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CONFORMAL $\mathcal{F}$-HARMONIC MAPS FOR FINSLER MANIFOLDS

BY

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Abstract. By introducing the $\mathcal{F}$-stress energy tensor of maps from an n-dimensional Finsler manifold to a Finsler manifold and assuming that $(n-2)\mathcal{F}(t)' - 2t\mathcal{F}(t)'' \neq 0$ for any $t \in [0, \infty)$, we prove that any conformal strongly $\mathcal{F}$-harmonic map must be homothetic. This assertion generalizes the results by He and Shen for harmonics map and by Ara for the Riemannian case.

1. Introduction. Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M$, $Y \in T_xM$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ with the following properties:

(i) Regularity: $F(x, Y)$ is smooth in $TM \setminus \{0\}$;
(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$;
(iii) Strong convexity: the fundamental quadratic form $g = g_{ij} dx^i \otimes dx^j$ is positive definite, where $g_{ij} = \frac{\partial^2(F^2)}{2\partial Y^i \partial Y^j}$.

Let $\phi : M \to \overline{M}$ be a non-degenerate smooth map between Finsler manifolds, i.e. $\ker(d\phi) = \{0\}$. Harmonic maps between Finsler manifolds are defined as the critical points of energy functionals. They are important in both classical and modern differential geometry. In the last decade, some results on harmonic maps between Finsler manifolds have been obtained ([4], [5], [7], etc.). In [3], He and Shen introduced the stress energy tensor for ϕ and proved that any conformal strongly harmonic map from an n-dimensional $(n > 2)$ Finsler manifold to a Finsler manifold must be homothetic.

In this paper, we are concerned with $\mathcal{F}$-harmonic maps between Finsler manifolds, which is a natural generalization of harmonic maps. Let $\mathcal{F} : [0, \infty) \to [0, \infty)$ be a C^2 function such that $\mathcal{F}' > 0$ on $(0, \infty)$. An $\mathcal{F}$-harmonic map is a harmonic map, a p-harmonic map and an exponential harmonic map when $\mathcal{F}(t) = t, (2t)^{p/2}/p$ and e^t, respectively. We deal with

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the $\mathcal{F}$-stress energy tensor of maps between Finsler manifolds and discuss conformal $\mathcal{F}$-harmonic maps. We prove the following

MAIN THEOREM. *Assume that* $(n-2)\mathcal{F}(t)' - 2t\mathcal{F}(t)'' \neq 0$ *for any* $t \in [0,\infty)$. *If* ϕ *is a conformal strongly* $\mathcal{F}$*-harmonic map from an* n*-dimensional Finsler manifold* (M,F) *to a Finsler manifold, then the map* ϕ *must be homothetic.*

REMARK. This theorem was obtained by He and Shen [3] for strongly harmonic maps and by Ara [1] in the Riemannian case.

COROLLARY. *Any conformal strongly* p*-harmonic map from an* n*-dimensional Finsler manifold to a Finsler manifold must be homothetic* $(n>p\geq 2)$.

2. Preliminaries. We shall use the following convention for index ranges, unless otherwise stated:

$$1 \leq i,j,\ldots \leq n; \quad 1 \leq \alpha,\beta,\ldots \leq m; \quad 1 \leq a,b,\ldots \leq n-1.$$

Let (M,F) be an n-dimensional Finsler manifold. Then F determines the Hilbert form and the Cartan tensor as follows:

$$\omega^n = \frac{\partial F}{\partial Y^i}\,dx^i, \quad A = A_{ijk}dx^i\otimes dx^j\otimes dx^k, \quad A_{ijk} = \frac{F\partial g_{ij}}{2\partial Y^k}, \quad g_{ij} = \frac{\partial^2 F^2}{2\partial Y^i\partial Y^j}.$$

It is well known that there exists a unique Chern connection ∇ on π^*TM with $\nabla\frac{\partial}{\partial x^i} = \omega_i^j\frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j dx^k$ satisfying

$$\begin{cases} d(dx^i) - dx^j\wedge\omega_j^i = -dx^j\wedge\omega_j^i = 0, \\ dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\dfrac{\delta Y^k}{F}, \end{cases}$$

where $\delta Y^i = dY^i + N_j^i dx^j$, $N_j^i = \gamma_{jk}^i Y^k - \frac{1}{F}A_{jk}^i\gamma_{st}^k Y^s Y^t$ and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}.

