COLLOQUIUM MATHEMATICUM

VOL. 134 2014 NO. 2

PRIME AND SEMIPRIME RINGS WITH SYMMETRIC SKEW n-DERIVATIONS

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Abstract. Let $n \geq 3$ be a positive integer. We study symmetric skew n-derivations of prime and semiprime rings and prove that under some certain conditions a prime ring with a nonzero symmetric skew n-derivation has to be commutative.

1. Introduction. Throughout the paper, $\mathcal{R}$ will represent a ring with a center $\mathcal{Z}$ and α an automorphism of $\mathcal{R}$. For a positive integer $n > 1$, we say that a ring $\mathcal{R}$ is *n-torsion free* if $nx = 0$, $x \in \mathcal{R}$, implies $x = 0$. As usual, the commutator $xy - yx$, $x, y \in \mathcal{R}$, will be denoted by $[x, y]$. Recall that a ring $\mathcal{R}$ is *prime* if $x\mathcal{R}y = 0$, $x, y \in \mathcal{R}$, implies $x = 0$ or $y = 0$, and it is *semiprime* if $x\mathcal{R}x = 0$, $x \in \mathcal{R}$, implies $x = 0$.

An additive map $d : \mathcal{R} \to \mathcal{R}$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$ and it is called a *skew derivation* (or an *α-derivation*) associated with the automorphism α if $d(xy) = d(x)y + \alpha(x)d(y)$ for $x, y \in \mathcal{R}$. Of course, skew derivations are generalizations of the usual derivations (corresponding to $\alpha = \mathrm{id}$, the identity map on $\mathcal{R}$). A map $f : \mathcal{R} \to \mathcal{R}$ is said to be *centralizing* if $[f(x), x] \in \mathcal{Z}$ for all $x \in \mathcal{R}$. In the special case when $[f(x), x] = 0$ for all $x \in \mathcal{R}$, the map f is said to be *commuting*.

The study of commuting mappings is closely connected with the notion of biderivations. A biadditive map $D : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is called a *biderivation* if for all $x, y \in \mathcal{R}$, the maps $x \mapsto D(x, y)$ and $y \mapsto D(x, y)$ are derivations. In particular, $D(xu, y) = D(x, y)u + xD(u, y)$ and $D(x, yv) = D(x, y)v + yD(x, v)$ for all $x, y, u, v \in \mathcal{R}$. It turns out that every commuting map gives rise to a biderivation. Namely, let f be a commuting map of $\mathcal{R}$ and let $D : \mathcal{R} \to \mathcal{R}$ be a map defined by

$$D(x, y) = [f(x), y], \qquad x, y \in \mathcal{R}.$$

By the linearization of $[f(x), x] = 0$, we get

$$[f(x), y] + [f(y), x] = 0, \qquad x, y \in \mathcal{R}.$$

2010 *Mathematics Subject Classification*: Primary 16W25; Secondary 16N60.

Key words and phrases: prime ring, semiprime ring, symmetric skew n-derivation, centralizing mapping, commuting mapping.

DOI: 10.4064/cm134-2-8

Thus, we have

$$\begin{aligned} D(xu,y) &= [f(xu),y] = [xu,f(y)] = [x,f(y)]u + x[u,f(y)] \\ &= [f(x),y]u + x[f(u),y] = D(x,y)u + xD(u,y) \end{aligned}$$

for all $x, y, u \in \mathcal{R}$. Similarly,

$$D(x,yv) = [f(x),y]v + y[f(x),v] = D(x,y)v + yD(x,v)$$

for all $x, y, v \in \mathcal{R}$. Hence, D is a biderivation. Brešar, Martindale, and Miers [3] proved that every biderivation D of a noncommutative prime ring $\mathcal{R}$ is of the form $D(x,y) = \lambda[x,y]$, $x, y \in \mathcal{R}$, where λ is a fixed element from the extended centroid of $\mathcal{R}$. Using certain functional identities, Brešar [1] extended this result to semiprime rings.

