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ON THE SPACING BETWEEN TERMS OF GENERALIZED FIBONACCI SEQUENCES

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Abstract. For $k \geq 2$, the k-generalized Fibonacci sequence $(F_n^{(k)})_n$ is defined to have the initial k terms $0, 0, \ldots, 0, 1$ and be such that each term afterwards is the sum of the k preceding terms. We will prove that the number of solutions of the Diophantine equation $F_m^{(k)} - F_n^{(\ell)} = c > 0$ (under some weak assumptions) is bounded by an effectively computable constant depending only on c.

1. Introduction. The problem of studying the spacing between terms of some sequences has attracted the attention of mathematicians for decades. For instance, the equation related to the spacing between perfect powers, is called *Pillai's equation*:

$$m^k - n^\ell = c, \tag{1.1}$$

for a fixed positive constant c. *Pillai's conjecture* [13] is that for any given $c \geq 1$, the number of positive integer solutions to the Diophantine equation (1.1), with $\min\{k, \ell\} \geq 2$, is finite. To the best of our knowledge, this conjecture remains open (there are several related results, some of them ineffective; see the nice survey [15]).

We recall that the case $c = 1$ was already considered by E. Catalan who, in 1844, conjectured that the only consecutive perfect powers are 8 and 9. Recently, this conjecture was confirmed by P. Mihăilescu [11]. We refer the reader to [1] for a better discussion of this subject.

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [7] together with its very extensive annotated bibliography for additional references and history). It is a simple matter to deduce that if $F_n \neq F_m$, then

$$|F_m - F_n| > \left(\frac{1+\sqrt{5}}{2}\right)^{\max\{m,n\}-4}.$$

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There are several generalizations of Fibonacci numbers in the literature. For instance, the *Fibonomial coefficient* is defined, for $1 \leq k \leq m$, as

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m \cdots F_{m-k+1}}{F_k \cdots F_1}. \tag{1.2}$$

Clearly, $(\begin{bmatrix} m \\ 1 \end{bmatrix}_F)_m$ is the Fibonacci sequence. In 2010, Luca, Marques and Stănică [8] studied the spacing between Fibonomial coefficients. In particular, they proved that the difference

$$\left| \begin{bmatrix} m \\ k \end{bmatrix}_F - \begin{bmatrix} n \\ \ell \end{bmatrix}_F \right|$$

tends to infinity when (m, k, n, ℓ) are such that $1 \leq k \leq m/2,\ 1 \leq \ell \leq n/2$, $(m, k) \neq (n, \ell)$ and $\max\{m, n\}$ tends to infinity in an effective way.

Another known generalization is, for $k \geq 2$, the *k-generalized Fibonacci sequence* $F^{(k)} := (F_n^{(k)})_{n \geq -(k-2)}$, which is the sequence whose terms satisfy the kth order recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)}, \tag{1.3}$$

with initial conditions $0, 0, \ldots, 0, 1$ (k terms) and such that the first nonzero term is $F_1^{(k)} = 1$. Clearly for $k = 2$, we obtain the Fibonacci numbers $F_n^{(2)} = F_n$, and for $k = 3$, the Tribonacci numbers $F_n^{(3)} = T_n$.

The aim of this paper is to prove a related result (in the spirit of Pillai) about the spacing between terms of distinct k-generalized Fibonacci sequences. That is, we study the Diophantine equation

$$F_m^{(k)} - F_n^{(\ell)} = c. \tag{1.4}$$

This equation could be considered as a "Fibonacci version" of Pillai's equation (where we replace the powers ℓ and k by the respective order of a generalized Fibonacci sequence, that is, by the superscripts (ℓ) and (k)). We point out that equation (1.4) for $c = 0$ was solved independently by Bravo and Luca [2] and Marques [9].

Our main results are the following.

Theorem 1.1. *Let c be an integer. Then there exists an effectively computable constant $M = M(c)$ such that if (m, n, ℓ, k) is a positive integer solution of (1.4) with $\ell \geq k \geq 2$, $n > \ell + 2, m > k + 2$ and $m \neq n$, then $\max\{m, n, \ell, k\} < M$. A suitable choice for M is*

$$M := \max\{c_1, 1.9 \cdot 10^{146} c_2^{24} \log^{27} c_2, 8 \cdot 10^{246}\}, \tag{1.5}$$

where $c_1 := 5 \log(|c| + 1) + 2$ and $c_2 := 4 \log(|c| + 5)/\log 2$.