The curvature 2-forms of the Chern connection ∇ are

$$d\omega_j^i - \omega_j^k\wedge\omega_k^i = \Omega_j^i = \frac{1}{2}R_{jkl}^i dx^k\wedge dx^l + \frac{1}{F}P_{jkl}^i dx^k\wedge\delta Y^l,$$

where R_{jkl}^i and P_{jkl}^i are the components of the hh-curvature tensor and the hv-curvature tensor of the Chern connection, respectively.

Take a g-orthonormal frame $\{e_i = u_i^j\frac{\partial}{\partial x^j}\}$ with $e_n = \frac{Y^i}{F}\frac{\partial}{\partial x^i}$ for each fibre of π^*TM and let $\{\omega^i\}$ be its dual coframe, where $\pi: TM\to M$ denotes the natural projection. The collection $\{\omega^i,\omega_n^i\}$ forms an orthonormal basis for $T^*(TM\setminus\{0\})$ with respect to the Sasaki-type metric $g_{ij}dx^i\otimes dx^j + g_{ij}\delta Y^i\otimes\delta Y^j$. The pull-back of the Sasaki metric from $TM\setminus\{0\}$ to the sphere bundle SM is a Riemannian metric $\widehat{g} = g_{ij}dx^i\otimes dx^j + \delta_{ab}\omega_n^a\otimes\omega_n^b$. Thus the volume element dV_{SM} of SM may be defined as

$$dV_{SM} = dv\wedge\omega_n^1\wedge\cdots\wedge\omega_n^{n-1} = \Omega\,dx\wedge d\tau,$$

where $dv = \sqrt{g_{ij}}\, dx$, $\Omega = \det(g_{ij}/F)$, $d\tau = \sum_i (-1)^{i-1} Y^i dY^1 \wedge \cdots \wedge \widehat{dY^i} \wedge \cdots \wedge dY^n$, $dx = dx^1 \wedge \cdots \wedge dx^n$.

The volume form dV_M of an n-dimensional Finsler manifold (M, F) can be defined by

$$dV_M = \sigma(x)\, dx, \qquad \sigma(x) = \frac{1}{C_{n-1}} \int\limits_{S_x M} \Omega\, d\tau,$$

where $S_x M = \{Y \in T_x M : F(Y) = 1\}$ is the fibre of SM at x and C_{n-1} denotes the volume of the unit Euclidean sphere S^{n-1} .

We quote the following lemmas:

Lemma 2.1 ([7]). *For $\psi = \psi_i \omega^i \in \Gamma(\pi^* T^* M)$ and $T = T_{ij} \omega^i \otimes \omega^j \in \Gamma(\odot^2 \pi^* T^* M)$, we have*

$$\operatorname{div}_{\widehat{g}} \psi = \sum_i (\nabla_{e_i^H} \psi) e_i + \sum_{a,b} \psi_a P_{bba},$$

$$\operatorname{div}_{\widehat{g}} T(e_i) = \sum_j \nabla_{e_j^H} (T)(e_i, e_j) + \sum_{a,b} T_{ib} P_{aab},$$

where $e_i^H = u_i^j \frac{\delta}{\delta x^j} = u_i^j \big(\frac{\partial}{\partial x^j} - N_j^k \frac{\partial}{\partial Y^k} \big)$ denotes the horizontal part of e_i and $P_{bba} = P_{bba}^n$.

Let $\phi : M^n \to \overline{M}^m$ be a non-degenerate smooth map, i.e. $\ker(d\phi) = \{0\}$, and $\widetilde{\nabla}$ be the pullback of the Chern connection on $\pi^*(\phi^{-1} T\overline{M})$. We have

Lemma 2.2 ([5]).

$$\begin{aligned} X \langle d\phi U, d\phi V \rangle = {} & \langle \widetilde{\nabla}_X (d\phi U), d\phi V \rangle + \langle d\phi U, \widetilde{\nabla}_X (d\phi V) \rangle \\ & + 2\overline{C}(d\phi U, d\phi V, (\widetilde{\nabla}_X (d\phi F e_n)), \end{aligned}$$

where $\overline{C} = \frac{1}{F} \overline{A}$ and $X, U, V \in \Gamma(\pi^* TM)$.