The famous result of Posner [8] states that the existence of a nonzero centralizing derivation on a prime ring implies that the ring is commutative. In fact, this result initiated the study of centralizing and commuting mappings in rings. In the last few decades a number of mathematicians have done a great deal of work concerning commutativity of prime and semiprime rings admitting different kind of mappings which are centralizing or commuting on some appropriate subset of a ring (see [2] for further references). Moreover, also biderivations and related mappings of prime and semiprime rings as well as of some certain algebras have been studied a lot. Let us just mention the work of Vukman [9, 10] who investigated symmetric bi-derivations on prime and semiprime rings in connection with centralizing mappings. In [6], Jung and Park studied symmetric 3-derivations and commutativity of prime rings and in [7] Park generalized the results obtained in [6] to symmetric n-derivations ($n \geq 3$).

Recently we obtained similar results to Posner's and Vukman's for symmetric skew 3-derivations on prime and semiprime rings [5]. The main purpose of this paper is to generalize these results and to apply Posner's theorem [8, Theorem 2] to symmetric skew n-derivations for $n \geq 3$.

2. Preliminaries. In the following, n will be a positive integer. Before stating our main theorems, let us recall some basic definitions and well-known results which we will need.

Let $\mathcal{R}^n = \mathcal{R} \times \cdots \times \mathcal{R}$. A map $D : \mathcal{R}^n \to \mathcal{R}$ is *n-additive* if it is additive in each argument, and it is *symmetric* if $D(x_1, \ldots, x_n) = D(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all $x_1, \ldots, x_n \in \mathcal{R}$ and every permutation $\pi \in \mathcal{S}_n$. Now, let D be a symmetric n-additive map. Then it is easy to see that

$$D(-x_1, x_2, \ldots, x_n) = -D(x_1, x_2, \ldots, x_n) \tag{1}$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{R}$. Thus, for all elements $x_2, \ldots, x_n \in \mathcal{R}$, the map $D(\cdot, x_2, \ldots, x_n) : \mathcal{R} \to \mathcal{R}$ is an endomorphism of the additive group of $\mathcal{R}$.

Furthermore, the map $\tau : \mathcal{R} \to \mathcal{R}$ defined by

$$\tau(x) = D(x, \ldots, x), \qquad x \in \mathcal{R},$$

is called the *trace* of D. It is easy to compute that

$$\tau(x+y) = \tau(x) + \tau(y) + \sum_{k=1}^{n-1} \binom{n}{k} D(\underbrace{x, \ldots, x}_{k\,\text{times}}, \underbrace{y, \ldots, y}_{n-k\,\text{times}})$$

for all $x, y \in \mathcal{R}$. Note also that, by (1), τ is an odd function if n is odd, and an even function if n is even.

Motivated by the notion of n-derivations we introduce the following definition.

Definition. An n-additive map $D : \mathcal{R}^n \to \mathcal{R}$ is called a *skew n-derivation* associated with the automorphism α if for every $k = 1, \ldots, n$ and all $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in \mathcal{R}$, the map $x \mapsto D(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n)$ is a skew derivation of $\mathcal{R}$ associated with α. In particular, for all $x, y, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in \mathcal{R}$ we have

$$\begin{aligned} &D(x_1, \ldots, x_{k-1}, xy, x_{k+1}, \ldots, x_n) \\ &\ = D(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n)y + \alpha(x)D(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n). \end{aligned}$$

The above definition covers the notion of skew derivations as well as the notion of skew biderivations. Namely, a skew 1-derivation is a skew derivation and a skew 2-derivation is a skew biderivation. Moreover, this definition generalizes the notion of n-derivations (the case when $\alpha = \mathrm{id}$).

Let us end this section with two simple examples.

