ge 1123, \tag{5.8}$$

so that $2^{48} \mid a_{m_2}$. If a_{m_1} is a Fibonacci number, then, by (I), $a_{m_1} \in \{1, 2, 8\}$. Since

$$F_n \equiv 1, 2, 8 \pmod{2^{48}},$$

it follows that $n \equiv 3 \pmod 6$ if $a_{m_1} = 2$ and $n \equiv 0 \pmod 6$ if $a_{m_1} = 8$. Using formula (5.2), we get

$$a_{m_2} = F_n - a_{m_1} = F_{(n\pm k)/2} L_{(n\mp k)/2} \quad \text{for some } k \in \{1, 2, 3, 6\},$$

where, of course, $n \equiv k \pmod 2$. Since L_m is never a multiple of 8 for any positive integer m, we deduce, by Lemmas 3.1(ii) and 5.2(ii), that

$$\begin{aligned} \frac{3m_2}{10 \log m_2} &< \text{ord}_2(a_{m_2}) = \text{ord}_2(F_{(n\pm k)/2}) + \text{ord}_2(L_{(n\mp k)/2}) \\ &\le \frac{\log((n+6)/6)}{\log 2} + 4. \end{aligned}$$

Thus, by (5.7), we get

$$(n-2)\log\gamma < \log 2 + \frac{3m_2}{\log m_2} < \log 2 + \frac{10\log((n+6)/6)}{\log 2} + 40,$$

giving $n < 192$, a contradiction. Thus, a_{m_1} is not a Fibonacci number.

We next find an upper bound for m_1. Assume that $m_1 \ge 286$. We rewrite our equation as

$$F_n - a_{m_1} = a_{m_2}.$$

We compute the 2-adic valuation of the left-hand side. By Lemma 5.1, we see, assuming $n > 23000 > e^{10}$, that

$$\text{ord}_2(F_n - a_{m_1}) < 1730(\log n)^2 \log(6a_{m_1}^2). \tag{5.9}$$

Now, by Lemma 3.1(iv) and inequality (5.6), we have

$$\log(6a_{m_1}^2) = \log 6 + 2\log a_{m_1} < \log 6 + \frac{6m_1}{\log m_1} < \log 6 + \frac{20\log(n/3)}{\log 2} + 40.$$

However, Lemma 3.1(ii) shows that

$$\text{ord}_2(F_n - a_{m_1}) = \text{ord}_2(a_{m_2}) > \frac{3m_2}{10\log m_2},$$

which together with (5.9) gives

$$\frac{m_2}{\log m_2} < \frac{17300}{3}(\log n)^2\left(\log 6 + \frac{20\log(n/3)}{\log 2} + 40\right). \tag{5.10}$$

Inserting (5.10) into (5.7), we get

$$(n-2)\log\gamma < \log 2 + 17300(\log n)^2\left(\log 6 + \frac{20\log(n/3)}{\log 2} + 40\right),$$

which leads to $n < 1.4 \cdot 10^{10}$. In particular, $\text{ord}_2(F_n) \le 34$. This implies that $\text{ord}_2(a_{m_1}) \le 34$, so, by Lemma 3.1(ii), we have

$$\frac{3m_1}{10\log m_1} < \text{ord}_2(a_{m_1}) \le 34,$$

giving $m_1 \le 750$. Computations show that in fact $m_1 \le 202$, so the assumption that $m_1 \ge 286$ does not hold. Hence, $m_1 \le 285$, and therefore $\log(6a_{m_1}^2) \le \log(6a_{285}^2) \le 181$. Thus, by Lemmas 3.1(ii) and 5.1, assuming still that $n > 23000 > e^{10}$, we have

$$\begin{aligned}\frac{3m_2}{10\log m_2} &< \operatorname{ord}_2(a_{m_2}) = \operatorname{ord}_2(F_n - a_{m_1}) \le 1730(\log n)^2 \log(6a_{m_1}^2)\\ &< 1730 \cdot 181(\log n)^2,\end{aligned}$$

and consequently

$$\frac{m_2}{\log m_2} < 1.05 \cdot 10^6 (\log n)^2.$$

The above inequality together with (5.7) gives

$$(n-2)\log\gamma < \log 2 + 3.15 \cdot 10^6 (\log n)^2,$$

which implies that $n < 3.2 \cdot 10^9$. This shows that $\operatorname{ord}_2(F_n) \le 31$, therefore $\operatorname{ord}_2(a_{m_1}) \le 31$, which gives $m_1 \le 181$.

