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RELATIONSHIPS BETWEEN GENERALIZED HEISENBERG ALGEBRAS AND THE CLASSICAL HEISENBERG ALGEBRA

BY

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Abstract. A Lie algebra is called a *generalized Heisenberg algebra of degree n* if its centre coincides with its derived algebra and is n-dimensional. In this paper we define for each positive integer n a generalized Heisenberg algebra $\mathcal{H}_n$. We show that $\mathcal{H}_n$ and $\mathcal{H}_1^n$, the Lie algebra which is the direct product of n copies of $\mathcal{H}_1$, contain isomorphic copies of each other. We show that $\mathcal{H}_n$ is an indecomposable Lie algebra. We prove that $\mathcal{H}_n$ and $\mathcal{H}_1^n$ are not quotients of each other when $n \geq 2$, but $\mathcal{H}_1$ is a quotient of $\mathcal{H}_n$ for each positive integer n. These results are used to obtain several families of $\mathcal{H}_n$-modules from the Fock space representation of $\mathcal{H}_1$. Analogues of Verma modules for $\mathcal{H}_n$, $n \geq 2$, are also constructed using the set of rational primes.

1. Introduction. The classical Heisenberg algebra plays an important role in representations of affine algebras. Vertex operator representations of the simplest affine algebra $A_1^{(1)}$ arise from a canonical representation of a Heisenberg subalgebra $\mathcal{H}_1$ with generators $\{x_k, z : k \in \mathbb{Z}-\{0\}\}$ and relations

$$\begin{cases} [x_k, x_l] = k\delta_{k+l,0}z, \\ z \text{ is a central symbol,} \end{cases}$$

where $\delta_{x,y}$ is Kronecker delta for any symbols x, y. In analogy with the notation in $\mathcal{H}_n$ defined below we write kz as z_k and assume that z_k is $\mathbb{Z}$-linear in the subscript with $z_1 = z$ and $0z = z_0 = 0$.

The importance of the Heisenberg algebra in this context is reflected in [KP] and [L] where it is shown that each vertex operator realization of the basic module depends on the choice of the Heisenberg subalgebra of the corresponding affine Lie algebra. In other words, the number of distinct realizations is equal to the number of inequivalent (i.e., non-conjugate under the adjoint action of the associated Kac–Moody group) Heisenberg subalgebras of the corresponding affine Lie algebra. This precise formulation is taken from [M]. We refer the reader to [D], [FJ], [J], [LW], and [KKLW] for more on the role of Heisenberg algebras in representations of infinite-dimensional algebras.

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Toroidal algebras are defined in [MRY] for $n \geq 1$ and for $n = 1$ they are precisely the affine algebras. Related objects have also been studied in a physical context (see [B]). In contrast to the one-dimensional centre for affine algebras, the toroidal algebras have an infinite-dimensional centre for $n \geq 2$. One property of the toroidal algebras that distinguishes them from many of the infinite-dimensional Lie algebras in the literature is that they have vertex operator representations.

Another interesting feature of the toroidal algebras is that they contain *Generalized Heisenberg Algebras* (GHA's) as subalgebras (see for instance [F], [FO2], [BA], and [RB]). Like the Heisenberg algebra $\mathcal{H}_1$, these have the property that their derived algebra is equal to the centre, but the centre is now infinite-dimensional. We distinguish the generalized Heisenberg algebras that have a finite-dimensional centre from *Generalized Heisenberg Algebras* by calling the former *generalized Heisenberg algebras of degree* n, where n is the dimension of the centre. Unlike the Heisenberg algebra $\mathcal{H}_1$, the simplest Generalized Heisenberg Algebra does not have a canonical irreducible representation. It is given by generators $\{x_{\underline{r}}, z_{\underline{r}}(\underline{s}) : \underline{r}, \underline{s} \in \mathbb{Z}^n - \{0\}\}$ and relations

$$\begin{cases} [x_{\underline{r}}, x_{\underline{s}}] = z_{\underline{r}}(\underline{r} + \underline{s}), \\ z_{\underline{r}}(\underline{s}) \text{ is central and } \mathbb{Z}\text{-linear in the subscript,} \end{cases}$$

and $z_{(0,\ldots,0)} = 0$.

Guided by the lesson of affine algebras we feel that a better understanding of GHA's would contribute to the study of toroidal algebras. One direction of pursuit is to follow the route of [BC] and work with a central quotient of the toroidal algebra. In [BC] only the central terms of homogeneous degree zero remain. This means that the resulting toroidal algebra τ_n has an n-dimensional centre.