Example 1. Let $\mathcal{R}$ be a commutative ring, α an automorphism of $\mathcal{R}$, and $d : \mathcal{R} \to \mathcal{R}$ a skew derivation of $\mathcal{R}$ associated with α. Then the map $D : \mathcal{R}^n \to \mathcal{R}$ defined by

$$D(x_1, \ldots, x_n) = d(x_1) \cdots d(x_n), \qquad x_1, \ldots, x_n \in \mathcal{R},$$

is a symmetric skew n-derivation associated with α.

Example 2 ([7]). Let $\mathbb{F}$ be a field and

$$\mathcal{R} = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in \mathbb{F} \right\}.$$

It is easy to see that the set $\mathcal{R}$ with matrix addition and matrix multiplication is a noncommutative ring. Let $A_k = \left[\begin{smallmatrix} x_k & y_k \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{R}$, $k = 1, \ldots, n$, and define a map $D : \mathcal{R}^n \to \mathcal{R}$ by

$$D(A_1, \ldots, A_n) = \begin{bmatrix} 0 & x_1 \cdots x_n \\ 0 & 0 \end{bmatrix}.$$

Obviously, D is a symmetric skew n-derivation of $\mathcal{R}$ associated with id.

3. The results. From now on we will always assume that $n \geq 3$. For a positive integer k with $1 \leq k \leq n$ and for $x, y \in \mathcal{R}$, we will write

$$D_k(x,y) = D(\underbrace{x,\ldots,x}_{k\,\text{times}}, \underbrace{y,\ldots,y}_{n-k\,\text{times}}).$$

In the proofs of our results, we will use the following simple lemmas. The first one was proved by Chung and Luh [4].

Lemma 1 ([4, Lemma 1]). *Let $\mathcal{R}$ be an $n!$-torsion free ring and $x_1, \ldots, x_n \in \mathcal{R}$ such that*

$$tx_1 + t^2x_2 + \cdots + t^nx_n = 0$$

for all positive integers $1 \leq t \leq n$. Then $x_i = 0$ for all $1 \leq i \leq n$.

The next lemma is an immediate consequence of Lemma 1.

Lemma 2. *Let $\mathcal{R}$ be a $n!$-torsion free ring and $x_1, \ldots, x_n \in \mathcal{R}$ such that*

$$tx_1 + t^2x_2 + \cdots + t^nx_n \in \mathcal{Z}$$

for all positive integers $1 \leq t \leq n$. Then $x_i \in \mathcal{Z}$ for all $1 \leq i \leq n$.

Proof. Let $y \in \mathcal{R}$. Then, according to our assumptions,

$$0 = [tx_1 + t^2x_2 + \cdots + t^nx_n, y] = t[x_1, y] + t^2[x_2, y] + \cdots + t^n[x_n, y]$$

for all positive integers $1 \leq t \leq n$. By Lemma 1, it follows that $[x_k, y] = 0$ for $k = 1, \ldots, n$. Thus, $x_1, \ldots, x_n \in \mathcal{Z}$, as desired. ∎

Lemma 3. *Let $\mathcal{R}$ be a prime ring and $a, b \in \mathcal{R}$. If $a[x, b] = 0$ for all $x \in \mathcal{R}$, then either $a = 0$ or $b \in \mathcal{Z}$.*

Proof. Note that

$$0 = a[xy, b] = ax[y, b] + a[x, b]y = ax[y, b]$$

for all $x, y \in \mathcal{R}$. Thus, $a\mathcal{R}[y, b] = 0$ for all $y \in \mathcal{R}$, and, since $\mathcal{R}$ is prime, either $a = 0$ or $b \in \mathcal{Z}$. ∎

Lemma 4. *If $\mathcal{I}$ is a nonzero two-sided ideal of a prime ring $\mathcal{R}$ and $D : \mathcal{R}^n \to \mathcal{R}$ a symmetric skew n-derivation associated with an automorphism α such that $D(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in \mathcal{I}$, then $D(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \mathcal{R}$.*

Proof. It is sufficient to show that for any integer k with $1 \leq k \leq n$,

$$D(r_1, \ldots, r_k, x_{k+1}, \ldots, x_n) = 0$$

for all $r_1, \ldots, r_k \in \mathcal{R}$ and $x_{k+1}, \ldots, x_n \in \mathcal{I}$. We use induction on k.

