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Abstract. We show the consistency of "there is a nice σ-ideal $\mathcal{I}$ on the reals with $\mathrm{add}(\mathcal{I}) = \aleph_1$ which cannot be represented as the union of a strictly increasing sequence of length ω_1 of σ-subideals". This answers [Borodulin-Nadzieja and Głąb, Math. Logic Quart. 57 (2011), 582–590, Problem 6.2(ii)].

1. Introduction. Borodulin-Nadzieja and Głąb [3] studied generalizations of the Mokobodzki ideal and they showed that those σ-ideals do not have Borel bases of bounded Borel complexity. In [3, Section 5] they noticed that the unbounded Borel complexity of bases implies that the additivity of the σ-ideal under consideration is $\aleph_1$. This observation exposed the heart of a result of Cichoń and Pawlikowski [5, Corollary 2.4] and showed the importance of the existence of a strictly increasing ω_1-sequence of σ-subideals which add up to the whole ideal.

Therefore Borodulin-Nadzieja and Głąb introduced a new cardinal invariant $\mathrm{cofin}(\mathcal{I})$ associated with non-trivial σ-ideals $\mathcal{I}$: the minimal length of a strictly increasing sequence of σ-subideals with union $\mathcal{I}$ (see Definition 2.1). They showed that the additivity of the σ-ideal $\mathcal{I}$ is not larger than $\mathrm{cofin}(\mathcal{I})$ (see [3, Proposition 5.2] or Theorem 2.2 here), and in [3, Problem 6.2(ii)] they asked if the two invariants can be different. In the present paper we answer this question in the affirmative.

In the second section we define the relevant cardinal invariants, and we point out situations when $\mathrm{cofin}(\mathcal{I}) < \mathrm{cof}(\mathcal{I})$ for the meager and the null ideals. In Section 3 we introduce a nicely definable σ-ideal $\mathcal{I}_f$ with a Borel basis consisting of Π^0_2 sets. Then we show that, consistently, $\mathrm{add}(\mathcal{I}_f) = \aleph_1$ while $\mathrm{cofin}(\mathcal{I}_f) = \aleph_2$ (Corollary 3.15).

Notation. Most of our notation is standard and compatible with that of classical textbooks (like Bartoszyński and Judah [1]). However, in forcing we keep the older convention that *a stronger condition is the larger one.*

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• Ordinals will be denoted with initial letters of the Greek alphabet (α–ζ) and integers (finite ordinals) will be denoted by i, j, k, ℓ, m, n. Letters κ, λ, μ will denote uncountable cardinals.

• By a *sequence* we mean a function whose domain is a set of ordinals. Sequences will be denoted by letters $\eta, \nu, \rho, \sigma, \varsigma, \varphi, \psi$ (with possible indices).

For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The *length* of a sequence η is the order type of its domain and it is denoted by $\ell g(\eta)$.

• The power set of a set X is denoted by $\mathcal{P}(X)$, the collection of all subsets of X of size m is denoted by $[X]^m$, and the collection of all finite subsets of X is denoted by $[X]^{<\aleph_0}$.

• The Cantor space ${}^{\omega}2$ is the space of all functions from ω to 2, equipped with the product topology generated by sets of the form $\{\eta \in {}^{\omega}2 : \nu \vartriangleleft \eta\}$ for $\nu \in {}^{\omega>}2$.

• A family $\mathcal{I}$ of subsets of X which is closed under finite unions and taking subsets is called an *ideal on* X. It is a *proper* ideal if $X \notin \mathcal{I}$ (i.e., $\mathcal{I} \neq \mathcal{P}(X)$) and it is a *$\sigma$-ideal* if it is closed under countable unions. The σ-ideal of meager subsets of the Cantor space ${}^{\omega}2$ is called $\mathcal{M}$ and the σ-ideal of Lebesgue null sets is $\mathcal{N}$.

• For a forcing notion $\mathbb{P}$, all $\mathbb{P}$-names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g. $\underaccent{\tilde}{A}$, $\underaccent{\tilde}{\eta}$). The canonical name for a $\mathbb{P}$-generic filter over $\mathbf{V}$ is denoted $\underaccent{\tilde}{G}_{\mathbb{P}}$. The Cohen forcing for adding κ many Cohen reals in ${}^{\omega}2$ is called $\mathbb{C}_\kappa$ (so a condition in $\mathbb{C}_\kappa$ is a finite function $p : \mathrm{dom}(p) \to 2$ with $\mathrm{dom}(p) \subseteq \kappa \times \omega$ and the order of $\mathbb{C}_\kappa$ is the inclusion). The forcing $\mathbb{C}$ is $\mathbb{C}_1$.

2. cofin and $\mathcal{M}$, $\mathcal{N}$

DEFINITION 2.1. Let $\mathcal{I}$ be an ideal on X. We define the following cardinal characteristics of $\mathcal{I}$:

(1) $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \&\ \bigcup \mathcal{A} \notin \mathcal{I}\}$;
(2) $\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{I} \ \&\ (\forall A \in \mathcal{I})(\exists B \in \mathcal{B})(A \subseteq B)\}$;
(3) $\mathrm{cofin}(\mathcal{I})$ is the minimal limit ordinal γ for which there exists a sequence $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \gamma \rangle$ such that
    (a) $\mathcal{I} = \bigcup_{\alpha<\gamma} \mathcal{I}_\alpha$,
    (b) $\mathcal{I}_\alpha \subsetneq \mathcal{I}_\beta$ for $\alpha < \beta < \gamma$, and
    (c) each $\mathcal{I}_\alpha$ is a σ-ideal

    (or ∞ if there is no sequence $\bar{\mathcal{I}}$ as above);
(4) $\mathrm{cofin}^-(\mathcal{I})$ and $\mathrm{cofin}^*(\mathcal{I})$ are defined similarly to $\mathrm{cofin}(\mathcal{I})$, but clause (c) is replaced by (c)$^-$ and (c)*, respectively, where

    (c)$^-$ each $\mathcal{I}_\alpha$ is an ideal;

(c)* each $\mathcal{I}_\alpha$ is closed under taking subsets (i.e., $B \subseteq A \in \mathcal{I}_\alpha$ implies $B \in \mathcal{I}_\alpha$);

(5) $\mathrm{cofin}^+(\mathcal{I})$ is the minimal limit ordinal γ for which there exists a sequence $\langle \mathcal{I}_\alpha : \alpha < \gamma \rangle$ such that clauses (a)–(c) of (3) above are satisfied and

(d) all singletons belong to $\mathcal{I}_0$.

If $\mathcal{I}$ is a non-principal ideal (i.e., $\mathrm{cof}(\mathcal{I}) \geq \aleph_0$), then $\mathrm{cofin}^-(\mathcal{I})$ is a well defined cardinal and $\mathrm{cofin}^-(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$. To see this, pick a basis $\{B_\zeta : \zeta < \mathrm{cof}(\mathcal{I})\} \subseteq \mathcal{I}$ for $\mathcal{I}$. Let ζ_0 be the first ordinal $\zeta \leq \mathrm{cof}(\mathcal{I})$ such that for some set $B \in \mathcal{I}$ every member of $\mathcal{I}$ can be covered by finitely many elements of $\{B_\varepsilon : \varepsilon < \zeta\} \cup \{B\}$. Necessarily, ζ_0 is a limit ordinal. Let $B^* \in \mathcal{I}$ be such that $\{B_\varepsilon : \varepsilon < \zeta_0\} \cup \{B^*\}$ generates $\mathcal{I}$, i.e., every set in $\mathcal{I}$ can be covered by B^* and finitely many sets B_ε with $\varepsilon < \zeta_0$. For $\zeta < \zeta_0$ let $\mathcal{I}_\zeta$ be the ideal generated by $\{B_\varepsilon : \varepsilon < \zeta\} \cup \{B^*\}$. Then $\mathcal{I} = \bigcup_{\zeta<\zeta_0} \mathcal{I}_\zeta$ and, by the minimality of ζ_0, the sequence $\langle \mathcal{I}_\zeta : \zeta < \zeta_0 \rangle$ does not stabilize. Consequently, we may choose an increasing sequence $\langle \zeta_\alpha : \alpha < \mathrm{cf}(\zeta_0) \rangle$ cofinal in ζ_0 and such that $\langle \mathcal{I}_{\zeta_\alpha} : \alpha < \mathrm{cf}(\zeta_0) \rangle$ is a strictly increasing sequence of ideals with union $\mathcal{I}$.

The cardinal invariant cofin was introduced by Borodulin-Nadzieja and Głąb in [3, Section 5]. It has the flavor of the altitude of Boolean algebras (see van Douwen, Monk and Rubin [9, p. 236]), but the two cardinal coefficients seem to be unrelated.