Observe that Theorem 1.1 implies, in particular, that the difference $|F_m^{(k)} - F_n^{(\ell)}|$ tends to infinity when (m, n, ℓ, k) are such that $m > k+2,\ n > \ell + 2,\ m \neq n$ and $\max\{m, n\}$ tends to infinity in an effective way.

Note that if (1.4) has infinitely many solutions, then $m = n$. Our next result treats this case.

THEOREM 1.2. *Set* $t_d = d \cdot 2^{d-3}$, *for* $0 \le d \ne 1$. *Given integers* $0 \le r < s$,

$$(m, n, \ell, k) = (k + s, k + s, k + s - r, k)$$

is a solution of (1.4) *when* $c = -(t_s - t_r)$, *for all* $k \ge s - 1$. *Moreover, for* $m = n$, *if* (1.4) *has a solution with* $m \le 2k + 1$, *then* $c = -(t_r - t_p)$ *for some integers* $r > p$.

The sequence $(t_d)_{d \ge 2}$ $(= 1, 3, 8, 20, 48, \ldots)$ is the OEIS [1] A001792 [14] and, for instance, t_d counts the number of parts in all compositions (ordered partitions) of $d + 1$. Also, the first values of the sequence $(t_s - t_r)_{r,s}$, with $s > r \ge 0$ and $s, r \ne 1$, are

$$1, 2, 3, 5, 7, 8, 12, 17, 19, 20, 28, 40, 45, 47, 48, 64, \ldots.$$

As another application of the method, we solve completely the case $c = 1$ ("Catalan–Fibonacci" version), that is, we find all consecutive numbers among $\bigcup_{k \ge 2} F^{(k)}$.

THEOREM 1.3. *The only solution of the Diophantine equation*

$$|F_m^{(k)} - F_n^{(\ell)}| = 1, \tag{1.6}$$

with $\ell \ge k \ge 2$, $n > \ell + 2$ *and* $m > k + 2$ *is* $(m, n, \ell, k) = (10, 8, 4, 2)$. *That is,*

$$F_8^{(4)} - F_{10}^{(2)} = 56 - 55 = 1.$$

We remark that the hypotheses $n > \ell + 2$ and $m > k + 2$ are necessary to avoid the trivial solutions

$$(m, n, \ell, k) = (k + 2, k + 2, k + 1, k)$$

for all $k \ge 2$.

Let us give a brief overview of our strategy for proving Theorem 1.1. First, we use a formula of Dresden [5, formula (2)] to get an upper bound for a linear form in three logarithms related to equation (1.4). Afterwards, we use a lower bound due to Matveev to obtain an upper bound for m and n in terms of ℓ. Very recently, Bravo and Luca solved the equation $F_n^{(k)} = 2^m$, using a nice argument combining some estimates with the Mean Value Theorem [3, pp. 77–78]. In our case, we must use this approach twice together with a reduction argument due to Dujella and Pethő. In the final section, we present a program for checking the "small" cases. The computations in the paper were performed using *Mathematica*.

We mention some differences between our work and the one by Bravo and Luca. In their paper, the equation $F_n^{(k)} = 2^m$ was studied. By applying a key

[1] On-Line Encyclopedia of Integer Sequences.

method, they get directly an upper bound for $|2^m - 2^{n-2}|$. In our case, the equation $F_m^{(k)} - F_n^{(\ell)} = c$ needs a little more work, because it is necessary to apply their method twice to get an upper bound for $|2^{n-2} - 2^{m-2}|$. Moreover, they used a reduction argument due to Dujella and Pethő to solve all small cases. In our work, we use a fast *Mathematica* routine to deal with the "very" small cases.

2. Auxiliary results. In order to avoid unnecessary repetitions, throughout the paper the integers m, n, k, ℓ are supposed to satisfy the conditions in the statement of Theorem 1.1.

Before proceeding, we shall recall some facts and properties of the relevant sequences.