Let $x_1, \ldots, x_n \in \mathcal{I}$ and $r_1 \in \mathcal{R}$. Then $r_1x_1 \in \mathcal{I}$ and

$$\begin{aligned} 0 &= D(r_1x_1, x_2, \ldots, x_n) = D(r_1, x_2, \ldots, x_n)x_1 + \alpha(r_1)D(x_1, x_2, \ldots, x_n) \\ &= D(r_1, x_2, \ldots, x_n)x_1. \end{aligned}$$

Thus, $D(r_1, x_2, \ldots, x_n)\mathcal{I} = 0$ and, since $\mathcal{R}$ is prime, $D(r_1, x_2, \ldots, x_n) = 0$. So, we proved our claim for $k = 1$.

Now, let $k \geq 1$ and assume that $D(r_1, \ldots, r_k, x_{k+1}, \ldots, x_n) = 0$ for all $r_1, \ldots, r_k \in \mathcal{R}$ and $x_{k+1}, \ldots, x_n \in \mathcal{I}$. Replacing x_{k+1} by $r_{k+1}x_{k+1}$, where $r_{k+1} \in \mathcal{R}$, we get

$$\begin{aligned} 0 &= D(r_1, \ldots, r_k, r_{k+1}x_{k+1}, \ldots, x_n) \\ &= D(r_1, \ldots, r_k, r_{k+1}, x_{k+2}, \ldots, x_n)x_{k+1} + \alpha(r_{k+1})D(r_1, \ldots, r_k, x_{k+1}, \ldots, x_n) \\ &= D(r_1, \ldots, r_k, r_{k+1}, x_{k+2}, \ldots, x_n)x_{k+1}. \end{aligned}$$

It follows that $D(r_1, \ldots, r_k, r_{k+1}, x_{k+2}, \ldots, x_n)\mathcal{I} = 0$ and, by primeness of $\mathcal{R}$, $D(r_1, \ldots, r_k, r_{k+1}, x_{k+2}, \ldots, x_n) = 0$, as desired. ∎

Our first theorem is a generalization of [5, Theorem 1] and [7, Theorem 2.3].

THEOREM 1. *Let $\mathcal{R}$ be a noncommutative $n!$-torsion free prime ring, $\mathcal{I}$ a nonzero two-sided ideal of $\mathcal{R}$, α an automorphism of $\mathcal{R}$, and $D : \mathcal{R}^n \to \mathcal{R}$ a symmetric skew n-derivation associated with α. Suppose that*

$$[\tau(x), \alpha(x)] = 0 \tag{2}$$

for all $x \in \mathcal{I}$. Then $D = 0$.

Proof. Let t be an integer with $1 \leq t \leq n$ and $x, y \in \mathcal{I}$. Substituting $x + ty$ for x in (2), we obtain

$$\begin{aligned} 0 = t&\left([\tau(x), \alpha(y)] + \binom{n}{n-1}[D_{n-1}(x, y), \alpha(x)]\right) \\ &+ t^2\left(\binom{n}{n-1}[D_{n-1}(x, y), \alpha(y)] + \binom{n}{n-2}[D_{n-2}(x, y), \alpha(x)]\right) \\ &\ \vdots \\ &+ t^n\left(\binom{n}{1}[D_1(x, y), \alpha(y)] + [\tau(y), \alpha(x)]\right). \end{aligned}$$