Theorem 2.2 (Borodulin-Nadzieja and Głąb [3, Section 5]). *Let $\mathcal{I}$ be a non-principal ideal of subsets of X. Then*

$$\mathrm{add}(\mathcal{I}) \leq \mathrm{cofin}^*(\mathcal{I}) \leq \mathrm{cofin}^-(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I}),$$
$$\mathrm{cofin}^-(\mathcal{I}) \leq \mathrm{cofin}(\mathcal{I}) \leq \mathrm{cofin}^+(\mathcal{I}).$$

Proposition 2.3. *Let $\kappa = \kappa^{\aleph_0}$ be an uncountable cardinal.*

(1) *The Cohen algebra $\mathbb{C}_\kappa$ for adding κ many Cohen reals forces that*

$$\mathrm{add}(\mathcal{M}) = \mathrm{cofin}(\mathcal{M}) = \mathrm{cofin}^+(\mathcal{M}) = \aleph_1 \leq \mathrm{cof}(\mathcal{M}) = \kappa = 2^{\aleph_0}.$$

(2) *The Solovay algebra $\mathbb{B}_\kappa$ for adding κ many random reals forces that*

$$\mathrm{add}(\mathcal{N}) = \mathrm{cofin}(\mathcal{N}) = \mathrm{cofin}^+(\mathcal{N}) = \aleph_1 \leq \mathrm{cof}(\mathcal{N}) = \kappa = 2^{\aleph_0}.$$

Proof. (2) In both cases the proof is essentially the same, so let us argue for the Solovay algebra only. Represent κ as the disjoint union $\kappa = \bigcup_{\varepsilon<\omega_1} K_\varepsilon$ where each K_ε is of size κ. For $\varepsilon < \omega_1$ set $\alpha_\varepsilon = \min(K_\varepsilon)$ and $A_\varepsilon = \bigcup_{\zeta<\varepsilon} K_\zeta$.

Suppose that $\bar{r} = \langle r_\alpha : \alpha < \kappa \rangle$ is a $\mathbb{B}_\kappa$-generic over $\mathbf{V}$, so $r_\alpha \in {}^\omega 2$ are random reals, and let us argue in $\mathbf{V}[\bar{r}]$. For each $\varepsilon < \omega_1$ let $\mathcal{I}_\varepsilon$ be the σ-ideal generated by singletons and the family of all Borel null sets coded in $\mathbf{V}[r_\alpha : \alpha \in A_\varepsilon]$. Then $\langle \mathcal{I}_\varepsilon : \varepsilon < \omega_1 \rangle$ is an increasing sequence of σ-ideals, $\mathcal{I}_0$ contains all singletons and $\mathcal{N} = \bigcup_{\varepsilon<\omega_1} \mathcal{I}_\varepsilon$. Moreover, for each $\varepsilon < \omega_1$,

$$B := \{x \in {}^\omega 2 : (\forall n < \omega)(x(2n) = r_{\alpha_\varepsilon}(2n))\} \in \mathcal{I}_{\varepsilon+1} \setminus \mathcal{I}_\varepsilon.$$

Why? Clearly, B is a Borel null set coded in $\mathbf{V}[r_\alpha : \alpha \in A_{\varepsilon+1}]$, so $B \in \mathcal{I}_{\varepsilon+1}$. Suppose B_i are Borel null sets coded in $\mathbf{V}[r_\alpha : \alpha \in A_\varepsilon]$ and $x_i \in ({}^\omega 2)^{\mathbf{V}[\bar{r}]}$ (for $i < \omega$). Choose $x^* \in \{x \in {}^\omega 2 \cap \mathbf{V} : (\forall n < \omega)(x(2n) = 0)\} \setminus \{x_i + r_{\alpha_\varepsilon} : i < \omega\}$. Then $x^* + r_{\alpha_\varepsilon}$ is a random real over $\mathbf{V}[r_\alpha : \alpha \in A_\varepsilon]$ and consequently $x^* + r_{\alpha_\varepsilon} \in B \setminus (\bigcup_{i<\omega} B_i \cup \{x_i : i < \omega\})$. Thus we may conclude that $B \notin \mathcal{I}_\varepsilon$. ∎

Definition 2.4 (Rosłanowski and Shelah [6, Definition 3.4]). Let $\mathcal{I}$ be an ideal of subsets of a space X and α^*, β^* be limit ordinals. *An $\alpha^* \times \beta^*$-base for $\mathcal{I}$* is an indexed family $\mathcal{B} = \{B_{\alpha,\beta} : \alpha < \alpha^* \ \&\ \beta < \beta^*\}$ of sets from $\mathcal{I}$ such that

(i) $\mathcal{B}$ is a basis for $\mathcal{I}$, i.e., $(\forall A \in \mathcal{I})(\exists B \in \mathcal{B})(A \subseteq B)$, and
(ii) for each $\alpha_0, \alpha_1 < \alpha^*$, $\beta_0, \beta_1 < \beta^*$ we have

$$B_{\alpha_0,\beta_0} \subseteq B_{\alpha_1,\beta_1} \quad \Leftrightarrow \quad \alpha_0 \le \alpha_1 \ \&\ \beta_0 \le \beta_1.$$

It follows from the results of Bartoszyński and Kada [2] (for the meager ideal) and Burke and Kada [4] (for the null ideal) that for any cardinals κ and λ of uncountable cofinality we may force that $\mathcal{M}$ has a $\kappa \times \lambda$-basis, and we may also force that $\mathcal{N}$ has a $\kappa \times \lambda$-basis. In [6, Theorem 3.7] we constructed a model in which both ideals have $\kappa \times \lambda$-bases.

Proposition 2.5. *Let κ, λ be regular uncountable cardinals, with $\kappa \le \lambda$.*

(1) *If $\mathcal{I}$ is a σ-ideal on a space X and $\mathcal{I}$ has a $\kappa \times \lambda$-base, then*

$$\kappa = \mathrm{add}(\mathcal{I}) = \mathrm{cofin}(\mathcal{I}) \quad \textit{and} \quad \mathrm{cof}(\mathcal{I}) = \lambda.$$

(2) *There is a ccc forcing notion $\mathbb{P}$ forcing that $2^{\aleph_0} = \lambda^{\aleph_0}$ and*

(i) *the σ-ideal $\mathcal{N}$ has a $\kappa \times \lambda$-base $\{A_{\alpha,\beta} : \alpha < \kappa,\ \beta < \lambda\}$ with the property that*

$$\alpha_0 > \alpha_1 \ \vee\ \beta_0 > \beta_1 \ \Rightarrow\ |A_{\alpha_0,\beta_0} \setminus A_{\alpha_1,\beta_1}| = 2^{\aleph_0},$$

(ii) *the σ-ideal $\mathcal{M}$ has a $\kappa \times \lambda$-base $\{B_{\alpha,\beta} : \alpha < \kappa,\ \beta < \lambda\}$ with the property that*

$$\alpha_0 > \alpha_1 \ \vee\ \beta_0 > \beta_1 \ \Rightarrow\ |B_{\alpha_0,\beta_0} \setminus B_{\alpha_1,\beta_1}| = 2^{\aleph_0}.$$

In particular,

$$\Vdash_{\mathbb{P}} \text{“}\mathrm{add}(\mathcal{M}) = \mathrm{add}(\mathcal{N}) = \mathrm{cofin}^+(\mathcal{M}) = \mathrm{cofin}^+(\mathcal{N}) = \kappa \textit{ and } \mathrm{cof}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \lambda\text{”}.$$

Proof. (1) Assume that $\{B_{\alpha,\beta} : \alpha < \kappa,\ \beta < \kappa\}$ is a $\kappa \times \lambda$-base for $\mathcal{I}$. It should be clear that then $\kappa = \mathrm{add}(\mathcal{I})$ and $\mathrm{cof}(\mathcal{I}) = \lambda$.

Let us argue that $\mathrm{cofin}(\mathcal{I}) \le \kappa$. For $\zeta < \kappa$ let $\mathcal{I}_\zeta$ be the σ-ideal generated by the family $\{B_{\alpha,\beta} : \alpha \le \zeta \ \&\ \beta < \lambda\}$. Plainly, $\langle \mathcal{I}_\zeta : \zeta < \kappa\rangle$ is an increasing sequence of σ-ideals such that $\mathcal{I} = \bigcup_{\zeta<\kappa} \mathcal{I}_\zeta$. We claim that $B_{\zeta+1,0} \in \mathcal{I}_{\zeta+1} \setminus \mathcal{I}_\zeta$. Suppose that $I \subseteq (\zeta+1)\times\lambda$ is countable. Then we may choose $\beta^* < \lambda$ such that $I \subseteq (\zeta+1)\times\beta^*$ and consequently $\bigcup\{B_{\alpha,\beta} : (\alpha,\beta)\in I\} \subseteq B_{\zeta,\beta^*}$. But $B_{\zeta+1,0} \nsubseteq B_{\zeta,\beta^*}$ and so $B_{\zeta+1,0} \nsubseteq \bigcup\{B_{\alpha,\beta} : (\alpha,\beta)\in I\}$. Now we may conclude that $B_{\zeta+1,0} \notin \mathcal{I}_\zeta$.

(2) The forcing notion $\mathbb{Q}^{\kappa,\lambda}$ constructed in the proof of [6, Theorem 3.7] has the desired properties (see [6, Remark 3.8]). ■

3. cofin **and** $\mathcal{I}_f$**.** We introduce here a nicely definable Borel ideal $\mathcal{I}_f$ for which, consistently, $\mathrm{add}(\mathcal{I}_f) < \mathrm{cofin}(\mathcal{I}_f)$. The proof of the consistency will resemble Shelah [8, Chapter II, Theorem 4.6] (and thus also [7]). The appropriate forcing notion is obtained by means of FS iteration of ccc forcing notions, however the iteration itself is forced too.

Context 3.1. Let us fix two strictly increasing functions $f, g : \omega \to \omega$ such that for each $n < \omega$ we have

$$2 < g(n) < f(n) \quad \text{and} \quad \frac{g(n)}{f(n)} \le \frac{1}{n+1}.$$

Definition 3.2.