Let $\mathcal{F} : [0, \infty) \to [0, \infty)$ be a C^2 function such that $\mathcal{F}' > 0$ on $(0, \infty)$. The $\mathcal{F}$-*energy density* of ϕ is the function $e_{\mathcal{F}}(\phi) : SM \to \mathbb{R}$ defined by

$$e_{\mathcal{F}}(\phi)(x, Y) = \mathcal{F}\big(\tfrac{1}{2} |d\phi|^2\big) = \mathcal{F}\big(\tfrac{1}{2} g^{ij}(x, Y) \phi_i^\alpha \phi_j^\beta \overline{g}_{\alpha\beta}(\overline{x}, \overline{Y})\big),$$

where $d\phi(\frac{\partial}{\partial x^i}) = \phi_i^\alpha \frac{\partial}{\partial \overline{x}^\alpha}$ and $\overline{Y} = \overline{Y}^\alpha \frac{\partial}{\partial \overline{x}^\alpha} = Y^i \phi_i^\alpha \frac{\partial}{\partial \overline{x}^\alpha}$.

We define the $\mathcal{F}$-*energy functional* $E_{\mathcal{F}}(\phi)$ by

$$E_{\mathcal{F}}(\phi) = \frac{1}{C_{n-1}} \int\limits_{SM} e_{\mathcal{F}}(\phi)\, dV_{SM}.$$

We call ϕ an $\mathcal{F}$-*harmonic map* if it is a critical point of the $\mathcal{F}$-energy functional.

PROPOSITION 2.3 ([6]). *ϕ is an $\mathcal{F}$-harmonic map if and only* if

$$\int_{SM} \langle V, \tau_{\mathcal{F}}\rangle\, dV_{SM} = 0$$

for any vector $V \in \Gamma(\phi^{-1}T\overline{M})$, where

$$\begin{aligned}\tau_{\mathcal{F}} = &\sum_i (\widetilde{\nabla}_{e_i^H}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)d\phi)e_i \\ &+ \sum_{i,\alpha}\big\{2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(\overline{e}_\alpha, d\phi e_i, \widetilde{\nabla}_{e_i^H} d\phi F e_n)\overline{e}_\alpha \\ &+ (\widetilde{\nabla}_{Fe_n^H}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2))\overline{C}(d\phi e_i, d\phi e_i, \overline{e}_\alpha)\overline{e}_\alpha \\ &+ \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{Fe_n^H}\overline{C})(d\phi e_i, d\phi e_i, \overline{e}_\alpha)\overline{e}_\alpha \\ &+ 2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(\widetilde{\nabla}_{Fe_n^H} d\phi e_i, d\phi e_i, \overline{e}_\alpha)\overline{e}_\alpha\big\} \\ &+ \sum_{a,b}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle \overline{e}_\alpha, d\phi e_b\rangle \overline{e}_\alpha P_{aab}.\end{aligned} \tag{2.1}$$

Here $\tau_{\mathcal{F}}$ is called the *$\mathcal{F}$-tension field* of ϕ.

DEFINITION 2.4. ϕ is called a *strongly $\mathcal{F}$-harmonic map* if $\tau_{\mathcal{F}} = 0$.

REMARK. For Riemannian manifolds any $\mathcal{F}$-harmonic map is actually a strongly $\mathcal{F}$-harmonic map.

3. The $\mathcal{F}$-stress energy tensor. Let ϕ be a non-degenerate map from a Finsler manifold (M, F) to a Finsler manifold $(\overline{M}, \overline{F})$. The *$\mathcal{F}$-stress energy tensor* $S_{\mathcal{F}}(\phi)$ is a tensor on SM defined by

$$S_{\mathcal{F}}(\phi) = \mathcal{F}(\tfrac{1}{2}|d\phi|^2)g - \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\phi^*\overline{g}.$$

THEOREM 3.1. *Let ϕ be a non-degenerate map from a Finsler manifold (M, F) to a Finsler manifold $(\overline{M}, \overline{F})$. Then*

$$\operatorname{div}_{\widehat{g}} S_{\mathcal{F}}(\phi)(Y) = -\langle \tau_{\mathcal{F}}, d\phi(Y)\rangle.$$

REMARK. This result was obtained by He and Shen [3] for harmonic maps and by Ara [1] in the Riemannian case.