Thus, by Lemma 1,

$$[\tau(x), \alpha(y)] + n[D_{n-1}(x, y), \alpha(x)] = 0 \tag{3}$$

for all $x, y \in \mathcal{I}$. Replacing y by xy in the above relation, we get

$$\begin{aligned} 0 &= \alpha(x)[\tau(x), \alpha(y)] + n\big(\tau(x)[y, \alpha(x)] + \alpha(x)[D_{n-1}(x, y), \alpha(x)]\big) \\ &= \alpha(x)\big([\tau(x), \alpha(y)] + n[D_{n-1}(x, y), \alpha(x)]\big) + n\tau(x)[y, \alpha(x)] \end{aligned}$$

and, according to (3), we have

$$\tau(x)[y, \alpha(x)] = 0, \qquad x, y \in \mathcal{I}.$$

First we would like to prove that $\tau(x) = 0$ for all $x \in \mathcal{I}$.

Recall that $\mathcal{I}$ is noncentral. Indeed, if $\mathcal{I}$ is central, then $0 = [\mathcal{R}\mathcal{I}, \mathcal{R}] = [\mathcal{R}, \mathcal{R}]\mathcal{I} + \mathcal{R}[\mathcal{I}, \mathcal{R}] = [\mathcal{R}, \mathcal{R}]\mathcal{I}$, and thus $[\mathcal{R}, \mathcal{R}] = 0$, i.e., $\mathcal{R}$ is commutative, a contradiction. So, suppose that $x \in \mathcal{I} \setminus \mathcal{Z}$. Then $\alpha(x) \notin \mathcal{Z}$ and, according to Lemma 3, $\tau(x) = 0$. Now, suppose that $x \in \mathcal{I} \cap \mathcal{Z}$ and choose $y \in \mathcal{I}$ such that $y \notin \mathcal{Z}$. Then $tx + y \in \mathcal{I} \setminus \mathcal{Z}$ for every integer $1 \le t \le n$ and

$$\begin{aligned} 0 = \tau(tx + y) &= t^n \tau(x) + \tau(y) + \sum_{k=1}^{n-1} t^k \binom{n}{k} D_k(x, y) \\ &= t^n \tau(x) + \sum_{k=1}^{n-1} t^k \binom{n}{k} D_k(x, y). \end{aligned}$$

Again using Lemma 1, we get $\tau(x) = 0$, as desired. So, we have proved that

$$\tau(x) = 0, \qquad x \in \mathcal{I}. \tag{4}$$

Next, we show that for any integer k with $1 \le k \le n$,

$$D(x_1, \ldots, x_k, \underbrace{x, \ldots, x}_{n-k\,\text{times}}) = 0$$

for all $x, x_1, \ldots, x_k \in \mathcal{I}$. We use induction on k.

Let $1 \le t \le n - 1$ be an integer and $x, x_1 \in \mathcal{I}$. Then, by (4), we have

$$\begin{aligned} 0 = \tau(tx + x_1) &= t^n \tau(x) + \tau(x_1) + \sum_{j=1}^{n-1} t^j \binom{n}{j} D_j(x, x_1) \\ &= \sum_{j=1}^{n-1} t^j \binom{n}{j} D_j(x, x_1) \end{aligned}$$

and, by Lemma 1,

$$D(\underbrace{x, \ldots, x}_{n-1\,\text{times}}, x_1) = D(x_1, \underbrace{x, \ldots, x}_{n-1\,\text{times}}) = 0$$

for all $x, x_1 \in \mathcal{I}$. So, we proved our claim for $k = 1$.