(1) *A null slalom below* f is a function $\varphi \in \prod_{n<\omega} \mathcal{P}(f(n))$ such that $\lim_{n\to\infty} |\varphi(n)|/f(n) = 0$.
(2) Let $\mathcal{S}_f$ be the collection of all null slaloms below f and let $\mathcal{X}_f = \prod_{n<\omega} f(n)$ be equipped with the natural product topology (so $\mathcal{X}_f$ is a Polish space).
(3) For $\varphi \in \mathcal{S}_f$ we define

$$[\varphi] = \{x \in \mathcal{X}_f : (\exists^\infty n < \omega)(x(n) \in \varphi(n))\}.$$

Observation 3.3. *Let* $\varphi_i \in \mathcal{S}_f$ *(for* $i < \omega$*).*

(1) $[\varphi_0] \subseteq [\varphi_1]$ *if and only if* $(\forall^\infty n < \omega)(\varphi_0(n) \subseteq \varphi_1(n))$.
(2) *There is* $\psi \in \mathcal{S}_f$ *such that* $\bigcup_{i<\omega} [\varphi_i] \subseteq [\psi]$.

Definition 3.4. Let $\mathcal{I}_f$ be the σ-ideal of subsets of $\mathcal{X}_f$ generated by all sets $[\varphi]$ for $\varphi \in \mathcal{S}_f$. Thus, by Observation 3.3,

$$\mathcal{I}_f = \{A \subseteq \mathcal{X}_f : (\exists \varphi \in \mathcal{S}_f)(A \subseteq [\varphi])\}.$$

We will construct a forcing notion $\mathbb{P}$ forcing that $\mathrm{add}(\mathcal{I}_f) < \mathrm{cofin}(\mathcal{I}_f)$, but first we need several technical ingredients.

Definition 3.5. For a cardinal κ we define a forcing notion $\mathbb{Q}_0^\kappa$.

A condition p *in* $\mathbb{Q}_0^\kappa$ is a finite function such that $\mathrm{dom}(p) \subseteq \kappa$ and for some $n = n^p < \omega$, for all $\varepsilon \in \mathrm{dom}(p)$ we have $p(\varepsilon) \in \prod_{i<n} [f(i)]^{g(i)}$.

The order $\leq = \leq_{\mathbb{Q}_0^\kappa}$ *of* $\mathbb{Q}_0^\kappa$ is defined by letting $p \leq q$ if and only if

$$\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \quad \text{and} \quad (\forall \varepsilon \in \mathrm{dom}(p))(p(\varepsilon) \trianglelefteq q(\varepsilon)).$$

For $\varepsilon < \kappa$, a $\mathbb{Q}_0^\kappa$-name $\underset{\sim}{\nu}(\varepsilon)$ is defined by

$$\Vdash_{\mathbb{Q}_0^\kappa} \underset{\sim}{\nu}(\varepsilon) = \bigcup \{p(\varepsilon) : \varepsilon \in \mathrm{dom}(p) \ \&\ p \in \underset{\sim}{G}_{\mathbb{Q}_0^\kappa}\}.$$

OBSERVATION 3.6.

(1) *The forcing notion* $\mathbb{Q}_0^\kappa$ *is equivalent to* $\mathbb{C}_\kappa$*, the forcing adding* κ *many Cohen reals.*
(2) $\Vdash_{\mathbb{Q}_0^\kappa}$ *"for every* $\varepsilon < \kappa$ *we have* $\underset{\sim}{\nu}(\varepsilon) \in \prod_{i<\omega} [f(i)]^{g(i)} \subseteq \mathcal{S}_f$ *".*

DEFINITION 3.7. Let μ be an infinite cardinal and $\bar{\varphi} = \langle \varphi_\zeta : \zeta < \mu \rangle$ be a sequence of null slaloms below f (so $\varphi_\zeta \in \mathcal{S}_f$ for $\zeta < \mu$). We define a forcing notion $\mathbb{Q}^*_\mu(\bar{\varphi})$.

A condition in $\mathbb{Q}^*_\mu(\bar{\varphi})$ is a tuple $p = (k^p, m^p, u^p, \sigma^p) = (k, m, u, \sigma)$ such that

(a) $k, m < \omega$, $\emptyset \neq u \in [\mu]^{<\aleph_0}$, $\sigma \in \prod_{i<k} \mathcal{P}(f(i))$, and
(b) for each $\ell \geq k$ and $\zeta \in u$ we have $|\varphi_\zeta(\ell)| < f(\ell)/(m \cdot |u|)$.

The order $\leq = \leq_{\mathbb{Q}^*_\mu(\bar{\varphi})}$ *of* $\mathbb{Q}^*_\mu(\bar{\varphi})$ is defined by $p \leq q$ if and only if ($p, q \in \mathbb{Q}^*_\mu(\bar{\varphi})$ and) $k^p \leq k^q$, $m^p \leq m^q$, $u^p \subseteq u^q$, $\sigma^p \trianglelefteq \sigma^q$, and for each $\ell \in [k^p, k^q)$ we have

$$|\sigma^q(\ell)| \leq f(\ell)/m^p \quad \text{and} \quad \bigcup \{\varphi_\zeta(\ell) : \zeta \in u^p\} \subseteq \sigma^q(\ell).$$

We also define a $\mathbb{Q}^*_\mu(\bar{\varphi})$-name $\underset{\sim}{\varsigma}$ by

$$\Vdash_{\mathbb{Q}^*_\mu(\bar{\varphi})} \underset{\sim}{\varsigma} = \bigcup \{\sigma^p : p \in \underset{\sim}{G}_{\mathbb{Q}^*_\mu(\bar{\varphi})}\}.$$

PROPOSITION 3.8. *Let* μ *be an infinite cardinal and* $\bar{\varphi} = \langle \varphi_\zeta : \zeta < \mu \rangle \subseteq \mathcal{S}_f$*. Then* $\mathbb{Q}^*_\mu(\bar{\varphi})$ *is a well defined ccc forcing notion of size* μ *and*

$$\Vdash_{\mathbb{Q}^*_\mu(\bar{\varphi})} \text{“} \underset{\sim}{\varsigma} \in \mathcal{S}_f \ \&\ \bigcup_{\zeta<\mu} [\varphi_\zeta] \subseteq [\underset{\sim}{\varsigma}] \in \mathcal{I}_f \text{”}.$$

Proof. First note that if $p \in \mathbb{Q}^*_\mu(\bar{\varphi})$ and $m = m^p$, $k = k^p + 1$, $u = u^p$ and $\sigma = \sigma^p {}^\frown \langle \bigcup_{\zeta \in u} \varphi_\zeta(k^p) \rangle$, then $(k, m, u, \sigma) \in \mathbb{Q}^*_\mu(\bar{\varphi})$ is a condition stronger than p. Hence we may conclude that $\Vdash_{\mathbb{Q}^*_\mu(\bar{\varphi})} \underset{\sim}{\varsigma} \in \prod_{i<\omega} \mathcal{P}(f(i))$.

Also, if $p \in \mathbb{Q}^*_\mu(\bar{\varphi})$ and $m > m^p$, then we may find $k > k^p$ such that $|\varphi_\zeta(\ell)| < f(\ell)/(m \cdot |u^p|)$ for all $\zeta \in u^p$ and $\ell \geq k$. Let $u = u^p$ and σ in $\prod_{i<k} \mathcal{P}(f(i))$ be such that $\sigma(\ell) = \sigma^p(\ell)$ for $\ell < k^p$ and $\sigma(\ell) = \bigcup_{\zeta \in u} \varphi_\zeta(\ell)$ for $\ell \in [k^p, k)$. Then $(k, m, u, \sigma) \in \mathbb{Q}^*_\mu(\bar{\varphi})$ is a condition stronger than p and it forces that $|\underset{\sim}{\varsigma}(\ell)| \leq f(\ell)/m$ for all $\ell \geq k$. Hence we may conclude that $\Vdash_{\mathbb{Q}^*_\mu(\bar{\varphi})} \underset{\sim}{\varsigma} \in \mathcal{S}_f$.

It follows from the definition of the order of $\mathbb{Q}^*_\mu(\bar\varphi)$ that

$$p \Vdash_{\mathbb{Q}^*_\mu(\bar\varphi)} (\forall \ell \geq k^p)(\forall \zeta \in u^p)(\varphi_\zeta(\ell) \subseteq \underset{\sim}{\varsigma}(\ell)),$$

and hence easily $\Vdash_{\mathbb{Q}^*_\mu(\bar\varphi)} \bigcup_{\zeta<\mu}[\varphi_\zeta] \subseteq [\underset{\sim}{\varsigma}]$.