In this paper we investigate the corresponding effect on the Heisenberg subalgebra. The resulting generalized Heisenberg algebra of degree n, denoted by $\mathcal{H}_n$, is given by generators $\{x_{\underline{r}}, z_{\underline{r}} : \underline{r} \in \mathbb{Z}^n - \{0\}\}$ and relations

$$\begin{cases} [x_{\underline{r}}, x_{\underline{s}}] = \delta_{\underline{r}+\underline{s},\underline{0}} z_{\underline{r}}, \\ z_{\underline{r}} \text{ is central and } \mathbb{Z}\text{-linear in the subscript.} \end{cases}$$

We show that although $\mathcal{H}_n$ and $\mathcal{H}_1^n = \mathcal{H}_1 \times \cdots \times \mathcal{H}_1$ are quite similar, they are not isomorphic and neither is a quotient of the other. On the other hand, they are related in the sense that one can obtain some representations of $\mathcal{H}_n$ from those of $\mathcal{H}_1^n$ because they are isomorphic to subalgebras of each other.

We hope that these results will stimulate interest in the several questions they raise. For instance, what is the relationship between the representations given by our results and the known constructions of representations for toroidal Lie algebras? In [F] Fabbri has given vertex representations of the simplest GHA given by our second set of relations above. Are there representations of $\mathcal{H}_n$ that can be manufactured from these vertex representations? There is an exchange between $\mathcal{H}_n$ and $\mathcal{H}_1^n$ in which each passes its representations to the other using Propositions 2.5–2.7. In particular, Proposition 2.5 permits us to regard every representation of the classical Heisenberg Lie algebra $\mathcal{H}_1$ as $\mathcal{H}_n$-representations for all $n \geq 2$.

The methods in this paper can be used to show that, for an arbitrary positive integer n, there are $\mathrm{p}(n)$ isomorphism classes of generalized Heisenberg algebras of degree n, where $\mathrm{p}(n)$ is the number of partitions of n. This paper deals only with the shortest partition of n and the longest partition of n.

There is a class of generalized Heisenberg algebras similar to those in this paper. It is the *d-fold Heisenberg algebra* on p. 98 of [KMPS]. The commutation relations are given by $[a_m^i, a_n^j] = m\delta^{ij}\delta_{m+n,0}$, $i, j = 1, \dots d$, $m, n \in \mathbb{Z}$. Unlike the generalized Heisenberg algebras in the present paper, the derived algebra of the d-fold Heisenberg algebra is one-dimensional. The significance of the d-fold Heisenberg algebra in elementary quantum mechanics is discussed on pp. 98 and 99 of [KMPS].

We end this introduction with the statement of our main theorem.

Main Theorem. *For each positive integer n, the generalized Heisenberg algebra $\mathcal{H}_n$ is indecomposable. The direct product $\mathcal{H}_n^1$ embeds in $\mathcal{H}_n$, and $\mathcal{H}_n$ embeds in $\mathcal{H}_n^1$. When $n \geq 2$, $\mathcal{H}_n$ is not a quotient of $\mathcal{H}_n^1$, and $\mathcal{H}_n^1$ is not a quotient of $\mathcal{H}_n$. Moreover, for every integer n, $\mathcal{H}_1$ is a quotient of $\mathcal{H}_n$.*

2. Relationships between the generalized Heisenberg algebras of degree n and the Heisenberg algebra. The definitions of the Lie algebras in this paper make sense for any integral domain of characteristic zero. However, for concreteness, one may assume that our Lie algebras are over the field of complex numbers, $\mathbb{C}$.

A basis for the centre, $Z(\mathcal{H}_n)$, of $\mathcal{H}_n$ is $\{z_{\underline{e}_1}, \dots, z_{\underline{e}_n}\}$ where $\{\underline{e}_1, \dots, \underline{e}_n\}$ is the standard basis for $\mathbb{Z}^n$. With z as the generator of $Z(\mathcal{H}_1)$ a basis for $Z(\mathcal{H}_1^n)$ is $\{z\underline{e}_1, \dots, z\underline{e}_n\}$

Remark. When $n \geq m$, the centre of $\mathcal{H}_m$ embeds in the centre of $\mathcal{H}_n$. For that reason not only does $\mathcal{H}_m$ embed in $\mathcal{H}_n$ but it is also a quotient of $\mathcal{H}_n$. The same statement applies to $\mathcal{H}_1^m$ and $\mathcal{H}_1^n$.

Definition 2.1. A Lie algebra, L, is *indecomposable* if whenever $L = L_1 \dot{+} L_2$ (internal direct sum), where L_1, L_2 are ideals, then L_1 or L_2 is 0.