Now, let $k \ge 1$ and assume that $D(x_1, \ldots, x_k, \underbrace{x, \ldots, x}_{n-k\,\text{times}}) = 0$ for all $x, x_1, \ldots, x_k \in \mathcal{I}$. Furthermore, let $1 \le t \le n - k - 1$ be an integer and $x_{k+1} \in \mathcal{I}$. Then, according to the induction hypothesis,

$$\begin{aligned} 0 &= D(\underbrace{tx + x_{k+1}, \ldots, tx + x_{k+1}}_{n-k\,\text{times}}, x_1, \ldots, x_k) \\ &= t^{n-k} D(\underbrace{x, \ldots, x}_{n-k\,\text{times}}, x_1, \ldots, x_k) + D(\underbrace{x_{k+1}, \ldots, x_{k+1}}_{n-k\,\text{times}}, x_1, \ldots, x_k) \end{aligned}$$

$$+\sum_{j=1}^{n-k-1} t^j \binom{n-k}{j} D(\underbrace{x,\ldots,x}_{j\text{ times}},\underbrace{x_{k+1},\ldots,x_{k+1}}_{n-k-j\text{ times}},x_1,\ldots,x_k)$$

$$=\sum_{j=1}^{n-k-1} t^j \binom{n-k}{j} D(\underbrace{x,\ldots,x}_{j\text{ times}},\underbrace{x_{k+1},\ldots,x_{k+1}}_{n-k-j\text{ times}},x_1,\ldots,x_k)$$

and, by Lemma 1,

$$D(\underbrace{x,\ldots,x}_{n-k-1\text{ times}},x_{k+1},x_1,\ldots,x_k) = D(x_1,\ldots,x_k,x_{k+1},\underbrace{x,\ldots,x}_{n-k-1\text{ times}}) = 0$$

for all $x, x_1, \ldots, x_k, x_{k+1} \in \mathcal{I}$, as desired. In particular, $D(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in \mathcal{I}$. Therefore, by Lemma 4, $D(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \mathcal{R}$. ∎

The next result concerns semiprime rings.

THEOREM 2. *Let $\mathcal{R}$ be a noncommutative $n!$-torsion free semiprime ring, α an automorphism of $\mathcal{R}$, and $D : \mathcal{R}^n \to \mathcal{R}$ a symmetric skew n-derivation associated with α. Suppose that the trace function τ is commuting on $\mathcal{R}$ and*

$$[\tau(x), \alpha(x)] \in \mathcal{Z} \tag{5}$$

for all $x \in \mathcal{R}$. Then $[\tau(x), \alpha(x)] = 0$ for all $x \in \mathcal{R}$.

Proof. Let t be an integer with $1 \le t \le n$, and $x, y \in \mathcal{R}$. Substituting $x + ty$ for x in (5), we obtain

$$\begin{aligned}\mathcal{Z} \ni t&\bigg([\tau(x), \alpha(y)] + \binom{n}{n-1}[D_{n-1}(x,y), \alpha(x)]\bigg)\\ &+ t^2\bigg(\binom{n}{n-1}[D_{n-1}(x,y), \alpha(y)] + \binom{n}{n-2}[D_{n-2}(x,y), \alpha(x)]\bigg)\\ &\vdots\\ &+ t^n\bigg(\binom{n}{1}[D_1(x,y), \alpha(y)] + [\tau(y), \alpha(x)]\bigg).\end{aligned}$$

Thus, by Lemma 2,

$$[\tau(x), \alpha(y)] + n[D_{n-1}(x,y), \alpha(x)] \in \mathcal{Z} \tag{6}$$

for all $x, y \in \mathcal{R}$.

Substituting xy for y in (6), we get

$$\begin{aligned}\mathcal{Z} \ni\ & [\tau(x), \alpha(xy)] + n[D_{n-1}(x,xy), \alpha(x)]\\ &= [\tau(x), \alpha(x)]\alpha(y) + \alpha(x)[\tau(x), \alpha(y)] + n[\tau(x)y + \alpha(x)D_{n-1}(x,y), \alpha(x)]\\ &= \alpha(x)([\tau(x), \alpha(y)] + n[D_{n-1}(x,y), \alpha(x)]) + (\alpha(y) + ny)[\tau(x), \alpha(x)]\\ &\quad + n\tau(x)[y, \alpha(x)].\end{aligned}$$