Let us argue now that $\mathbb{Q}^*_\mu(\bar\varphi)$ satisfies the ccc. Suppose $\langle p_\varepsilon : \varepsilon < \omega_1\rangle \subseteq \mathbb{Q}^*_\mu(\bar\varphi)$. For each $\varepsilon < \omega_1$, we may find $K^\varepsilon > k^{p_\varepsilon}$ such that

$(\oplus)_1$ $(\forall \ell \geq K^\varepsilon)(\forall \zeta \in u^{p_\varepsilon})(|\varphi_\zeta(\ell)| < f(\ell)/(2 \cdot |u^{p_\varepsilon}| \cdot m^{p_\varepsilon}))$,

and define $\rho^\varepsilon \in \prod_{i<K^\varepsilon} \mathcal{P}(f(i))$ so that $\rho^\varepsilon(\ell) = \sigma^{p_\varepsilon}(\ell)$ for $\ell < k^{p_\varepsilon}$ and $\rho^\varepsilon(\ell) = \bigcup_{\zeta \in u^{p_\varepsilon}} \varphi_\zeta(\ell)$ for $\ell \in [k^{p_\varepsilon}, K^\varepsilon)$. Then we may find an uncountable set $S \subseteq \omega_1$ and K^*, m^*, ρ^*, ℓ^* such that for all $\varepsilon \in S$:

$(\oplus)_2$ $K^* = K^\varepsilon$, $m^* = m^{p_\varepsilon}$, $\rho^* = \rho^\varepsilon$ and $|u^\varepsilon| = \ell^*$.

Consider distinct $\varepsilon_0, \varepsilon_1 \in S$: letting $u^* = u^{\varepsilon_0} \cup u^{\varepsilon_1}$ we get a condition $(K^*, m^*, u^*, \rho^*) \in \mathbb{Q}^*_\mu(\bar\varphi)$ stronger than both p_{ε_0} and p_{ε_1}. ■

Definition 3.9. Let $\kappa < \lambda$ be uncountable regular cardinals.

(1) A *Y-iteration for* κ, λ is a finite support iteration $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ of ccc forcing notions such that the following demands $(\otimes)_1$–$(\otimes)_3$ are satisfied.

$(\otimes)_1$ $0 < \alpha \leq \lambda$ and $\underset{\sim}{\mathbb{Q}}_0 = \mathbb{Q}^\kappa_0$ is the forcing notion adding κ Cohen reals as represented in Definition 3.5 with $\mathbb{Q}^\kappa_0$-names $\underset{\sim}{\nu}(\varepsilon)$ (for $\varepsilon < \kappa$) as defined there.

$(\otimes)_2$ For each $\beta < \alpha$ we have $\Vdash_{\mathbb{P}_\beta} |\underset{\sim}{\mathbb{Q}}_\beta| \leq \lambda$.

$(\otimes)_3$ Let $n < \omega$. Suppose that $\langle p_\zeta : \zeta < \kappa\rangle \subseteq \mathbb{P}_\alpha$ and $\langle \delta_\zeta : \zeta < \kappa\rangle \subseteq \kappa$ and $\delta_\zeta \neq \delta_{\zeta'}$ for $\zeta < \zeta' < \kappa$. Then there are $q \in \mathbb{P}_\alpha$, $m > n$, $v \subseteq \kappa$, and A_ζ (for $\zeta \in v$) such that

(i) $|v| \geq f(m)/(2 \cdot g(m))$,
(ii) $p_\zeta \leq q$ for all $\zeta \in v$,
(iii) $A_\zeta \in [f(m)]^{g(m)}$ (for $\zeta \in v$) are pairwise disjoint sets,
(iv) $q \Vdash_{\mathbb{P}_\alpha}$ "$\big(\forall \zeta \in v\big)\big(\underset{\sim}{\nu}(\delta_\zeta)(m) = A_\zeta\big)$".

(2) The collection of all (dense subsets of) Y-iterations for κ, λ of length $< \lambda$ which belong to $\mathcal{H}(\beth^+_\lambda)$ is denoted by $\mathbb{Y}^\lambda_\kappa$. It is ordered by the end-extension of iterations $\trianglelefteq$.

The condition 3.9(1)$(\otimes)_3$ implies that the null slaloms added at the first step of a Y-iteration provide a family of sets whose union is not included in any null slalom. Note that in 3.9(1)$(\otimes)_3$ necessarily $|v| \leq f(m)/g(m)$.

Lemma 3.10. *Assume $\kappa < \lambda$ are regular uncountable cardinals. Suppose that $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ is a Y-iteration for κ, λ. Then* $\Vdash_{\mathbb{P}_\alpha} \operatorname{add}(\mathcal{I}_f) \leq \kappa$.

Proof. We know that for each $\varepsilon < \kappa$ we have $\Vdash_{\mathbb{P}_\alpha} \underset{\sim}{\nu}(\varepsilon) \in \mathcal{S}_f$ (remember Observation 3.6), and therefore $\Vdash_{\mathbb{P}_\alpha} \{[\underset{\sim}{\nu}(\varepsilon)] : \varepsilon < \kappa\} \subseteq \mathcal{I}_f$. We are going to argue that

$$\Vdash_{\mathbb{P}_\alpha} \bigcup\{[\underset{\sim}{\nu}(\varepsilon)] : \varepsilon < \kappa\} \notin \mathcal{I}_f.$$

Suppose towards a contradiction that this is not the case. Then we may pick $p \in \mathbb{P}_\alpha$ and a $\mathbb{P}_\alpha$-name $\underset{\sim}{\varphi}$ such that

$$p \Vdash_{\mathbb{P}_\alpha} \underset{\sim}{\varphi} \in \mathcal{S}_f \ \& \ (\forall \varepsilon < \kappa)(\forall^\infty n < \omega)\big(\underset{\sim}{\nu}(\varepsilon)(n) \subseteq \underset{\sim}{\varphi}(n)\big)$$

(remember Observation 3.3). Now for each $\varepsilon < \kappa$ we pick a condition $p_\varepsilon \geq p$ and an integer $n_\varepsilon < \omega$ such that

$$p_\varepsilon \Vdash_{\mathbb{P}_\alpha} (\forall n \geq n_\varepsilon)\big(\underset{\sim}{\nu}(\varepsilon)(n) \subseteq \underset{\sim}{\varphi}(n) \ \& \ |\underset{\sim}{\varphi}(n)|/f(n) < 1/4\big).$$

For some $n^* < \omega$ the set $S = \{\varepsilon < \kappa : n_\varepsilon = n^*\}$ is of size κ. Apply 3.9(1)$(\otimes)_3$ to $\langle p_\varepsilon : \varepsilon \in S\rangle \subseteq \mathbb{P}_\alpha$ and $\langle \varepsilon : \varepsilon \in S\rangle \subseteq \kappa$ and $n = n^*$ to find $q \in \mathbb{P}_\alpha$, $m > n^*$, $v \subseteq S$, and A_ε (for $\varepsilon \in v$) such that conditions (i)–(iv) there hold. Then

$$q \Vdash_{\mathbb{P}_\alpha} \text{“} \bigcup_{\varepsilon \in v} A_\varepsilon = \bigcup_{\varepsilon \in v} \underset{\sim}{\nu}(\varepsilon)(m) \subseteq \underset{\sim}{\varphi}(m) \ \& \ |\underset{\sim}{\varphi}(m)| < f(m)/4\text{”}.$$

But $|\bigcup_{\varepsilon \in v} A_\varepsilon| = |v| \cdot g(m) \geq f(m)/2$, a contradiction. ∎

Context 3.11. For the rest of this section we fix uncountable regular cardinals $\kappa < \lambda$ such that $\lambda^\kappa = \lambda$. Also, instead of "Y-iteration for κ, λ" we will just say "Y-iteration".

Lemma 3.12.

(1) $\langle \mathbb{P}_0, \mathbb{Q}_0^\kappa\rangle$ *is a Y-iteration* (*of length* 1).
(2) *Assume that* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ *is a Y-iteration of length* $\alpha < \lambda$ *and* $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}_\alpha$*-name for a ccc forcing notion of size* $<\kappa$ (*i.e.,* $\Vdash_{\mathbb{P}_\alpha} |\underset{\sim}{\mathbb{Q}}| < \kappa$). *Then* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle^\frown\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}\rangle$ *is a Y-iteration of length* $\alpha + 1$. *In particular,* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle^\frown\langle \mathbb{P}_\alpha, \mathbb{C}\rangle$ *is a Y-iteration.*
(3) *If* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ *is a Y-iteration and* $\underset{\sim}{\mathbb{Q}}$ *is a* $\mathbb{P}_\alpha$*-name for a* σ*-centered forcing, then* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle^\frown\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}\rangle$ *is a Y-iteration.*
(4) *If* $\gamma \leq \lambda$ *is a limit ordinal and* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma\rangle$ *is an FS iteration such that* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ *is a Y-iteration for every* $\alpha < \gamma$, *then* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma\rangle$ *is a Y-iteration.*
(5) $(\mathbb{Y}^\lambda_\kappa, \trianglelefteq)$ *is a* $<\lambda$*-complete forcing notion* (*i.e., all chains of length* $<\lambda$ *have an upper bound in* $\mathbb{Y}^\lambda_\kappa$).

Proof. In all cases the only demand of Definition 3.9(1) that needs to be verified is $(\otimes)_3$.

(1) Let $\mathbb{Q}_0^\kappa$ be the forcing notion adding κ Cohen reals as described in Definition 3.5. Let $n < \omega$, $\delta_\zeta \in \kappa$ and $p_\zeta \in \mathbb{Q}_0^\kappa$ (for $\zeta < \kappa$) satisfy the assumptions of 3.9(1)$(\otimes)_3$. By making conditions p_ζ stronger and possibly passing to a subsequence, we may also assume that:

$(*)_1$ $\delta_\zeta \in \mathrm{dom}(p_\zeta)$ for all $\zeta < \kappa$,

$(*)_2$ for some $m > n+2$, for all $\zeta < \kappa$, we have $n^{p_\zeta} = m$ (so $p_\zeta(\varepsilon) \in \prod_{i<m}[f(i)]^{g(i)}$ for $\varepsilon \in \mathrm{dom}(p_\zeta)$),

$(*)_3$ the family $\{\mathrm{dom}(p_\zeta) : \zeta < \kappa\}$ forms a Δ-system of finite sets and for all $\zeta, \zeta' < \kappa$ the conditions $p_\zeta, p_{\zeta'}$ are compatible.