Proof. From

$$S_{\mathcal{F}}(\phi) = S_{ij}\omega^i \otimes \omega^j = \{\mathcal{F}(\tfrac{1}{2}|d\phi|^2)\delta_{ij} - \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle d\phi e_i, d\phi e_j\rangle\}\omega^i \otimes \omega^j$$

and Lemma 2.1, we have, for $X = x^i e_i \in \Gamma(\pi^*TM)$,

$$\begin{aligned}\operatorname{div}_{\widehat{g}} S_{\mathcal{F}}(\phi)(X) &= (\nabla_{e_j^H} S_{\mathcal{F}}(\phi))(X, e_j) + \sum_{a,b} x^i S_{ib}P_{aab} \\ &= \sum_k \big\{(\nabla_{e_j^H}\mathcal{F}(\tfrac{1}{2}|d\phi|^2))\omega^k \otimes \omega^k \\ &\quad + \mathcal{F}(\tfrac{1}{2}|d\phi|^2)[(\nabla_{e_j^H}\omega^k) \otimes \omega^k + \omega^k \otimes (\nabla_{e_j^H}\omega^k)]\big\}(X, e_j)\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&- \big\{(\nabla_{e_j^H}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2))\langle d\phi e_k, d\phi e_l\rangle\omega^k\otimes\omega^l \\
&+ [\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle(\nabla_{e_j^H}d\phi)e_k, d\phi e_l\rangle + \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle d\phi e_k, (\nabla_{e_j^H}d\phi)e_l\rangle \\
&+ 2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_k, d\phi e_l, \widetilde{\nabla}_{e_j^H}d\phi Fe_n)]\omega^k\otimes\omega^l\big\}(X, e_j) \\
&+ \sum_a \mathcal{F}(\tfrac{1}{2}|d\phi|^2)x^i P_{aai} - \sum_{a,b}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle d\phi X, d\phi e_b\rangle P_{aab} \\
= &\sum_i \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle\nabla_{X^H}d\phi e_i, d\phi e_i\rangle \\
&+ \sum_i \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{X^H}d\phi Fe_n) \\
&- \big\{(\nabla_{e_j^H}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2))\langle d\phi X, d\phi e_j\rangle + \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle(\nabla_{e_j^H}d\phi)X, d\phi e_j\rangle \\
&+ \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle d\phi X, (\nabla_{e_j^H}d\phi)e_j\rangle \\
&+ 2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi X, d\phi e_j, \widetilde{\nabla}_{e_j^H}d\phi Fe_n)\big\} \\
&+ \sum_a \mathcal{F}(\tfrac{1}{2}|d\phi|^2)x^i P_{aai} - \sum_{a,b}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle d\phi X, d\phi e_b\rangle P_{aab} \\
= &-\langle d\phi X, (\nabla_{e_j^H}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)d\phi)e_j\rangle \\
&+ \sum_i \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{X^H}d\phi Fe_n) \\
&- 2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi X, d\phi e_j, \widetilde{\nabla}_{e_j^H}d\phi Fe_n) \\
&+ \sum_a \mathcal{F}(\tfrac{1}{2}|d\phi|^2)x^i P_{aai} - \sum_{a,b}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle d\phi X, d\phi e_b\rangle P_{aab}.
\end{aligned}
$$

Let $\psi = \sum_j \mathcal{F}'(\frac{1}{2}|d\phi|^2)\overline{C}(d\phi e_j, d\phi e_j, d\phi X)F\omega^n$, which is a global section of π^*T^*M. By Lemma 2.1 and $P_{aan} = 0$, we know that