Commuting with $\alpha(x)$, we obtain

$$\begin{aligned}0 &= \big[\alpha(x)([\tau(x),\alpha(y)] + n[D_{n-1}(x,y),\alpha(x)]),\alpha(x)\big]\\ &\quad + \big[(\alpha(y)+ny)[\tau(x),\alpha(x)] + n\tau(x)[y,\alpha(x)],\alpha(x)\big]\\ &= [\alpha(y)+2ny,\alpha(x)][\tau(x),\alpha(x)] + n\tau(x)[[y,\alpha(x)],\alpha(x)]\end{aligned}$$

for all $x,y\in\mathcal{R}$. Replacing y by $\tau(x)[\tau(x),\alpha(x)]$, we obtain

$$\begin{aligned}0 &= \big[\alpha(\tau(x)[\tau(x),\alpha(x)]) + 2n\tau(x)[\tau(x),\alpha(x)],\alpha(x)\big][\tau(x),\alpha(x)]\\ &\quad + n\tau(x)\big[[\tau(x)[\tau(x),\alpha(x)],\alpha(x)],\alpha(x)\big]\\ &= \big[\alpha(\tau(x)[\tau(x),\alpha(x)]),\alpha(x)\big][\tau(x),\alpha(x)] + 2n[\tau(x),\alpha(x)]^3\\ &= [\alpha(\tau(x)),\alpha(x)]\alpha([\tau(x),\alpha(x)])[\tau(x),\alpha(x)] + 2n[\tau(x),\alpha(x)]^3\\ &= 2n[\tau(x),\alpha(x)]^3.\end{aligned}$$

Therefore,

$$[\tau(x),\alpha(x)]^3 = 0,$$

and consequently

$$[\tau(x),\alpha(x)]^2\mathcal{R}[\tau(x),\alpha(x)]^2 = 0$$

for all $x\in\mathcal{R}$. Since $\mathcal{R}$ is semiprime, it follows that

$$[\tau(x),\alpha(x)]^2 = 0,\qquad x\in\mathcal{R}.$$

Note that zero is the only nilpotent element in the center of a semiprime ring. Thus, $[\tau(x),\alpha(x)] = 0$ for all $x\in\mathcal{R}$. ∎

The last result is an analogue of Posner's theorem [8, Theorem 2].

Corollary 1. *Let $\mathcal{R}$ be an $n!$-torsion free prime ring and α an automorphism of $\mathcal{R}$. Suppose that there exists a nonzero symmetric skew n-derivation $D:\mathcal{R}^n\to\mathcal{R}$ associated with α such that the trace function τ is commuting on $\mathcal{R}$ and $[\tau(x),\alpha(x)]\in\mathcal{Z}$ for all $x\in\mathcal{R}$. Then $\mathcal{R}$ is commutative.*

Proof. Suppose that $\mathcal{R}$ is not commutative. Then, according to Theorem 2, $[\tau(x),\alpha(x)] = 0$ for all $x\in\mathcal{R}$ and, by Theorem 1, $D = 0$, a contradiction. ∎

Let us point out that in Theorem 2 we assumed that the trace function τ of a skew n-derivation D is commuting on $\mathcal{R}$. If we drop this assumption, we do not know whether the statement holds true as well. Even for $n = 3$ this is still an open question. So, let us end this paper with the following conjecture.

Conjecture. Let $\mathcal{R}$ be a prime ring with suitable torsion restrictions and α an automorphism of $\mathcal{R}$. Suppose that there exists a nonzero symmetric skew n-derivation $D:\mathcal{R}^n\to\mathcal{R}$ associated with α such that $[\tau(x),\alpha(x)]\in\mathcal{Z}$ for all $x\in\mathcal{R}$. Then $\mathcal{R}$ is commutative.

Acknowledgements. The author is sincerely thankful to the referee for careful reading of the manuscript and for comments and suggestions which helped the author to improve the paper.

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Received 27 July 2013;
revised 16 December 2013 (5993)