We know that the characteristic polynomial of $(F_n^{(k)})_n$ is

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1$$

and it is irreducible over $\mathbb{Q}[x]$ with just one zero outside the unit circle. That single zero is located between $2(1 - 2^{-k})$ and 2 (see [16]). Also, in a recent paper, G. Dresden [5, Theorem 1] gave a simplified "Binet-like" formula for $F_n^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{2.1}$$

for $\alpha = \alpha_1, \ldots, \alpha_k$ being the roots of $\psi_k(x)$. It was proved in [4, Lemma 1] that

$$\alpha^{n-2} \le F_n^{(k)} \le \alpha^{n-1} \quad \text{for all } n \ge 1, \tag{2.2}$$

where α is the dominant root of $\psi_k(x)$. The contribution of the roots inside the unit circle in formula (2.1) is almost trivial. More precisely, it was proved in [5] that

$$|F_n^{(k)} - g(\alpha, k)\alpha^{n-1}| < 1/2, \tag{2.3}$$

where $g(x, y) := (x - 1)/(2 + (y + 1)(x - 2))$.

Now, we wish to find a lower bound for m in terms of n. In fact, by (1.4) and (2.2),

$$2^{n-1} > \phi^{n-1} \ge F_n^{(\ell)} = F_m^{(k)} + 1 > \alpha^{m-2} > (\sqrt{2})^{m-2} \quad \text{and so} \quad 2n > m, \tag{2.4}$$

where in the last inequality we used that $\alpha > 3/2 > \sqrt{2}$.

Also, observe that $(F_n^{(\ell)})_n$ and $(F_n^{(\ell)})_\ell$ are nondecreasing sequences.

As another tool to prove Theorem 1.1, we use a lower bound for a linear forms in logarithms à la Baker given by the following result of Matveev [10].

LEMMA 2.1. *Let $\gamma_1, \ldots, \gamma_t$ be real algebraic numbers and let $b_1, \ldots, b_t$ be nonzero rational integers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_t)$*

over $\mathbb{Q}$ *and let* A_j *be a positive real number satisfying*

$$A_j \geq \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\} \quad \text{for } j = 1, \ldots, t.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_t^{b_t} \neq 1$, *then*

$$|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \geq \exp\big(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t\big).$$

As usual, in the above statement, the *logarithmic height* of an s-degree algebraic number γ is defined as

$$h(\gamma) = \frac{1}{s}\Big(\log|a| + \sum_{j=1}^{s} \log\max\{1, |\gamma^{(j)}|\}\Big),$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$) and $(\gamma^{(j)})_{1\leq j\leq s}$ are the conjugates of α (over $\mathbb{Q}$).

After finding an upper bound on n which is in general too large, the next step is to reduce it. For that, our last ingredient is a variant of the famous Baker–Davenport lemma, due to Dujella and Pethő [6, Lemma 5(a)]. For a real number x, we use $\|x\| = \min\{|x - n| : n \in \mathbb{N}\}$ for the distance from x to the nearest integer.

Lemma 2.2. *Suppose that* M *is a positive integer. Let* p/q *be a convergent of the continued fraction expansion of the irrational number* γ *such that* $q > 6M$ *and let* A, B *be some real numbers with* $A > 0$ *and* $B > 1$. *Let* $\epsilon = \|\mu q\| - M\|\gamma q\|$, *where* μ *is a real number. If* $\epsilon > 0$, *then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < A \cdot B^{-k}$$

in positive integers m, n *and* k *with*

$$m \leq M \quad \text{and} \quad k \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

3. The proof of Theorems 1.1 and 1.2

3.1. Proof of Theorem 1.1. First note that when $\ell = k$, then:

- If $c > 0$, then (1.4) implies $m > n$ and so
$$c = F_m^{(k)} - F_n^{(k)} \geq F_m^{(k)} - F_{m-1}^{(k)} \geq F_{m-2}^{(k)} > (1.5)^{m-4},$$
yielding $m < 3\log c + 4 < c_1$.
- When $c < 0$, we have $n > m$ and then
$$-c = F_n^{(k)} - F_m^{(k)} \geq F_{m+1}^{(k)} - F_m^{(k)} \geq F_{m-1}^{(k)} > (1.5)^{m-3},$$
yielding $m < 3\log(-c) + 3 < c_1$.

Thus, we may suppose that $\ell > k$. We also suppose, without loss of generality, that $m > n$ (the case $n > m$ follows exactly the same lines; for convenience of the reader, we shall indicate the substantial changes for $n > m$).

Note that in order to prove the Theorem 1.1, it suffices to show that (1.4) has no solution when $m > M$ (with M defined as in (1.5)). Thus suppose, towards a contradiction, that (m, n, ℓ, k) is a solution of (1.4) with $m > M$.