Proposition 2.2. *$\mathcal{H}_n$ is an indecomposable Lie algebra.*

Proof. Given any $f \in \mathcal{H}_n$ and a generator $x_{\underline{r}}$, it follows from the definition of the Lie bracket in $\mathcal{H}_n$ that $[f, x_{\underline{r}}]$ is non-zero only when f has $x_{-\underline{r}}$ as a non-zero component. In that case $[f, x_{\underline{r}}]$ is a non-zero multiple of $z_{\underline{r}}$. A consequence of this fact is that if $\mathcal{H}_n = A \dotplus B$, $A \neq 0$, $B \neq 0$, then for each $i \in \{1, \dots, n\}$, either A or B, but not both, has an element, f_i, with a non-zero $x_{\underline{e}_i}$-term. From $[f_i, x_{-\underline{e}_i}]$ we conclude that $z_{\underline{e}_i}$ is in A or B and does not split into two non-zero summands z_A and z_B with $z_A \in A$ and $z_B \in B$.

Suppose that $z_{\underline{e}_i} \in A$ for every $i \in \{1, \dots, n\}$. Then $Z(\mathcal{H}_n) \subseteq A$. This implies that every generator $x_{\underline{r}}$ is in A by the first paragraph of the proof. Hence $B = 0$. In that case we would be done. We therefore assume that the standard basis of $Z(\mathcal{H}_n)$ splits into two disjoint non-empty subsets $S_A \subseteq A$ and $S_B \subseteq B$. The subscripts $\underline{e}_i$ of the elements $z_{\underline{e}_i}$ give a corresponding partition, $T_A \cup T_B$, of the standard basis of $\mathbb{Z}^n$. We deduce from $[x_{\underline{r}}, x_{\underline{s}}] = \delta_{\underline{r}+\underline{s},0} z_{\underline{r}}$ that whenever $x_{\underline{r}}$ is a component of an element in A (respectively, B), then $\underline{r}$ is spanned by T_A (respectively, T_B.) Conversely, if every component of $\underline{s}$ is non-zero, then $x_{\underline{s}}$ is neither in A nor B. Moreover no member of A or B has $x_{\underline{s}}$ in any of its terms.

By the linear independence of $\{x_{\underline{r}} : \underline{r} \in \mathbb{Z}^n - \{0\}\}$ we deduce that if no component of $\underline{s}$ is 0, then $x_{\underline{s}} \notin A \dotplus B$. This contradicts the assumption that $\mathcal{H}_n = A \dotplus B$. Therefore, $\mathcal{H}_n$ is indecomposable. ∎

Corollary 2.3. *$\mathcal{H}_n$ is not isomorphic to $\mathcal{H}_1^n$ for $n \geq 2$.*

Apart from having isomorphic centres, what other relationships can be established between $\mathcal{H}_n$ and $\mathcal{H}_1^n$ when $n \geq 2$? Proposition 2.4 states that neither is a quotient of the other. But Propositions 2.6 and 2.7 show that each contains an isomorphic copy of the other. Recall that $Z(A)$ denotes the centre of a Lie algebra A, shortened to Z if A is clear from the context.

Proposition 2.4. *Let $n \geq 2$. Then $\mathcal{H}_n$ is not a quotient of $\mathcal{H}_1^n$, nor is $\mathcal{H}_1^n$ a quotient of $\mathcal{H}_n$.*

Proof. The proof is by contradiction. Denote the derived algebra, $[\mathcal{L}, \mathcal{L}]$, of a Lie algebra $\mathcal{L}$ by $\mathcal{L}'$. Suppose $\chi : \mathcal{H}_1^n \to \mathcal{H}_n$ is an epimorphism. Then $\chi(Z(\mathcal{H}_1^n)) = \chi((\mathcal{H}_1^n)') = \mathcal{H}_n' = Z(\mathcal{H}_n)$. Recall that $\dim Z(\mathcal{H}_1^n) = \dim Z(\mathcal{H}_n) = n$. Hence χ restricted to $Z(\mathcal{H}_1^n)$ is an isomorphism onto $Z(H_n)$. We now show that this implies that χ is injective. Suppose $g \in \mathcal{H}_1^n$ and $g \notin Z(\mathcal{H}_1^n)$. Then $g = f + z$ for some f that has no non-zero component in $Z(\mathcal{H}_1^n)$ and some $z \in Z(\mathcal{H}_1^n)$. Therefore some component of f has a non-zero x_r-term, r some non-zero integer. Say $f = (a_1, \dots, a_r + \alpha x_r, 0, \dots, 0)$ where $a_i \in \mathcal{H}_1$, a_r has no x_r-term and α is a non-zero scalar. If $\chi(f + z) = 0$, then $\chi(f) \in Z(\mathcal{H}_n)$. Now $[f, (0, \dots, 0, x_{-r}, 0, \dots, 0)] = (0, \dots, 0, \alpha z_r, 0, \dots, 0)$. Hence

$$[\chi(f), \chi((0, \dots, 0, x_r, 0, \dots, 0))] = \chi((0, \dots, 0, \alpha z_r, 0, \dots, 0)) \in Z(\mathcal{H}_n).$$

Since χ is injective on $Z(\mathcal{H}_1^n)$, this implies that $\chi(f) \notin Z(\mathcal{H}_n)$. Hence $\chi(f+z) \neq 0$. Therefore $\chi : \mathcal{H}_1^n \to \mathcal{H}_n$ is an isomorphism. This contradicts Corollary 2.3.