Pick any $v \subseteq \kappa$ of size $\lceil f(m)/(2 \cdot g(m)) \rceil$. Since

$$\left\lceil \frac{f(m)}{2 \cdot g(m)} \right\rceil \cdot g(m) \le \frac{f(m)}{2} + g(m) \le \frac{f(m)}{2} + \frac{f(m)}{m+1} < f(m),$$

we may choose pairwise disjoint sets $A_\zeta \in [f(m)]^{g(m)}$ (for $\zeta \in v$). Now define a condition $q \in \mathbb{Q}_0^\kappa$ so that $\mathrm{dom}(q) = \bigcup\{\mathrm{dom}(p_\zeta) : \zeta \in v\}$, $n^q = m+1$ and for $\varepsilon \in \mathrm{dom}(p_\zeta)$ the sequence $q(\varepsilon)$ extends $p_\zeta(\varepsilon)$ and $q(\delta_\zeta)(m) = A_\zeta$ (for $\zeta \in v$).

(2) Without loss of generality, for some ordinal $\gamma^* < \kappa$ we have $\Vdash_{\mathbb{P}_\alpha}$"the set of conditions in $\underset{\sim}{\mathbb{Q}}$ is γ^*". Let $n < \omega$ and $p_\zeta \in \mathbb{P}_{\alpha+1}$, $\delta_\zeta \in \kappa$ (for $\zeta < \kappa$) satisfy the assumptions of 3.9(1)$(\otimes)_3$. We may make our conditions stronger and we may pass to a subsequence, so we may assume that $\alpha \in \mathrm{dom}(p_\zeta)$ and $p_\zeta(\alpha) = \gamma < \gamma^*$ is an actual object, not a name (for $\zeta < \kappa$). Apply the assumption of 3.9(1)$(\otimes)_3$ for $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ to $n, p_\zeta{\restriction}\alpha, \delta_\zeta$ (for $\zeta < \kappa$) and choose $m > n$, $q^* \in \mathbb{P}_\alpha$, $v \subseteq \kappa$ and pairwise disjoint sets $A_\zeta \subseteq f(m)$ each of size $g(m)$ (for $\zeta \in v$) such that

- $|v| \ge f(m)/(2 \cdot g(m))$, and
- q^* is stronger than all $p_\zeta{\restriction}\alpha$ for $\zeta \in v$, and it forces that $\underset{\sim}{\nu}(\delta_\zeta)(m) = A_\zeta$ (for $\zeta \in v$).

Let $q \in \mathbb{P}_{\alpha+1}$ be such that $q{\restriction}\alpha = q^*$ and $q(\alpha) = \gamma$.

(3) Assume that $\Vdash_{\mathbb{P}_\alpha}$"$\underset{\sim}{\mathbb{Q}}$ is a σ-centered forcing notion", and fix a $\mathbb{P}_\alpha$-name $\underset{\sim}{F}$ such that

$$\begin{aligned}&\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{F} : \underset{\sim}{\mathbb{Q}} \to \omega \text{ is a function satisfying:}\\ &\quad \text{if } x_0, \ldots, x_k \in \underset{\sim}{\mathbb{Q}},\ k < \omega, \text{ and } \underset{\sim}{F}(x_0) = \cdots = \underset{\sim}{F}(x_k) = m,\\ &\quad \text{then the conditions } x_0, \ldots, x_k \text{ have a common upper bound in } \underset{\sim}{\mathbb{Q}}\text{”.}\end{aligned}$$

Suppose that $n < \omega$ and $p_\zeta \in \mathbb{P}_{\alpha+1}$, $\delta_\zeta \in \kappa$ (for $\zeta < \kappa$) satisfy the assumptions of 3.9(1)$(\otimes)_3$. By making the conditions stronger and passing to a subsequence we may demand that $\alpha \in \mathrm{dom}(p_\zeta)$ and for some $M < \omega$ we also have $p_\zeta{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha}$"$\underset{\sim}{F}(p_\zeta(\alpha)) = M$". Use the assumption of 3.9(1)$(\otimes)_3$ for $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ for $n, p_\zeta{\restriction}\alpha, \delta_\zeta$ (for $\zeta < \kappa$) to find $m > n$, $q^* \in \mathbb{P}_\alpha$, $v \subseteq \kappa$ and pairwise disjoint sets $A_\zeta \in [f(m)]^{g(m)}$ (for $\zeta \in v$) such that

- $|v| \ge f(m)/(2 \cdot g(m))$, and
- q^* is stronger than all $p_\zeta{\restriction}\alpha$ for $\zeta \in v$, and it forces that $\underset{\sim}{\nu}(\delta_\zeta)(m) = A_\zeta$ (for $\zeta \in v$).

Then also the condition q^* forces that $\underset{\sim}{F}(p_\zeta(\alpha)) = M$ for all $\zeta \in v$, and thus we may pick a $\mathbb{P}_\alpha$-name $\underset{\sim}{q}_\alpha$ such that $q^* \Vdash$ " $\underset{\sim}{q}_\alpha$ is a condition stronger than all $p_\zeta(\alpha)$ for $\zeta \in v$". Define $q \in \mathbb{P}_{\alpha+1}$ by $q{\restriction}\alpha = q^*$ and $q(\alpha) = \underset{\sim}{q}_\alpha$.

(4) Let n, p_ζ, δ_ζ (for $\zeta < \kappa$) be as in the assumptions of 3.9(1)$(\otimes)_3$. By passing to a subsequence we may also demand that $\{\mathrm{dom}(p_\zeta) : \zeta < \kappa\}$ is a Δ-system of finite subsets of γ with root D. Pick $\alpha < \gamma$ such that $D \subseteq \alpha$. Since $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ is a Y-iteration, we may apply 3.9(1)$(\otimes)_3$ to n, δ_ζ and $p_\zeta{\restriction}\alpha$ (for $\zeta < \kappa$). This will give us q^*, v and A_ζ (for $\zeta \in v$) satisfying (i)–(iv) there (with $p_\zeta{\restriction}\alpha$ in place of p_ζ and q^* in place of q). Let $q \in \mathbb{P}_\gamma$ be such that $\mathrm{dom}(q) = \mathrm{dom}(q^*) \cup \bigcup\{\mathrm{dom}(p_\zeta) : \zeta \in v\}$ and $q{\restriction}\alpha = q^*$ and $q(\beta) = p_\zeta(\beta)$ whenever $\zeta \in v$, $\beta \in \mathrm{dom}(p_\zeta) \setminus \alpha$.

(5) Follows from (4). ■

Lemma 3.13. *Assume that*

(a) $\aleph_0 \le \mu \le \kappa$ *is a regular cardinal,* $\alpha < \lambda$ *is a limit ordinal of cofinality* μ *and* $\langle \alpha(\zeta) : \zeta < \mu\rangle$ *is a strictly increasing sequence cofinal in* α*,*

(b) $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle$ *is a Y-iteration,*

(c) $\underset{\sim}{\bar{\varphi}} = \langle \underset{\sim}{\varphi}_\zeta : \zeta < \mu\rangle$ *is a* $\mathbb{P}_{\alpha(0)}$*-name for a* μ*-sequence of null slaloms below* f *(so* $\Vdash \underset{\sim}{\varphi}_\zeta \in \mathcal{S}_f$*),*

(d) *for each* $\zeta < \mu$ *we have* $\Vdash_{\mathbb{P}_{\alpha(\zeta)}} \underset{\sim}{\mathbb{Q}}_{\alpha(\zeta)} = \mathbb{C}$ *with* $\underset{\sim}{c}_\zeta$ *being the* $\mathbb{P}_{\alpha(\zeta)+1}$*-name for the Cohen real in* ${}^{\omega}2$ *added by* $\underset{\sim}{\mathbb{Q}}_{\alpha(\zeta)}$*,*

(e) $\underset{\sim}{\tau}_\zeta$ *is a* $\mathbb{P}_\alpha$*-name for an element of* 2 *(for* $\zeta < \mu$*),*

(f) *for* $\zeta < \mu$*,* $\underset{\sim}{\psi}_\zeta$ *is a* $\mathbb{P}_\alpha$*-name for a null slalom below* f *such that*

$$\Vdash_{\mathbb{P}_\alpha} \text{``} \underset{\sim}{\psi}_\zeta(i) = \begin{cases} \underset{\sim}{\varphi}_\zeta(i) & \text{if } \underset{\sim}{c}_\zeta(i) = \underset{\sim}{\tau}_\zeta, \\ \emptyset & \text{if } \underset{\sim}{c}_\zeta(i) = 1 - \underset{\sim}{\tau}_\zeta \end{cases} \quad \text{for each } i < \omega \text{''},$$

and $\underset{\sim}{\bar{\psi}} = \langle \underset{\sim}{\psi}_\zeta : \zeta < \mu\rangle$ *is the resulting* $\mathbb{P}_\alpha$*-name for a* μ*-sequence of null slaloms below* f*.*

Then $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle^\frown\langle \mathbb{P}_\alpha, \mathbb{Q}^*_\mu(\underset{\sim}{\bar{\psi}})\rangle$ *is a Y-iteration of length* $\alpha + 1$*.*

Proof. First we consider the case when $\mu = \kappa$ and let us argue that condition 3.9(1)$(\otimes)_3$ holds for $\mathbb{P}_{\alpha+1}$.