The first step is to find an upper bound for m (and so for n) in terms of ℓ.

For that, we use (2.3) to get

$$|F_m^{(k)} - g(\alpha, k)\alpha^{m-1}| < 1/2 \quad \text{and} \quad |F_n^{(\ell)} - g(\phi, \ell)\phi^{n-1}| < 1/2,$$

where α and ϕ are the dominant roots of the recurrences $(F_m^{(k)})_m$ and $(F_n^{(\ell)})_n$, respectively. Combining these inequalities with $|F_n^{(\ell)} - F_m^{(k)}| = |c|$, we obtain

$$|g(\phi, \ell)\phi^{n-1} - g(\alpha, k)\alpha^{m-1}| < |c| + 1 \tag{3.1}$$

and so

$$\left|\frac{g(\phi, \ell)\phi^{n-1}}{g(\alpha, k)\alpha^{m-1}} - 1\right| < \frac{|c| + 1}{g(\alpha, k)\alpha^{m-1}} < \frac{4(|c| + 1)}{\alpha^{m-1}} < \frac{1}{\alpha^{m/2}}, \tag{3.2}$$

where we have used the facts that $\alpha^{(m-2)/2} > 4(|c| + 1)$ (since $m > c_1$) and $g(\alpha, k) > 1/4$ (here we would divide (3.1) by $g(\phi, \ell)\phi^{n-1}$ when $n > m$). Thus (3.2) becomes

$$|e^{\Lambda} - 1| < 1/\alpha^{m/2}, \tag{3.3}$$

where $\Lambda := (n - 1)\log\phi + \log(g(\phi, \ell)/g(\alpha, k)) - (m - 1)\log\alpha$.

Now, we shall apply Lemma 2.1. To this end, take $t := 3$,

$$\gamma_1 := \phi, \quad \gamma_2 := \frac{g(\phi, \ell)}{g(\alpha, k)}, \quad \gamma_3 := \alpha$$

and

$$b_1 := n - 1, \quad b_2 := 1, \quad b_3 := m - 1.$$

For this choice, we have $D = [\mathbb{Q}(\alpha, \phi) : \mathbb{Q}] \leq k\ell < \ell^2$. Also $h(\gamma_1) = (\log\phi)/\ell < (\log 2)/\ell < 0.7/\ell$ and similarly $h(\gamma_3) < 0.7/k$. In [3, p. 73], an estimate for $h(g(\alpha, k))$ was given. More precisely, it was proved that

$$h(g(\alpha, k)) < \log(k + 1) + \log 4.$$

Analogously,

$$h(g(\phi, \ell)) < \log(\ell + 1) + \log 4.$$

Thus

$$h(\gamma_2) \leq h(g(\phi, \ell)) + h(g(\alpha, k)) \leq \log(\ell + 1) + \log(k + 1) + 2\log 4,$$

where we have used the well-known facts that $h(xy) \leq h(x) + h(y)$ and $h(x) = h(x^{-1})$. Also, in [3] it was proved that $|g(\alpha_i, k)| < 2$ for all $i = 1, \ldots, k$.

Since $\ell > k$ and $m > n$, we can take $A_1 = A_3 := 0.7\ell$, $A_2 := 2\ell^2 \log(4\ell+4)$ and $B := m - 1$.

Before applying Lemma 2.1, it remains to prove that $e^{\Lambda} \neq 1$. Suppose the contrary, i.e., $g(\alpha, k)\alpha^{m-1} = g(\phi, \ell)\phi^{n-1} \in \mathbb{Q}(\phi)$. We can conjugate this relation in $\mathbb{Q}(\phi)$ to get

$$g(\alpha_{s_i}, k)\alpha_{s_i}^{m-1} = g(\phi_i, \ell)\phi_i^{n-1} \quad \text{for } i = 1, \ldots, \ell,$$

where α_{s_i} are the ℓ conjugates of α over $\mathbb{Q}(\phi)$. Since $g(\alpha, k)\alpha^{m-1}$ has at most k conjugates (over $\mathbb{Q}$), each number in the list $\{g(\alpha_{s_i}, k)\alpha_{s_i}^{m-1} : 1 \leq i \leq \ell\}$ is repeated at least $\ell/k > 1$ times. In particular, there exists $t \in \{2, \ldots, \ell\}$ such that $g(\alpha_{s_1}, k)\alpha_{s_1}^{m-1} = g(\alpha_{s_t}, k)\alpha_{s_t}^{m-1}$. Thus, $g(\phi, \ell)\phi^{n-1} = g(\phi_t, \ell)\phi_t^{n-1}$ and then