Next, we split the remaining range in two according to the value of m_2. For $q \in \{11, 13, 17, 19, 23\}$, assume that the interval $(m_2/q, m_2/(q-1)]$ does not contain a prime number. Put $x = m_2/q$, and let k be the largest positive integer such that $p_k \le m_2/q$, where $2 = p_1 < p_2 < \cdots$ is the sequence of all prime numbers. Then, since the interval $(x, x + x/q(q-1)]$ does not contain a prime, it follows that $p_{k+1} > (1 + 1/506)p_k$. A result of Schoenfeld [8] (even with the number 16597 instead of 506) shows that $x < 2010760$, so $k \le \pi(2010760) = 149689$. We checked numerically that in fact the largest such k is $k = 3385$, so $m_2 \le qp_{3386} = 23 \cdot 31469 = 723787$. Thus, assume that $m_2 \le 750000$. Then

$$\log(2a_{m_2}) \le \log 2 + \sum_{p \le 750000} \log\left(\frac{750000}{p-1} + 1\right) < 90000,$$

which yields

$$(n-2)\log\gamma \le \log F_n = \log(a_{m_1} + a_{m_2}) \le \log(2a_{m_2}) < 90000,$$

giving $n < 190000$. Next we computed

$$\{F_n \pmod{2^{40}}\}_{1 \le n \le 190000} \quad \text{and} \quad \{a_{m_1} \pmod{2^{40}}\}_{1 \le m_1 \le 200},$$

and we intersected the above two lists. Unsurprisingly, the only common values were found for $(n, m_1) = (1,1), (2,1), (3,2), (6,4)$, which are the cases for which $a_{m_1} \in \{1, 2, 8\}$ is a Fibonacci number; we have already treated these.

Thus, from now on, we assume $m_2 > 750000$. We next prove that

$$2^{48} \cdot 3^{28} \cdot 5^9 \cdot 7^4 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \mid a_{m_2}.$$

The fact that 2^{48} divides a_{m_2} has been shown in (5.8) and the fact that each of the primes $q \in \{11, 13, 17, 19, 23\}$ divides a_{m_2} follows from the above arguments because we are in the case when $m_2 > 750000$. For the exponents of the primes 3, 5 and 7, we used Lemma 3.1(ii) in the following way. For the power of 3, we checked first that

$$\frac{m_2/2}{\log(m_2/2) - 0.5} - \frac{m_2/3}{\log(m_2/3) - 1.5} \geq 28$$

for all $m_2 > 2500$. In light of Lemma 3.1, this shows that the interval $(m_2/3, m_2/2]$ contains at least 28 primes for $m_2 > 2500$, which is acceptable for us. Similar arguments work for the other two primes. For example, the inequality

$$\frac{m_2/4}{\log(m_2/4) - 0.5} - \frac{m_2/5}{\log(m_2/5) - 1.5} > 9$$

holds for all $m_2 > 8000$, and the inequality

$$\frac{m_2/6}{\log(m_2/6) - 0.5} - \frac{m_2/7}{\log(m_2/7) - 1.5} > 4$$

holds for all $m_2 > 41000$.

From now on, we start eliminating some values of m_1:

- Assume that $4 \,\|\, a_{m_1}$. Then $4 \,\|\, F_n$, which is impossible.
- Assume that $11 \mid a_{m_1}$ but $5 \nmid a_{m_1}$. Then $10 = z(11) \mid n$, so $5 \mid F_{10} \mid F_n$, and $5 \mid a_{m_2}$, but $5 \nmid a_{m_1}$, a contradiction.
- Assume that $15 \mid a_{m_1}$ but $11 \nmid a_{m_1}$. Then $20 = z(15) \mid n$, so that $11 \mid F_{10} \mid F_n$, and $11 \mid a_{m_2}$, but $11 \nmid a_{m_1}$, a contradiction.
- Assume that $16 \mid a_{m_1}$ but $9 \nmid a_{m_1}$. Then $12 = z(16) \mid n$, so $9 \mid F_{12} \mid F_n$, and $9 \mid a_{m_2}$, but $9 \nmid a_{m_1}$, a contradiction.
- Assume that $27 \mid a_{m_1}$ but $17 \nmid a_{m_1}$. Then $9 \mid 36 = z(27) \mid n$, so $17 \mid F_9 \mid F_n$, and $17 \mid a_{m_2}$, but $17 \nmid a_{m_1}$, a contradiction.
- Assume that $27 \mid a_{m_1}$ but $19 \nmid a_{m_1}$. Then $18 \mid 36 = z(27) \mid n$, so that $19 \mid F_{18} \mid F_n$, and $19 \mid a_{m_2}$, but $19 \nmid a_{m_1}$, a contradiction.
- Assume that $32 \mid a_{m_1}$ but $7 \nmid a_{m_1}$. Then $8 = z(32) \mid n$, so $7 \mid F_8 \mid F_n$, and $7 \mid a_{m_2}$, but $7 \nmid a_{m_1}$, a contradiction.
- Assume that $32 \mid a_{m_1}$ but $23 \nmid a_{m_1}$. Then $24 = z(32) \mid n$, so $23 \mid F_n$, and $23 \mid a_{m_2}$, but $23 \nmid a_{m_1}$, a contradiction.
- Assume that $40 \mid a_{m_1}$ but $11 \nmid a_{m_1}$. Then $30 = z(40) \mid n$, so $z(40) \mid n$, therefore $11 \mid F_{30} \mid F_n$, and $11 \mid a_{m_2}$, but $11 \nmid a_{m_1}$, a contradiction.

After all the above tests, no $m_1 \in [1, 181] \setminus \{1, 2, 4\}$ survived. This completes the proof of Theorem 1.3. ■

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