The proof that $\mathcal{H}_1^n$ is not a quotient of $\mathcal{H}_n$ is similar, *mutatis mutandis.* ∎

REMARK. The obvious projection from $\mathcal{H}_n$ onto $\mathcal{H}_1$ is not a Lie algebra map, as we now show. Let $j \in \{1,\dots,n\}$. The map $x_{\underline{r}} \mapsto x_{r_j}$, $z_{\underline{r}} \mapsto z_{r_j}$ is not a Lie algebra epimorphism from $\mathcal{H}_n$ to $\mathcal{H}_1$. For instance, let $n=2$. Then $[x_{(1,-1)}, x_{(1,1)}] = 0$, while $[x_1, x_{-1}] \neq 0$. So an epimorphism from $\mathcal{H}_n$ onto $\mathcal{H}_1$ cannot be obtained in the natural way.

To get around the difficulty pointed out in the above remark, we let $S_j = \{(r_1,\dots,r_j,r_{j+1},\dots,r_n) : r_1,\dots,r_n \in \mathbb{Z},\ r_j > 0\}$ and let $\psi_j : S_j \to \mathbb{N}$ be a bijection between S_j and the set of natural numbers. In our use of ψ_j in the proof of Proposition 2.5 the subscript j in ψ_j will be suppressed.

PROPOSITION 2.5. *Let $n \geq 2$. For each $j \in \{1,\dots,n\}$ there is a Lie algebra epimorphism $\phi_j : \mathcal{H}_n \to \mathcal{H}_1$.*

Proof. Let $\underline{r} = (r_1,\dots,r_n)$. If $r_j = 0$, set $\phi_j(x_{\underline{r}}) = \phi_j(z_{\underline{r}}) = 0$. If $r_j > 0$, then $\underline{r} \in S_j$; set

$$\phi_j(x_{\underline{r}}) = \frac{r_j}{\psi(\underline{r})} x_{\psi(\underline{r})}, \qquad \phi_j(x_{-\underline{r}}) = x_{-\psi(\underline{r})}, \qquad \phi_j(z_{\underline{r}}) = r_j z.$$

If $r_j < 0$, then $-\underline{r} \in S_j$. In that case, set $\phi(x_{\underline{r}}) = x_{-\psi(-\underline{r})}$. Set $\phi_j(z_{-\underline{r}}) = -r_j z$. Proposition 2.5 now follows from the definitions of ψ and $\mathcal{H}_n$ and the assumptions that $z_{\underline{r}}$ and z_k are $\mathbb{Z}$-linear in the arbitrary subscripts $\underline{r}$ and k. ∎

The natural map $(x_{r_1},\dots,x_{r_n}) \mapsto x_{(r_1,\dots,r_n)}$ and $(z_{r_1},\dots,z_{r_n}) \mapsto z_{(r_1,\dots,r_n)}$ looks like a good candidate for an embedding of $\mathcal{H}_1^n$ into $\mathcal{H}_n$. However it does not yield a Lie algebra map. For example, let $n=2$. Then $[(x_1,x_2),(x_{-1},x_2)] = (z,0) \neq (0,0)$ but $[x_{(1,2)}, x_{(-1,2)}] = 0$. So for the non-zero element $(z,0)$ we have $(z,0) \mapsto 0$. Hence we do not have an embedding.

In order to get an embedding, we let G_1^n and G_n be the respective generators of $\mathcal{H}_1^n$ and $\mathcal{H}_n$.

PROPOSITION 2.6. *The map ϕ from G_1^n to G_n given by $\phi((x_{r_1},\dots,x_{r_n})) = \sum_{i=1}^n x_{r_i \underline{e}_i}$ and $\phi((r_1 z,\dots,r_n z)) = z_{(r_1,\dots,r_n)}$ yields a Lie algebra embedding of $\mathcal{H}_1^n$ into $\mathcal{H}_n$.*

Proof. Let $\phi : \mathcal{H}_1^n \to \mathcal{H}_n$ be the hoped-for embedding. We have

$$\phi((x_{r_1},\dots,x_{r_n})) = \sum_{i=1}^n x_{r_i \underline{e}_i}, \qquad \phi((r_1 z,\dots,r_n z)) = z_{(r_1,\dots,r_n)}$$

and

$$[\phi((x_{r_1},\dots,x_{r_n})),\phi((x_{s_1},\dots,x_{s_n}))]=\Big[\sum_{i=1}^{n}x_{r_i\underline{e}_i},\sum_{i=1}^{n}x_{s_i\underline{e}_i}\Big].$$