$$\left(\frac{7}{4}\right)^{n-1} < \phi^{n-1} = \left|\frac{g(\phi_t, \ell)}{g(\phi, \ell)}\right| |\phi_t|^{n-1} < 8,$$

where we have used that $\phi > 2(1 - 2^{-\ell}) \geq 7/4$, $|g(\phi_t, \ell)| < 2 < 8|g(\phi, \ell)|$ and $|\phi_t| < 1$ for $t > 1$. However, the inequality $(7/4)^{n-1} < 8$ holds only for $n = 1, 2, 3, 4$, $n > \ell + 1 \geq 3 + 1 = 4$. Therefore $e^{\Lambda} \neq 1$.

Now, the conditions to apply Lemma 2.1 are fulfilled and hence

$$|e^{\Lambda} - 1| > \exp\big(-1.5 \cdot 10^{11}\ell^8(1 + 2\log\ell)\log(4\ell + 4)(1 + \log(m - 1))\big).$$

Since $1 + 2\log\ell \leq 3\log\ell$, $4\ell + 4 < \ell^{2.6}$ (for $\ell \geq 3$) and $m - 1 < m^{1.1}$, we have

$$|e^{\Lambda} - 1| > \exp(-2.64 \cdot 10^{12}\ell^8 \log^2\ell \log m)). \tag{3.4}$$

By combining (3.3) and (3.4), we get

$$\frac{m}{\log m} < 1.33 \cdot 10^{13}\ell^8 \log^2\ell,$$

where we have used that $\log\alpha > 0.4$. Since the function $x/\log x$ is increasing for $x > e$, it is a simple matter to prove that

$$\frac{x}{\log x} < A \quad \text{implies that} \quad x < 2A\log A. \tag{3.5}$$

A proof can be found in [3, p. 74].

Thus, by using (3.5) for $x := m$ and $A := 1.33 \cdot 10^{13}\ell^8 \log^2\ell$, we have

$$m < 2(1.33 \cdot 10^{13}\ell^8 \log^2\ell)\log(1.33 \cdot 10^{13}\ell^8 \log^2\ell).$$

Now, the inequality $31 + 2\log\log\ell < 29\log\ell$ for $\ell \geq 3$ yields

$$\log(1.33 \cdot 10^{13}\ell^8 \log^2\ell) < 31 + 8\log\ell + 2\log\log\ell < 37\log\ell.$$

Therefore,

$$m < 9.9 \cdot 10^{14} \ell^8 \log^3 \ell. \tag{3.6}$$

The next step is to find an upper bound for ℓ in terms of k. For that, consider $\ell \leq 240$; then the inequality (3.6) yields $m < 1.8 \cdot 10^{36}$, contradicting $m > M$. Thus, we may assume that $\ell > 240$. Therefore

$$n < 9.9 \cdot 10^{14} \ell^8 \log^3 \ell < 2^{\ell/2}, \tag{3.7}$$

where we have used (3.6) and the fact that $n < m$. Now, we shall use a key argument due to Bravo and Luca [3, pp. 77–78]. For the sake of completeness and because one needs a slight modification in its final part, we shall present their nice idea.

Setting $\lambda = 2 - \phi$, we deduce that $0 < \lambda < 1/2^{\ell-1}$ (because $2(1 - 2^{-\ell}) < \phi < 2$). So

$$\phi^{n-1} = (2 - \lambda)^{n-1} = 2^{n-1}(1 - \lambda/2)^{n-1} > 2^{n-1}(1 - (n-1)\lambda),$$

since $(1 - x)^n > 1 - 2nx$ for all $n \geq 1$ and $0 < x < 1$. Moreover, $(n-1)\lambda < 2^{\ell/2}/2^{\ell-1} = 2/2^{\ell/2}$ and hence

$$2^{n-1} - \frac{2^n}{2^{\ell/2}} < \phi^{n-1} < 2^{n-1} + \frac{2^n}{2^{\ell/2}},$$

yielding

$$|\phi^{n-1} - 2^{n-1}| < \frac{2^n}{2^{\ell/2}}. \tag{3.8}$$

Now, we define for $x > 2(1 - 2^{-\ell})$ the function $f(x) := g(x, \ell)$ which is differentiable in the interval $[\phi, 2]$. So, by the Mean Value Theorem, there exists $\xi \in (\phi, 2)$ such that $f(\phi) - f(2) = f'(\xi)(\phi - 2)$. Thus