$$
\begin{aligned}
\operatorname{div}_{\widehat{g}}\psi = &\sum_j\big\{(\widetilde{\nabla}_{Fe_n^H}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2))\overline{C}(d\phi e_j, d\phi e_j, d\phi X) \\
&+ \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{Fe_n^H}\overline{C})(d\phi e_j, d\phi e_j, d\phi X) \\
&+ 2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{Fe_n^H}d\phi e_j), d\phi e_j, d\phi X) \\
&+ \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_j, d\phi e_j, (\widetilde{\nabla}_{Fe_n^H}d\phi X))\big\},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\text{(3.2)}\qquad &\sum_i \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{X^H}d\phi Fe_n) \\
&= \operatorname{div}_{\widehat{g}}\psi - \sum_j(\widetilde{\nabla}_{Fe_n^H}\mathcal{F}'(\tfrac{1}{2}|d\phi|^2))\overline{C}(d\phi e_j, d\phi e_j, d\phi X)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_j \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{Fe_n^H}\overline{C})(d\phi e_j, d\phi e_j, d\phi X) \\
&- \sum_j 2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{Fe_n^H} d\phi e_j), d\phi e_j, d\phi X) \\
&- \sum_i \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, d\phi \nabla_{Fe_n^H} X).
\end{aligned}
$$

Substituting (3.2) into (3.1), we obtain

$$
\begin{aligned}
(3.3)\qquad & \operatorname{div}_{\widehat{g}} S_{\mathcal{F}}(\phi)(X) \\
&= \operatorname{div}_{\widehat{g}} \psi - \sum_j \big\{ (\widetilde{\nabla}_{Fe_n^H} \mathcal{F}'(\tfrac{1}{2}|d\phi|^2))\overline{C}(d\phi e_j, d\phi e_j, d\phi X) \\
&\quad + \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)(\widetilde{\nabla}_{Fe_n^H}\overline{C})(d\phi e_j, d\phi e_j, d\phi X) \\
&\quad + 2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}((\widetilde{\nabla}_{Fe_n^H} d\phi e_j), d\phi e_j, d\phi X) \\
&\quad + \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_j, d\phi e_j, d\phi \nabla_{Fe_n^H} X) \big\} \\
&\quad - \langle d\phi X, (\nabla_{e_j^H} \mathcal{F}'(\tfrac{1}{2}|d\phi|^2) d\phi) e_j \rangle \\
&\quad - 2\mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi X, d\phi e_j, \widetilde{\nabla}_{e_j^H} d\phi F e_n) \\
&\quad + \sum_a \mathcal{F}(\tfrac{1}{2}|d\phi|^2) x^i P_{aai} - \sum_{a,b} \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\langle d\phi X, d\phi e_b\rangle P_{aab} \\
&= -\langle \tau_{\mathcal{F}}, d\phi(X)\rangle + \operatorname{div}_{\widehat{g}} \psi \\
&\quad - \sum_i \mathcal{F}'(\tfrac{1}{2}|d\phi|^2)\overline{C}(d\phi e_i, d\phi e_i, d\phi \nabla_{Fe_n^H} X) + \sum_a \mathcal{F}(\tfrac{1}{2}|d\phi|^2) x^i P_{aai}.
\end{aligned}
$$

Setting $X = Y = Fe_n$ in (3.3), using the fact that $\overline{C}(d\phi Y, \cdot, \cdot) = 0$, $\nabla_{e_i^H} Y = 0$ and $P_{aan} = 0$, we get

$$
(3.4)\qquad \operatorname{div}_{\widehat{g}} S_{\mathcal{F}}(\phi)(Y) = -\langle \tau_{\mathcal{F}}, d\phi(Y)\rangle. \ \blacksquare
$$

COROLLARY 3.2. *Let ϕ be a non-degenerate map from a Finsler manifold (M, F) to a Finsler manifold $(\overline{M}, \overline{F})$. Then*

$$
\operatorname{div}_{\widehat{g}} S_p(\phi)(Y) = -\langle \tau_p, d\phi(Y)\rangle.
$$

REMARK. This result was obtained by Takeuchi [8] in the Riemannian case.

DEFINITION 3.3. A map $\phi : (M, F) \to (\overline{M}, \overline{F})$ is said to be *conformal* if $\phi^*\overline{g} = \mu g$, where $\mu \in C(SM)$ and $\mu > 0$.

It is well known from [3] that μ must be independent of Y, that is, $\mu = \mu(x)$.

DEFINITION 3.4. A map ϕ is called *homothetic* if μ is a positive constant.