Using $[x_{r_i\underline{e}_i},x_{s_i\underline{e}_j}]=0$ if $i\neq j$, we get

$$\Big[\sum_{i=1}^{n}x_{r_i\underline{e}_i},\sum_{i=1}^{n}x_{s_i\underline{e}_i}\Big]=\sum_{i=1}^{n}\delta_{r_i+s_i,0}z_{r_i\underline{e}_i}.$$

On the other hand,

$$[(x_{r_1},\dots,x_{r_n}),(x_{s_1},\dots,x_{s_n})]=(\delta_{r_1+s_1,0}z_{r_1},\dots,\delta_{r_n+s_n,0}z_{r_n}).$$

By the definition of ϕ and $\mathbb{Z}$-linearity in subscripts, we find that

$$\phi((\delta_{r_1+s_1,0}z_{r_1},\dots,\delta_{r_n+s_n,0}z_{r_n}))=z_{(\delta_{r_i+s_i,0}r_1,\dots,\delta_{r_n+s_n,0}r_n)}=\sum_{i=1}^{n}\delta_{r_i+s_i,0}z_{r_i\underline{e}_i}.$$

Therefore ϕ is Lie algebra map. Moreover, $\phi(Z(\mathcal{H}_1^n))=Z(\mathcal{H}_n)$. This proves that ϕ is a Lie algebra map with $\phi(Z(\mathcal{H}_1^n))=Z(\mathcal{H}_n)$. This puts ϕ in the same situation as χ in Proposition 2.4. Replacing χ by ϕ there shows that ϕ is an embedding. ∎

Remark. The obvious embedding $x_{(r_1,\dots,r_n)}\mapsto(x_{r_1},\dots,x_{r_n})$ of $\mathcal{H}_n$ into $\mathcal{H}_1^n$ is not a Lie algebra map. For example, $[x_{(1,1)},x_{(-1,1)}]=0$ in $\mathcal{H}_2$, but $[(x_1,x_1),(x_{-1},x_1)]=(z_1,0)\neq0$ in $\mathcal{H}_1^2$. The first embedding we present is not explicit as it relies on Proposition 2.5 and the remark that the kernel of ϕ_j in Proposition 2.5 is generated by $\{x_{\underline{r}}:r_j=0\}$. We give two proofs of Proposition 2.7. The first proof shows that $\mathcal{H}_n$ is a *subdirect product* of $\mathcal{H}_1^n$.

Proposition 2.7. *$\mathcal{H}_1^n$ contains a subalgebra isomorphic to $\mathcal{H}_n$.*

First proof of Proposition 2.7. Define $\prod_{j=1}^n\phi_j:\mathcal{H}_n\to\mathcal{H}_1^n$, where ϕ_j is the map in Proposition 2.5, by $(\prod_{j=1}^n\phi_j)(x_{\underline{r}})=(\phi_1(x_{\underline{r}}),\dots,\phi_n(x_{\underline{r}}))$. This map is an embedding because the kernel of ϕ_j is generated by $\{x_{\underline{r}}:r_j=0\}$ and the convention is that $x_{(0,\dots,0)}=0$. ∎

We have already noted that the natural map from $\mathcal{H}_n$ given by $x_{(r_1,\dots,r_n)}\mapsto(x_{r_1},\dots,x_{r_n})$ is not an embedding. We now remedy the situation in a way that allows us to get intrinsic representations of $\mathcal{H}_n$ in Section 2. First we define positive elements in $\mathbb{Z}^n$, $n\geq2$. An element $x_{\underline{r}}=x_{(r_1,\dots,r_n)}$ is *positive* (respectively *negative*) if its first non-zero entry is positive (respectively, negative). Consequently, if an element $x_{\underline{r}}=x_{(r_1,\dots,r_n)}$ is positive (respectively, negative), then for some positive integer k, $r_k>0$ (respectively $r_k<0$) and $r_m=0$ for $1\leq m<k$.

Example. Let $n=4$. The elements $x_{(10,0,0)}$, $x_{(0,2,-1,1)}$, and $x_{(0,0,1,-1)}$ are positive while $x_{(-1,1,2,3)}$, $x_{(0,-1,2,4)}$ and $x_{(0,0,0,-2)}$ are negative.

We have a countably infinite set of positive generators. We impose an enumeration on them as $\{n_1, n_2, \ldots\}$. We need an infinite set of disjoint infinite subsets of the set of natural numbers. For that, we let P_i be the set of positive powers of the ith positive prime p_i. We match $\{P_1, P_2, \ldots\}$ with $\{n_1, n_2, \ldots\}$. Suppose $n_i = x_{(r_1,\ldots,r_n)}$. We match $(r_1, \ldots, r_n)$ with $(p_i, p_i^2, \ldots, p_i^n)$, the first n elements of P_i . Let $\phi(n_i) = P_i$. For $j = 1, \ldots, n$, let $\phi_j(r_j) = p_i^j$. We use these bijections and notation to give a second proof of an embedding Φ of $\mathcal{H}_1^n$ into $\mathcal{H}_n$. We suppress j in ϕ_j.