$$|f(\phi) - f(2)| < \frac{2\ell}{2^\ell}, \tag{3.9}$$

where we have used the bounds $|\phi - 2| < 1/2^{\ell-1}$ and $|f'(\xi)| < \ell$. For simplicity, we denote $\delta = \phi^{n-1} - 2^{n-1}$ and $\eta = f(\phi) - f(2) = f(\phi) - 1/2$. After some calculations, we arrive at

$$2^{n-2} = f(\phi)\phi^{n-1} - 2^{n-1}\eta - \delta/2 - \delta\eta.$$

Therefore

$$\begin{aligned} |2^{n-2} - g(\alpha, k)\alpha^{m-1}| &\leq |f(\phi)\phi^{n-1} - g(\alpha, k)\alpha^{m-1}| + 2^{n-1}|\eta| + |\delta/2| + |\delta\eta| \\ &\leq |c| + 1 + \frac{2^n \ell}{2^\ell} + \frac{2^{n-1}}{2^{\ell/2}} + \frac{2^{n+1}\ell}{2^{3\ell/2}}, \end{aligned}$$

where we have used (3.8) and (3.9). Since $n > \ell + 2$, one has $1 < 2^{n-2}/2^{\ell/2}$ and we rewrite the above inequality as

$$|2^{n-2} - g(\alpha, k)\alpha^{m-1}| < (|c| + 1)\frac{2^{n-2}}{2^{\ell/2}} + \frac{4\ell}{2^{\ell/2}}\frac{2^{n-2}}{2^{\ell/2}} + 2 \cdot \frac{2^{n-2}}{2^{\ell/2}} + \frac{8\ell}{2^\ell}\frac{2^{n-2}}{2^{\ell/2}}.$$

Since the inequalities $4\ell < 8\ell < 2^{\ell/2} < 2^{\ell}$ hold for all $\ell > 240$ (actually, they hold for $\ell > 13$), we have

$$|2^{n-2} - g(\alpha,k)\alpha^{m-1}| < \frac{(|c|+5)\cdot 2^{n-2}}{2^{\ell/2}} < \frac{2^{n-2}}{2^{\ell/4}}, \tag{3.10}$$

where we have used that $2^{\ell/4} > |c| + 5$. This follows because $\ell > c_2$ (in fact, otherwise we can use (3.6) to get $M < m < 9.9 \cdot 10^{14} c_2^8 \log^3 c_2$).

Equivalently, we have

$$|1 - g(\alpha,k)\alpha^{m-1}2^{-(n-2)}| < \frac{1}{2^{\ell/4}}. \tag{3.11}$$

In order to apply Lemma 2.1, it remains to prove that the left-hand side of (3.11) is nonzero, or equivalently, $2^{n-2} \neq g(\alpha,k)\alpha^{m-1}$. Suppose the contrary, i.e., $2^{n-2} = g(\alpha,k)\alpha^{m-1}$. By conjugating this relation in the splitting field of $\psi_k(x)$, we obtain $2^{n-2} = g(\alpha_i,k)\alpha_i^{m-1}$ for $i = 1,\dots,k$. However, when $i > 1$, we have $|\alpha_i| < 1$ and $|g(\alpha_i,k)| < 2$, which leads to the absurdity

$$2^{n-2} = |g(\alpha_i,k)|\,|\alpha_i|^{m-1} < 2,$$

since $n > 4$. Therefore $g(\alpha,k)\alpha^{m-1}2^{-(n-2)} \neq 1$ and so we are in a position to apply Lemma 2.1. For that, take $t := 3$,

$$\gamma_1 := g(\alpha,k), \qquad \gamma_2 := \alpha, \qquad \gamma_3 := 2$$

and

$$b_1 := 1, \qquad b_2 := m-1, \qquad b_3 := -(n-2).$$

By calculations performed in Section 2, we see that $A_1 := k\log(4k+4)$, $A_2 = A_3 := 0.7$ are suitable choices. Moreover $D = k$ and $B = m-1$. Thus

$$\begin{aligned} &|1 - g(\alpha,k)\alpha^{m-1}2^{-(n-2)}| \\ &\qquad > \exp\bigl(-D' \cdot k^3(1+\log k)(1+\log(m-1))\log(4k+4)\bigr), \end{aligned} \tag{3.12}$$