Next we have

Proposition 3.5. *Let ϕ be a non-degenerate map from an n-dimensional Finsler manifold (M,F) to $(\overline{M},\overline{F})$. Then $S_{\mathcal{F}}(\phi)=0$ if and only if ϕ is conformal. In this case the conformal factor μ of ϕ satisfies $\mathcal{F}(n\mu/2)-\mu\mathcal{F}(n\mu/2)'=0$.*

Proof. (1) If $\mathcal{F}(t)-\frac{2t}{n}\mathcal{F}(t)'=0$ and ϕ is conformal, then we have $|d\phi|^2=\sum_i\langle d\phi e_i,d\phi e_i\rangle=n\mu$, which implies that

$$S_{\mathcal{F}}(\phi)=(\mathcal{F}(n\mu/2)-\mu\mathcal{F}'(n\mu/2))g=\left(\mathcal{F}(t)-\frac{2t}{n}\mathcal{F}(t)'\right)g=0.$$

(2) If $S_{\mathcal{F}}(\phi)=0$, we have $\mathcal{F}(\frac{1}{2}|d\phi|^2)g=\mathcal{F}'(\frac{1}{2}|d\phi|^2)\phi^*\overline{g}$, i.e. $\phi^*\overline{g}=\mu g$, where $\mu=\mathcal{F}(\frac{1}{2}|d\phi|^2)/\mathcal{F}'(\frac{1}{2}|d\phi|^2)$. Then ϕ is conformal and $\mathcal{F}(n\mu/2)=\mu\mathcal{F}'(n\mu/2)$. ∎

Corollary 3.6. *Let ϕ be a non-degenerate map from an n-dimensional Finsler manifold (M,F) to $(\overline{M},\overline{F})$. Then $S_p(\phi)=0$ if and only if $n=p$ and ϕ is conformal.*

Remark. This result was obtained by He–Shen [3] for harmonic maps.

Main Theorem. *Assume that $(n-2)\mathcal{F}(t)'-2t\mathcal{F}(t)''\neq 0$ for any $t\in[0,\infty)$. If ϕ is a conformal strongly $\mathcal{F}$-harmonic map from an n-dimensional Finsler manifold (M,F) to a Finsler manifold, then ϕ must be homothetic.*

Proof. Since ϕ is conformal, we have $S_{\mathcal{F}}(\phi)=(\mathcal{F}(n\mu/2)-\mu\mathcal{F}'(n\mu/2))g$; from this, together with Lemma 2.1 and Theorem 3.1, it can be seen that

$$\begin{aligned}
0 &= \operatorname{div} S_{\mathcal{F}}(\phi)(Y)\\
&= \nabla_{e_k^H}\{(\mathcal{F}(n\mu/2)-\mu\mathcal{F}'(n\mu/2))\delta_{ij}\omega^i\otimes\omega^j\}(Y,e_k)+\sum_{a,b}S_{\mathcal{F}}(\phi)(Y,e_b)P_{aab}\\
&= \nabla_{Y^H}(\mathcal{F}(n\mu/2)-\mu\mathcal{F}'(n\mu/2))\\
&\quad -\sum_i(\mathcal{F}(n\mu/2)-\mu\mathcal{F}'(n\mu/2))[(\nabla_{e_k^H}\omega^i)\otimes\omega^i+\omega^i\otimes(\nabla_{e_k^H}\omega^i)](Y,e_k)\\
&= (n/2)\mathcal{F}'(n\mu/2)Y(\mu)-\mathcal{F}'(n\mu/2)Y(\mu)-(n\mu/2)\mathcal{F}''(n\mu/2)Y(\mu)\\
&= \tfrac{1}{2}\{(n-2)\mathcal{F}(t)'-2t\mathcal{F}(t)''\}Y(\mu).
\end{aligned}$$

It is obvious that μ is constant, which finishes the proof of the theorem. ∎

For a p-harmonic map ϕ, i.e. $\mathcal{F}(t)=(2t)^{p/2}/p$, we have

$$(n-2)\mathcal{F}(t)'-2t\mathcal{F}(t)''=(n-p)|d\phi|^{p-2}.$$

Thus we obtain immediately

Corollary 3.7. *Any conformal strongly p-harmonic map from an n-dimensional Finsler manifold to a Finsler manifold must be homothetic $(n>p\geq 2)$.*

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