Second proof of Proposition 2.7. In order to define $\Phi : \mathcal{H}_n \to \mathcal{H}_1^n$, we note that any $x_{\underline{r}}$ is either positive or negative. Suppose that it is positive. Then it is equal to n_i for some positive integer i. Hence $n_i = x_{(r_1,\ldots,r_n)}$. Set

$$\Phi(x_{(r_1,\ldots,r_n)}) = \left(\frac{r_1}{\phi(r_1)} x_{\phi(r_1)}, \ldots, \frac{r_n}{\phi(r_n)} x_{\phi(r_n)}\right)$$

and $\Phi(x_{-\underline{r}}) = (x_{-\phi(r_1)}, \ldots, x_{-\phi(r_n)})$. Let $\Phi(z_{\underline{r}}) = (r_1 z, \ldots, r_n z)$, where z is the generator of $Z(\mathcal{H}_1)$.

This takes care of every generator $x_{\underline{r}}$. We note that $\Phi(Z(\mathcal{H}_n)) \cong Z(\mathcal{H}_1^n)$. To check that we have a Lie algebra map, we have to verify that $\phi[x_{\underline{r}}, x_{\underline{s}}] = [\phi(x_{\underline{r}}), \phi(x_{\underline{s}})]$. If $\underline{r} = -\underline{s}$, the definitions of ϕ and of the respective Lie brackets guarantee the validity of the above equation.

For the rest of the verification, we use the convention that $z_0 = z_{\underline{0}} = \underline{0}$. Suppose $\underline{r} + \underline{s} \neq \underline{0}$. Then $[x_{\underline{r}}, x_{\underline{s}}] = \underline{0}$. If $\underline{r}$ and $\underline{s}$ are both positive, then

$$\begin{aligned}
&[\phi(x_{\underline{r}}), \phi(x_{\underline{s}})] \\
&\quad = \left[\left(\frac{r_1}{\phi(r_1)} x_{\phi(r_1)}, \ldots, \frac{r_n}{\phi(r_n)} x_{\phi(r_n)}\right), \left(\frac{s_1}{\phi(s_1)} x_{\phi(s_1)}, \ldots, \frac{s_n}{\phi(s_n)} x_{\phi(s_n)}\right)\right] \\
&\quad = (0, \ldots, 0)
\end{aligned}$$

because the subscripts are all positive and $\delta_{\phi(r_j)+\phi(s_j),0} = 0$. Suppose $\underline{r}$ is positive and $\underline{s}$ is negative. Then $\underline{r}$ and $-\underline{s}$ are both positive variables. Say $\underline{r} = n_i$. Then $-\underline{s} = n_j$, and $i \neq j$, because $\underline{r} \neq -\underline{s}$. Hence $\delta_{\phi(r_j)+\phi(s_j),0} = 0$. This proves that $\Phi : \mathcal{H}_n \to \mathcal{H}_1^n$ is a Lie algebra map.

We observe that $\Phi(Z(\mathcal{H}_n)) = Z(\mathcal{H}_1^n)$. We now use the technique in the proof of Proposition 2.4 to show that ϕ is an embedding. Let $g \notin Z(\mathcal{H}_1^n)$. Then $g = f + f_1 + z$ where $f = \alpha x_{(r_1,\ldots,r_n)}$, α is a non-zero scalar, f_1 has no non-zero $x_{(r_1,\ldots,r_n)}$ component, and $z \in (\mathcal{H}_1^n)$. Just as in Proposition 2.4, $\Phi(f + f_1 + z) = 0$ leads to $\Phi(f + f_1) \in Z(\mathcal{H}_n)$. Now use $[f + f_1, x_{-(r_1,\ldots,r_n)}] = z_{(r_1,\ldots,r_n)}$ to conclude as in Proposition 2.4 that $\Phi(f + f_1) \notin Z(\mathcal{H}_n)$. Hence $\Phi(f + f_1 + z) \neq 0$ and Φ is a Lie algebra embedding.

REMARK. The methods in this section can be applied to the following questions:

Suppose m and n are positive integers with $1 < m < n$. Given the ordered pair $(\mathcal{H}_n, \mathcal{H}_m)$, is there an epimorphism from $\mathcal{H}_n$ to $\mathcal{H}_m$? Analogous questions can be posed for the ordered pairs $(\mathcal{H}_1^n, \mathcal{H}_m)$ and $(\mathcal{H}_n, \mathcal{H}_1^m)$.