Let $n < \omega$, $p_\zeta \in \mathbb{P}_{\alpha+1}$ and $\delta_\zeta < \kappa$ (for $\zeta < \kappa$) be such that $\delta_\zeta \ne \delta_{\zeta'}$ for $\zeta < \zeta' < \kappa$. For each $\zeta < \kappa$ pick a condition $p'_\zeta \in \mathbb{P}_{\alpha+1}$ stronger than p_ζ and such that

$(*)_1$ $\alpha \in \mathrm{dom}(p'_\zeta)$, and for some $k^\zeta, m^\zeta, u^\zeta$ and σ^ζ (objects, not names) we have $p'_\zeta{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha}$ " $p'_\zeta(\alpha) = (k^\zeta, m^\zeta, u^\zeta, \sigma^\zeta)$".

Choose conditions $p''_\zeta \in \mathbb{P}_{\alpha+1}$ stronger than p'_ζ (so also $p''_\zeta \ge p_\zeta$), and such that $p''_\zeta(\alpha) = p'_\zeta(\alpha)$ and for all ζ:

$(*)_2$ for each $\varepsilon \in u^\zeta$ we have $p''_\zeta{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha}$ " $\underset{\sim}{\tau}_\varepsilon = \mathrm{t}^\zeta_\varepsilon$" for some $\mathrm{t}^\zeta_\varepsilon$ (an object, not name),

$(*)_3$ for some $i^\zeta < \omega$ and all $\varepsilon \in u^\zeta$,

$$\alpha(\varepsilon) \in \mathrm{dom}(p''_\zeta) \text{ and } p''_\zeta(\alpha(\varepsilon)) \in {}^{i^\zeta}2 \text{ are actual objects, not names.}$$

Since each $\underset{\sim}{\varphi}_\varepsilon$ is a $\mathbb{P}_{\alpha(0)}$-name, we may decide the initial segments of $\underset{\sim}{\varphi}_\varepsilon$ by strengthening $p''_\zeta{\restriction}\alpha(0)$ only (i.e., without changing $p''_\zeta{\restriction}[\alpha(0),\alpha]$). Therefore, after using a procedure similar to that in the proof of 3.8, for each $\zeta < \kappa$ we may find a condition $p^*_\zeta \in \mathbb{P}_{\alpha+1}$, $K^\zeta > k^\zeta + i^\zeta$ and a sequence $\rho^\zeta \in \prod_{i<K^\zeta} \mathcal{P}(f(i))$ such that

$(*)_4$ $p_\zeta \leq p''_\zeta \leq p^*_\zeta$ and $p''_\zeta{\restriction}[\alpha(0),\alpha) = p^*_\zeta{\restriction}[\alpha(0),\alpha)$, and

$(*)_5$ $p^*_\zeta \Vdash_{\mathbb{P}_\alpha}$ " $p^*_\zeta(\alpha) = (K^\zeta, m^\zeta, u^\zeta, \rho^\zeta)$ ".

Next we may find a set $S \subseteq \kappa$ of size κ, and K^*, m^*, ρ^*, i^* and ℓ^* such that

$(*)_6$ $K^* = K^\zeta$, $m^* = m^\zeta$, $\rho^* = \rho^\zeta$, $|u^\zeta| = \ell^*$ and $i^\zeta = i^*$, for all $\zeta \in S$,

$(*)_7$ $\{u^\zeta : \zeta \in S\}$ is a Δ-system of finite subsets of κ with root U,

$(*)_8$ $\{\mathrm{dom}(p^*_\zeta) : \zeta \in S\}$ is a Δ-system of finite subsets of $\alpha+1$ with root D,

$(*)_9$ for some $\varepsilon^* < \kappa$ we have $D \setminus \{\alpha\} \subseteq \alpha(\varepsilon^*)$ and $U = u^\zeta \cap \varepsilon^*$ for all $\zeta \in S$.

Since $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha(\varepsilon^*)\rangle$ is a Y-iteration, we may apply 3.9(1)$(\otimes)_3$ to $\langle p^*_\zeta{\restriction}\alpha(\varepsilon^*), \underset{\sim}{\delta}_\zeta : \zeta \in S\rangle$ and n. This will give us $v \subseteq S$, $q_0 \in \mathbb{P}_{\alpha(\varepsilon^*)}$, $m > n$ and $A_\zeta \in [f(m)]^{g(m)}$ for $\zeta \in v$ such that

$(*)_{10}$
- $|v| \geq f(m)/(2\cdot g(m))$ and $p^*_\zeta{\restriction}\alpha(\varepsilon^*) \leq q_0$ for all $\zeta \in v$,
- $A_\zeta \cap A_{\zeta'} = \emptyset$ for distinct $\zeta, \zeta' \in v$, and
- $q_0 \Vdash_{\mathbb{P}_{\alpha(\varepsilon^*)}}$ " $(\forall \zeta \in v)(\underset{\sim}{\nu}(\delta_\zeta)(m) = A_\zeta)$ ".

Next, since $\underset{\sim}{\varphi}_\varepsilon$ are $\mathbb{P}_{\alpha(0)}$-names, we may pick $q_1 \in \mathbb{P}_{\alpha(\varepsilon^*)}$, $q_1 \geq q_0$, $K > K^* \geq i^*$ and $\rho_\varepsilon \in \prod_{i<K} \mathcal{P}(f(i))$ (for $\varepsilon \in U$) such that

$$q_1 \Vdash_{\mathbb{P}_{\alpha(\varepsilon^*)}} \text{" } (\forall \varepsilon \in U)(\underset{\sim}{\varphi}_\varepsilon{\restriction}K = \rho_\varepsilon) \text{ "}$$

and

$$q_1 \Vdash_{\mathbb{P}_{\alpha(\varepsilon^*)}} \text{" } (\forall j \geq K)(\forall \zeta \in v)(\forall \varepsilon \in u^\zeta)(|\underset{\sim}{\varphi}_\varepsilon(j)| < f(j)/(|v|\cdot \ell^* \cdot m^*)) \text{ ".}$$

Define $q \in \mathbb{P}_{\alpha+1}$ so that

- $\mathrm{dom}(q) = \mathrm{dom}(q_1) \cup \bigcup\{\mathrm{dom}(p^*_\zeta) : \zeta \in v\}$,
- $q{\restriction}\alpha(\varepsilon^*) = q_1$,
- if $\zeta \in v$ and $\beta \in \mathrm{dom}(p^*_\zeta) \setminus (\alpha(\varepsilon^*) \cup \{\alpha(\varepsilon) : \varepsilon \in u^\zeta\})$, then $q(\beta) = p^*_\zeta(\beta)$,
- if $\zeta \in v$ and $\varepsilon \in u^\zeta \setminus \varepsilon^*$, then $q(\alpha(\varepsilon)) \in {}^K2$ is such that $p''_\zeta(\alpha(\varepsilon)) = p^*_\zeta(\alpha(\varepsilon)) \lhd q(\alpha(\varepsilon))$ and for $i \in [i^*, K)$ we have $q(\alpha(\varepsilon))(i) = 1 - \mathfrak{t}^\zeta_\varepsilon$,
- $q(\alpha) = (K, m^*, u^+, \sigma^+)$, where $u^+ = \bigcup\{u^\zeta : \zeta \in v\}$ and $\sigma^+ \in \prod_{i<K} \mathcal{P}(f(i))$ is such that $\rho^* \lhd \sigma^+$ and $\sigma^+(i) = \bigcup_{\varepsilon \in U} \rho_\varepsilon(i)$ for $i \in [K^*, K)$.

The rest, when $\mu = \kappa$, should be clear.