As we shall see in the next section, there are several families of non-isomorphic representations of $\mathcal{H}_n$ when $n \geq 2$. Let M be any representation of $\mathcal{H}_n$. Then successive applications of Propositions 2.6 and 2.7 lead to a descending chain of $\mathcal{H}_n$-submodules of M.

3. An abundance of representations. The goal this section is to use the results of Section 2 to obtain representations of $\mathcal{H}_n$, $n \geq 2$, from the representations of the Heisenberg algebra $\mathcal{H}_1$. We also give representations that are intrinsic to $\mathcal{H}_n$ when $n \geq 2$. For compatibility with our references, in this section we work over a field K of characteristic zero. Our references for representations of $\mathcal{H}_1$ are [FLM], [KR], and [MP]. We find from any of the above references that the Fock space, i.e. the associative algebra $B = K[x_{-1}, \ldots, x_{-n}, \ldots]$ (the ring of polynomials, with coefficients in a field K of characteristic zero, in the variables, $x_{-1}, x_{-2}, \ldots$), is an irreducible representation of $\mathcal{H}_1$. In fact it is also an irreducible representation of a quantum Heisenberg algebra, $\mathcal{U}_q(\mathcal{A})$, when q is not a root of unity (see [FO1]).

Proposition 2.5 and the following evident result yield irreducible representations of $\mathcal{H}_n$.

PROPOSITION 3.1. *An epimorphism from $\mathcal{H}_n$ to $\mathcal{H}_1$ makes every irreducible representation of $\mathcal{H}_1$ an irreducible representation of $\mathcal{H}_n$ for every positive integer n.*

Denote by Φ_1 and Φ_2 respectively the first and second embedding in Proposition 2.7 of $\mathcal{H}_n$ into $\mathcal{H}_1^n$. The projection of $\mathcal{H}_1^n$ onto $\mathcal{H}_1$ restricted to $\Phi_1(\mathcal{H}_n)$ is onto $\mathcal{H}_1$, but not onto $\mathcal{H}_1$ when restricted to $\Phi_2(\mathcal{H}_n)$.

PROPOSITION 3.2. *When $n \geq 2$, there are irreducible representations of $\mathcal{H}_1$ that are also representations of $\mathcal{H}_n$ but are not irreducible as $\mathcal{H}_n$-representations.*

Proof. Let $\Phi_2 : \mathcal{H}_n \to \mathcal{H}_1^n$ be the second embedding in Proposition 2.7. Denote the projection of $\mathcal{H}_1^n$ onto the ith copy of $\mathcal{H}_1$ restricted to $\Phi_2(\mathcal{H}_n)$ by $\pi_i \circ \Phi_2$. It is not onto $\mathcal{H}_1$ because Φ_2 used only powers of positive primes. Let B be the Fock space. It is an irreducible $\mathcal{H}_1$-module. Consider it an $\mathcal{H}_n$-module through the map $\pi_i \circ \Phi_2$. Let $x_l \in \mathcal{H}_1$ for some positive integer that is not in the image of $\pi_i \circ \Phi_2$. The submodule $x_{-l}B$ is a proper submodule of B as an $\mathcal{H}_n$-module because the element x_l that would have differentiated x_{-l}^2 down to x_{-l} is missing from the image of $\pi_i \circ \Phi_2$. ∎

One consequence of $(\mathcal{H}_n)' = Z(\mathcal{H}_n)$ is the construction of analogues of Verma modules without relying on $\mathcal{H}_1$-modules. All we need is a partition of the set of generators $\{x_{\underline{r}}, z_{\underline{r}} : \underline{r} \in \mathbb{Z}^n - \{0\}\}$ into two disjoint subsets $\mathcal{P} \cup (-\mathcal{P})$. For our first example, we recall the terminology in preceding Proposition 2.7 (Definition 2.1). An element $x_{\underline{r}} = x_{(r_1,\dots,r_n)}$ is declared *positive* (respectively *negative*) if its first non-zero entry is positive (respectively, negative).

Let $\mathcal{H}_n(+)$, $\mathcal{H}_n(0)$, and $\mathcal{H}_n(-)$ denote the positive generators $\mathcal{P}$, the centre of $\mathcal{H}_n$, and $-\mathcal{P}$. Let $\mathcal{B} = K[x_{-\underline{r}} : \underline{r} \in \mathcal{P}]$. This is our analogue of the Fock space in [FLM], [KR], and [MP].

In order to make $\mathcal{B}$ a representation of $\mathcal{H}_n$, we let α be a non-zero element of K. An element $z_{\underline{r}}$ in the centre of $\mathcal{H}_n$ acts on $\mathcal{B}$ as multiplication by α. In (3.1) the right hand sides show how the left hand sides act as operators on $\mathcal{B}$, and $\underline{r}$ is assumed positive.

$$\begin{aligned} x_{\underline{r}} &= \alpha \partial_{x_{\underline{r}}}, \\ x_{-\underline{r}} &= l_{x_{-\underline{r}}}, \end{aligned} \tag{3.1}$$

where $\partial_{x_{\underline{r}}}$ is the partial derivative with respect to the variable $x_{\underline{r}}$, and $l_{x_{-\underline{r}}}$ is left multiplication on $\mathcal{B}$ by $x_{-\underline{r}}$. When $\partial_{x_{\underline{r}}}$ acts on $\mathcal{B}$ the negative sign in $x_{-\underline{s}}$ is ignored. We now show that (3.1) yields a representation of $\mathcal{H}_n$.

Let $x_{\underline{r}}$ and $x_{\underline{t}}$ be positive. Hence $x_{-\underline{t}} \in \mathcal{B}$. In $\mathcal{H}_n$, $[x_{\underline{r}}, x_{-\underline{r}}] = z_{\underline{r}}$ and $[x_{\underline{r}}, x_{-\underline{r}}]$ acts on $x_{-\underline{t}}$ as $(\alpha\partial_{x_{\underline{r}}} l_{x_{-\underline{r}}} - l_{x_{-\underline{r}}} \alpha \partial_{x_{\underline{r}}})(x_{-\underline{t}})$. This is equal to $\alpha[\partial_{x_{\underline{r}}}(x_{-\underline{r}} x_{-\underline{t}}) - x_{-\underline{r}} \partial_{x_{\underline{r}}}(x_{-\underline{t}})] = \alpha x_{-\underline{t}} = z_{\underline{r}} x_{-\underline{t}}$, as required. Let $\underline{r}$ and $\underline{s}$ be arbitrary elements of $\mathbb{Z}^n - \{0\}$ with $\underline{r} + \underline{s} \neq 0$. A similar computation shows that $[x_{\underline{r}}, x_{-\underline{s}}]$ acts as the zero operator on $\mathcal{B}$.

Notation. Denote by $\mathcal{A}_n(\alpha)$ the representation of $\mathcal{H}_n$ constructed using (3.1) and the non-zero scalar α. Let $\mathcal{A}(j)$ be the representation of $\mathcal{H}_n$ obtained from Proposition 3.1 using S_j, $j \in \{1, \dots, n\}$, in Proposition 2.5.

We can appeal to the methods in [FLM], [KR], or [MP] to see that $\mathcal{A}_n(\alpha)$ is an irreducible representation of $\mathcal{H}_n$. Unlike $\mathcal{A}(j)$, $\mathcal{A}_n(\alpha)$ is *faithful*, that is, it has no non-zero annihilator. Consequently, we have the next proposition.

Proposition 3.3 ([FM]). *Let $n \geq 2$. For every non-zero element α in K, $\mathcal{A}_n(\alpha)$ is an irreducible representation of $\mathcal{H}_n$ that is not isomorphic to $\mathcal{A}(j)$.*

We now want to give examples of representations of $\mathcal{H}_n$ that have no irreducible submodules, using the same definition of positive elements of $\mathbb{Z}^n - \{0\}$ used in the second proof of Proposition 2.7. Hence the Fock space is again $\mathcal{B}$. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ be any non-zero vector in K^n. Assuming that $x_{\underline{r}}$ is positive we specify the following action of $\mathcal{H}_n$ on $\mathcal{B}$. Let $z_{\underline{r}}$ act as multiplication by $\underline{\alpha} \cdot \underline{r}$, the usual dot product. We have

$$x_{\underline{r}} = (\underline{\alpha} \cdot \underline{r}) \partial_{x_{\underline{r}}}, \qquad x_{-\underline{r}} = l_{x_{-\underline{r}}}, \tag{3.2}$$

Denote the resulting module by $\mathcal{A}_n(\underline{\alpha})$.

PROPOSITION 3.4. *Let $n \geq 2$. Then $\mathcal{A}_n(\underline{\alpha})$ is a representation of $\mathcal{H}_n$ that has no non-zero irreducible submodule.*

Proof. Let $\underline{r} \in \underline{\alpha}^{\perp}$, the orthogonal complement of $\underline{\alpha}$. If f is any non-zero element in a submodule, N, of $\mathcal{A}_n(\underline{\alpha})$, then $x_{-\underline{r}}f$ generates a proper non-zero submodule of N because the element $x_{\underline{r}}$ needed to differentiate down acts as zero on $\mathcal{B}$ by (3.2). ∎

We have not exhausted all the possible ways of obtaining representations of $\mathcal{H}_n$ when $n \geq 2$. For instance, it is likely that new representations of $\mathcal{H}_n$ can be obtained from the vertex representations of the generalized Heisenberg algebras given in [F] and [FM].

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