Let us assume now that $\mu < \kappa$ and again, to argue for 3.9(1)$(\otimes)_3$, suppose that $n < \omega$, $p_\zeta \in \mathbb{P}_{\alpha+1}$ and $\delta_\zeta < \kappa$ (for $\zeta < \kappa$) are such that $\delta_\zeta \neq \delta_{\zeta'}$ for $\zeta < \zeta' < \kappa$. Passing to stronger conditions we may assume that, for each $\zeta < \kappa$,

$$\alpha \in \mathrm{dom}(p_\zeta) \quad \text{and} \quad p_\zeta{\restriction}\alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\, p_\zeta(\alpha) = (k^\zeta, m^\zeta, u^\zeta, \sigma^\zeta)\,\text{”}$$

(where $k^\zeta, m^\zeta, u^\zeta, \sigma^\zeta$ are actual objects). For some $\varepsilon^* < \mu$ and k, m, u, σ the set

$$S = \big\{\zeta < \kappa : \mathrm{dom}(p_\zeta) \subseteq \alpha(\varepsilon^*) \cup \{\alpha\} \ \&\ (k^\zeta, m^\zeta, u^\zeta, \sigma^\zeta) = (k, m, u, \sigma)\big\}$$

is of size κ. As before, $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha(\varepsilon^*)\rangle$ is a Y-iteration, so we may find $v \subseteq S$, $q_0 \in \mathbb{P}_{\alpha(\varepsilon^*)}$, $m > n$ and $A_\zeta \in [f(m)]^{g(m)}$ for $\zeta \in v$ such that demands listed in $(*)_{10}$ are satisfied. Let $q \in \mathbb{P}_{\alpha+1}$ be such that $\mathrm{dom}(q) = \mathrm{dom}(q_0) \cup \{\alpha\}$ and $q{\restriction}\alpha \Vdash q(\alpha) = (k, m, u, \sigma)$. ■

THEOREM 3.14. *Assume $\kappa < \lambda$ are uncountable regular cardinals such that $\lambda^\kappa = \lambda$. Let $H \subseteq \mathbb{Y}^\lambda_\kappa$ be generic over $\mathbf{V}$ and let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda\rangle = \bigcup H \in \mathbf{V}[H]$ and $\mathbb{P}_\lambda = \lim(\bar{\mathbb{Q}})$. Then $\mathbb{P}_\lambda$ is a ccc forcing notion with a dense subset of size λ and*

$$\Vdash_{\mathbb{P}_\lambda} \text{“}\mathbf{MA}_{<\kappa}(\text{ccc}) \ \textit{and}\ \mathbf{MA}(\sigma\text{-centered}) \ \textit{and}$$
$$\mathrm{add}(\mathcal{I}_f) = \kappa \ \textit{and}\ \mathrm{cofin}^-(\mathcal{I}_f) \geq \kappa^+ \ \textit{and}\ 2^{\aleph_0} = \lambda\,\text{”}.$$

Proof. First note that the forcing with $\mathbb{Y}^\lambda_\kappa$ does not add sequences of ordinals of length $<\lambda$ (by Lemma 3.12(5)). Hence in $\mathbf{V}[H]$ κ, λ are still regular cardinals and $\lambda^\kappa = \lambda$.

Let us work in $\mathbf{V}[H]$.

Clearly $\bar{\mathbb{Q}}$ is a Y-iteration for κ, λ of length λ. Hence $\mathbb{P}_\lambda$ is a ccc forcing notion, it has a dense subset of size λ and forces that $2^{\aleph_0} = \lambda$ (remember 3.9(1)$(\otimes)_2$, 3.12(2)). *A canonical $\mathbb{P}_\lambda$-name* $\underset{\sim}{\eta}$ for a real in $\prod_{n<\omega} \mathcal{Z}_n$ (where $\langle \mathcal{Z}_n : n < \omega\rangle \in \mathbf{V}$, $\mathcal{Z}_n \neq \emptyset$) is a sequence $\langle A_n, \pi_n : n < \omega\rangle$ such that each A_n is a maximal antichain in $\mathbb{P}_\lambda$, $\pi_n : A_n \to \mathcal{Z}_n$ and $q \Vdash_{\mathbb{P}_\lambda}$“ $\underset{\sim}{\eta}(n) = \pi_n(q)$” for $q \in A_n$, $n < \omega$. For every $\mathbb{P}_\lambda$-name $\underset{\sim}{\rho}$ for an element of $\prod_{n<\omega} \mathcal{Z}_n$ there is a canonical name $\underset{\sim}{\eta}$ such that $\Vdash \underset{\sim}{\eta} = \underset{\sim}{\rho}$. Also, if $\underset{\sim}{\eta}$ is a canonical $\mathbb{P}_\lambda$-name for a real, then it is a $\mathbb{P}_\alpha$-name for some $\alpha < \lambda$.

Let us argue that $\Vdash_{\mathbb{P}_\lambda} \mathrm{cofin}^-(\mathcal{I}_f) \geq \kappa^+$. If not, then for some infinite regular cardinal $\mu \leq \kappa$ and $\mathbb{P}_\lambda$-names $\underset{\sim}{\mathcal{I}}_\zeta, \underset{\sim}{\varphi}_\zeta$ (for $\zeta < \mu$) we have

$(\circledast)_1$ $\Vdash_{\mathbb{P}_\lambda}$ “ $\underset{\sim}{\varphi}_\zeta \in \mathcal{S}_f$ and $\underset{\sim}{\mathcal{I}}_\zeta \subseteq \mathcal{I}_f$ is an ideal”,

and for some $p \in \mathbb{P}_\lambda$,

$(\circledast)_2$ $p \Vdash_{\mathbb{P}_\lambda}$ “ $\bigcup_{\zeta<\mu} \underset{\sim}{\mathcal{I}}_\zeta = \mathcal{I}_f$ and $(\forall \zeta < \mu)([\underset{\sim}{\varphi}_\zeta] \notin \underset{\sim}{\mathcal{I}}_\zeta)$”.

We may assume that all $\underset{\sim}{\varphi}_\zeta$ are canonical $\mathbb{P}_{\alpha_0}$-names for some $\alpha_0 < \lambda$.

Suppose now that $\zeta < \mu$ and $\underset{\sim}{c}_\zeta$ is a canonical $\mathbb{P}_\lambda$-name for a real in ${}^\omega 2$. Let $\underset{\sim}{\psi}^0_\zeta, \underset{\sim}{\psi}^1_\zeta$ be $\mathbb{P}_\lambda$-names for elements of $\mathcal{S}_f$ such that for each $i < 2$ we have

$$\Vdash_{\mathbb{P}_\lambda} \text{“ } \underset{\sim}{\psi}^i_\zeta(n) = \begin{cases} \underset{\sim}{\varphi}_\zeta(n) & \text{if } \underset{\sim}{c}_\zeta(n) = i, \\ \emptyset & \text{if } \underset{\sim}{c}_\zeta(n) = 1 - i \end{cases} \quad \text{for each } n < \omega\text{”}.$$

Then $\Vdash_{\mathbb{P}_\lambda} [\underset{\sim}{\varphi}_\zeta] = [\underset{\sim}{\psi}^0_\zeta] \cup [\underset{\sim}{\psi}^1_\zeta]$, so $p \Vdash_{\mathbb{P}_\lambda}$“$[\underset{\sim}{\psi}^0_\zeta] \notin \underset{\sim}{\mathcal{I}}_\zeta$ or $[\underset{\sim}{\psi}^1_\zeta] \notin \underset{\sim}{\mathcal{I}}_\zeta$”. Let $\underset{\sim}{\tau} = \tau(\zeta, \underset{\sim}{c}_\zeta)$ be a canonical $\mathbb{P}_\lambda$-name for a member of $\{0,1\}$ such that $p \Vdash$ “$[\underset{\sim}{\psi}^{\underset{\sim}{\tau}}_\zeta] \notin \underset{\sim}{\mathcal{I}}_\zeta$”.

Claim 3.14.1. *For some sequence $\langle \alpha(\zeta), \underset{\sim}{c}_\zeta, \underset{\sim}{\psi}_\zeta : \zeta < \mu\rangle$ we have*

(i) *$\langle \alpha(\zeta) : \zeta < \mu\rangle \subseteq \lambda$ is strictly increasing with $\alpha_0 \le \alpha(0)$, and for each $\zeta < \mu$:*

(ii) *$\Vdash_{\mathbb{P}_{\alpha(\zeta)}} \underset{\sim}{\mathbb{Q}}_{\alpha(\zeta)} = \mathbb{C}$ and $\underset{\sim}{c}_\zeta$ is the canonical $\mathbb{P}_{\alpha(\zeta)+1}$-name for the Cohen real in ${}^\omega 2$ added by $\underset{\sim}{\mathbb{Q}}_{\alpha(\zeta)}$, and $\tau(\zeta, \underset{\sim}{c}_\zeta)$ is a $\mathbb{P}_{\alpha(\zeta+1)}$-name (for a member of $\{0,1\}$),*

(iii) *$\underset{\sim}{\psi}_\zeta$ is a $\mathbb{P}_{\alpha(\zeta+1)}$-name for an element of $\mathcal{S}_f$ such that*

$$\Vdash_{\mathbb{P}_{\alpha(\zeta+1)}} \text{“}\underset{\sim}{\psi}_\zeta(n) = \begin{cases} \underset{\sim}{\varphi}_\zeta(n) & \textit{if } \underset{\sim}{c}_\zeta(n) = \tau(\zeta, \underset{\sim}{c}_\zeta), \\ \emptyset & \textit{if } \underset{\sim}{c}_\zeta(n) = 1 - \tau(\zeta, \underset{\sim}{c}_\zeta) \end{cases} \quad \textit{for all } n < \omega\text{”},$$

(iv) *if $\alpha^* = \sup(\alpha(\zeta) : \zeta < \mu)$, then $\Vdash_{\mathbb{P}_{\alpha^*}} \underset{\sim}{\mathbb{Q}}_{\alpha^*} = \mathbb{Q}^*_\mu(\bar{\underset{\sim}{\psi}})$, where $\bar{\underset{\sim}{\psi}} = \langle \underset{\sim}{\psi}_\zeta : \zeta < \mu\rangle$.*

Proof of the Claim. We move back to $\mathbf{V}$ and we use a density argument in $\mathbb{Y}^\lambda_\kappa$ above $P = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha_0 + 1\rangle \in \mathbb{Y}^\lambda_\kappa$. Let $\underset{\sim}{T}$ be a $\mathbb{Y}^\lambda_\kappa$-name for the function $\tau(\cdot,\cdot)$ introduced (in $\mathbf{V}[H]$) earlier. Note that if $\underset{\sim}{c}$ is a canonical $\mathbb{P}^*_\gamma$-name, $Q^* = \langle \mathbb{P}^*_\beta, \underset{\sim}{\mathbb{Q}}^*_\beta : \beta < \gamma\rangle \in \mathbb{Y}^\lambda_\kappa$ and $\zeta < \mu$, then Q^* forces that $(\zeta, \underset{\sim}{c})$ belongs to the domain of $\underset{\sim}{T}$ and $\underset{\sim}{T}(\zeta, \underset{\sim}{c})$ is a $\mathbb{Y}^\lambda_\kappa$-name for an element of $\mathbf{V}$.

Suppose that $Q = \langle \mathbb{P}'_\beta, \underset{\sim}{\mathbb{Q}}'_\beta : \beta < \alpha\rangle \in \mathbb{Y}^\lambda_\kappa$ is a condition stronger than P (so $\alpha_0 + 1 \le \alpha$ and $\underset{\sim}{\mathbb{Q}}'_\beta = \underset{\sim}{\mathbb{Q}}_\beta$ for $\beta \le \alpha_0$).

By induction on $\zeta < \mu$ we build a sequence $\langle Q_\zeta, \alpha(\zeta), \underset{\sim}{c}_\zeta : \zeta < \mu\rangle$ such that

$(\boxtimes)_1$ $Q_\zeta = \langle \mathbb{P}'_\beta, \underset{\sim}{\mathbb{Q}}'_\beta : \beta \le \alpha(\zeta)\rangle \in \mathbb{Y}^\lambda_\kappa$ (so $\ell g(Q_\zeta) = \alpha(\zeta) + 1 < \lambda$),

$(\boxtimes)_2$ for $\zeta < \varepsilon < \mu$ we have

$$Q \le_{\mathbb{Y}^\lambda_\kappa} Q_\zeta \le_{\mathbb{Y}^\lambda_\kappa} Q_\varepsilon \quad \text{and} \quad \alpha < \alpha(\zeta) < \alpha(\varepsilon) < \lambda,$$

$(\boxtimes)_3$ $\Vdash_{\mathbb{P}'_{\alpha(\zeta)}} \underset{\sim}{\mathbb{Q}}'_{\alpha(\zeta)} = \mathbb{C}$ and $\underset{\sim}{c}_\zeta$ is the canonical $\mathbb{P}'_{\alpha(\zeta)+1}$-name for the Cohen real in ${}^\omega 2$ added by $\underset{\sim}{\mathbb{Q}}'_{\alpha(\zeta)}$,

$(\boxtimes)_4$ $Q_{\zeta+1}$ decides the value of $\underset{\sim}{T}(\zeta, \underset{\sim}{c}_\zeta)$ and forces (in $\mathbb{Y}^\lambda_\kappa$) that it is a $\mathbb{P}'_{\alpha(\zeta+1)}$-name.

The construction is clearly possible by Lemma 3.12(2), (4). Then letting $\alpha^* = \sup(\alpha(\zeta) : \zeta < \mu)$ we find that $Q_\mu = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}'_\beta : \beta < \alpha^* \rangle \in \mathbb{Y}^\lambda_\kappa$ is a condition stronger than all Q_ζ (for $\zeta < \mu$); remember 3.12(4) again. Moreover, if names $\underset{\sim}{\psi}_\zeta$ are defined as in clause (iii), and $\underset{\sim}{\tau}_\zeta$ is the value forced to $\underset{\sim}{T}(\zeta, \underset{\sim}{c}_\zeta)$ by $Q_{\zeta+1}$ (see $(\boxtimes)_4$ above), and $\underset{\sim}{c}_\zeta$ are as described in $(\boxtimes)_3$, then the assumptions of Lemma 3.13 are satisfied. Therefore $Q^* = Q_\mu {}^\frown \langle \mathbb{P}_{\alpha^*}, \mathbb{Q}^*_\mu(\bar{\underset{\sim}{\psi}}) \rangle \in \mathbb{Y}^\lambda_\kappa$ is a condition stronger than Q. This condition forces in $\mathbb{Y}^\lambda_\kappa$ that $\langle \alpha(\zeta), \underset{\sim}{c}_\zeta, \underset{\sim}{\psi}_\zeta : \zeta < \mu \rangle$ satisfies the demands (i)–(iv). ∎

Let $\alpha(\zeta), \underset{\sim}{c}_\zeta, \underset{\sim}{\psi}_\zeta$ (for $\zeta < \mu$) and α^* be as in Claim 3.14.1(i)–(iv), so in particular $\Vdash_{\mathbb{P}_{\alpha^*}} \underset{\sim}{\mathbb{Q}}_{\alpha^*} = \mathbb{Q}^*_\mu(\bar{\underset{\sim}{\psi}})$. Let $\underset{\sim}{\varsigma}$ be a $\mathbb{P}_{\alpha^*+1}$-name for the null slalom added by $\underset{\sim}{\mathbb{Q}}_{\alpha^*}$ (see Definition 3.7). It follows from Proposition 3.8 that

$$\Vdash_{\mathbb{P}_\lambda} \bigcup_{\zeta<\mu} [\underset{\sim}{\psi}_\zeta] \subseteq [\underset{\sim}{\varsigma}] \in \mathcal{I}_f,$$

and hence, by $(\circledast)_2$, $p \Vdash_{\mathbb{P}_\lambda} (\exists \varepsilon < \mu)(\bigcup_{\zeta<\mu}[\underset{\sim}{\psi}_\zeta] \in \underset{\sim}{\mathcal{I}}_\varepsilon)$. Pick $\varepsilon^* < \mu$ and a condition $q \in \mathbb{P}_\lambda$ stronger than p such that $q \Vdash_{\mathbb{P}_\lambda} \bigcup_{\zeta<\mu}[\underset{\sim}{\psi}_\zeta] \in \underset{\sim}{\mathcal{I}}_{\varepsilon^*}$. Then also $q \Vdash [\underset{\sim}{\psi}_{\varepsilon^*}] \in \underset{\sim}{\mathcal{I}}_{\varepsilon^*}$ (remember $(\circledast)_1$), but this contradicts the choice of $\tau(\varepsilon^*, \underset{\sim}{c}_{\varepsilon^*})$ and $\underset{\sim}{\psi}_{\varepsilon^*}$.

To argue that $\Vdash_{\mathbb{P}_\lambda}$ $\mathbf{MA}_{<\kappa}$(ccc) note that every $\mathbb{P}_\lambda$-name $\underset{\sim}{\mathbb{Q}}$ for a ccc forcing notion on some $\gamma^* < \kappa$ is actually a $\mathbb{P}_\alpha$-name for some $\alpha < \lambda$. Therefore by the standard density argument in $\mathbb{Y}^\lambda_\kappa$, for unboundedly many $\beta < \lambda$ we have $\Vdash_{\mathbb{P}_\beta} \underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathbb{Q}}$ (remember 3.12(2)). Similarly we may justify that $\Vdash_{\mathbb{P}_\lambda}$ $\mathbf{MA}$(σ-centered).

It follows from Lemma 3.10 that $\Vdash_{\mathbb{P}_\lambda} \mathrm{add}(\mathcal{I}_f) \leq \kappa$. Since $\Vdash_{\mathbb{P}_\lambda}$ $\mathbf{MA}_{<\kappa}$(ccc), we easily see that the equality is forced. ∎

COROLLARY 3.15. *It is consistent that* $\mathrm{add}(\mathcal{I}_f) = \aleph_1$ *and* $\mathrm{cofin}^-(\mathcal{I}_f) = \mathrm{cofin}(\mathcal{I}_f) = \aleph_2$.

4. Open problems. Can we get a result parallel to Corollary 3.15 for the null and/or meager ideals? Or even better:

PROBLEM 4.1. Let $\mathcal{I}$ be either the meager ideal $\mathcal{M}$ or the null ideal $\mathcal{N}$. Is it consistent that

$$\mathrm{add}(\mathcal{I}) < \mathrm{cofin}(\mathcal{I}) < \mathrm{cof}(\mathcal{I})\,?$$

The method used in the proof of Theorem 3.14 gives the consistency of $\mathrm{add}(\mathcal{I}_f) \leq \kappa$ & $\kappa^+ \leq \mathrm{cofin}^-(\mathcal{I}_f)$. Can the gap be bigger?

PROBLEM 4.2. Is it consistent that $\mathrm{add}(\mathcal{I}_f) = \aleph_\alpha < \aleph_{\alpha+\omega} \leq \mathrm{cofin}^-(\mathcal{I}_f)\,$?

The cardinal invariant cofin introduced by Borodulin-Nadzieja and Głąb has several natural relatives (or variants), some were listed in Definition 2.1.

Are those coefficients distinct or are they equivalent within the realm of nice σ-ideals?

PROBLEM 4.3. Is it consistent that for some Borel σ-ideal $\mathcal{I}$ on ${}^{\omega}2$ we have $\mathrm{cofin}^*(\mathcal{I}) < \mathrm{cofin}^-(\mathcal{I})$? Or $\mathrm{cofin}^-(\mathcal{I}) < \mathrm{cofin}(\mathcal{I})$? Or $\mathrm{cofin}(\mathcal{I}) < \mathrm{cofin}^+(\mathcal{I})$?

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