where we can take $D' = 0.75 \cdot 10^{11}$. Combining (3.11) and (3.12) with a straightforward calculation, we get

$$\ell < 2.16 \cdot 10^{12} k^3 \log^2 k \log m. \tag{3.13}$$

On the other hand, $m < 9.9 \cdot 10^{14} \ell^8 \log^3 \ell$ (by (3.6)) and so

$$\log m < \log(9.9 \cdot 10^{14} \ell^8 \log^3 \ell) < 41 \log \ell, \tag{3.14}$$

where we have used that $35 + 3\log\log\ell < 33\log\ell$. Turning back to (3.13), we obtain

$$\frac{\ell}{\log \ell} < 8.9 \cdot 10^{13} k^3 \log^2 k,$$

which implies, by (3.5), that

$$\ell < 2(8.9 \cdot 10^{13} k^3 \log^2 k)\log(8.9 \cdot 10^{13} k^3 \log^2 k).$$

Since $\log(8.9\cdot 10^{13}k^3\log^2 k) < 47\log k$, we finally get

$$\ell < 8.4\cdot 10^{15}k^3\log^3 k. \tag{3.15}$$

Now, if $k\leq 1640$, then $\ell < 2\cdot 10^{28}$ (by (3.15)). Thus, by (3.6), one sees that $m < 7.1\cdot 10^{246}$, which is not possible, because $m > M$.

Therefore, we may suppose that $k > 1640$. Then the inequality $\ell < 8.9\cdot 10^{15}k^3\log^3 k$ together with (3.6) yields

$$\begin{aligned} m &< 9.9\cdot 10^{14}(8.9\cdot 10^{15}k^3\log^3 k)^8\log^3(8.9\cdot 10^{15}k^3\log^3 k)\\ &< 1.9\cdot 10^{146}k^{24}\log^{27}k < 2^{k/2}, \end{aligned}$$

where the last inequality holds because $k > 1640$. Now, we use again the key argument of Bravo and Luca to conclude that

$$|2^{m-2} - g(\phi,\ell)\phi^{n-1}| < \frac{2^{m-2}}{2^{k/4}}, \tag{3.16}$$

because $k > c_2$ (to get a contradiction, we substitute $k\leq c_2$ in (3.15) and (3.6) to obtain an upper bound for m less than M). Combining (3.10), (3.16) and (3.1), we get

$$\begin{aligned} |2^{n-2} - 2^{m-2}| &\leq |2^{n-2} - g(\alpha,k)\alpha^{n-1}| + |g(\alpha,k)\alpha^{n-1} - g(\phi,\ell)\phi^{n-1}|\\ &\quad + |2^{m-2} - g(\phi,\ell)\phi^{n-1}|\\ &< \frac{2^{n-2}}{2^{\ell/4}} + |c| + 1 + \frac{2^{m-2}}{2^{k/4}} < \frac{3\cdot 2^{m-2}}{2^{k/4}}, \end{aligned}$$

since $n < m,\ k < \ell, m > k+1$ and $|c|+1 < 2^{k/2}$ (otherwise we have $k\leq 2\log(|c|+1)/\log 2 < c_2$). Therefore

$$|2^{n-m} - 1| < \frac{3}{2^{k/4}}. \tag{3.17}$$

Since $n\leq m-1$, we have

$$\frac{1}{2}\leq 1 - 2^{n-m} = |2^{n-m} - 1| < \frac{3}{2^{k/4}}.$$

Thus $2^{k/4} < 6$, yielding $k\leq 10$, which is absurd, since $k > 1640$.

In the case of $n > m$, we have $|2^{n-m} - 1| > 1$ and the contradiction is the same. In fact, in any case, one has $|2^{n-m} - 1| > 1/2$. ∎

3.2. Proof of Theorem 1.2. The proof is based on the key identity

$$F_{k+a} = 2^{k+a-2} - t_a, \tag{3.18}$$

which holds for all $2\leq a\leq k+1$, where $t_a = a\cdot 2^{a-3}$. We shall leave the proof of this fact as an exercise to the reader (by induction on k).

For $s > r$, one has $s = r+j$ for some $j > 0$. Thus (3.18) gives

$$F^{(k)}_{k+(r+j)} = 2^{k+r+j-2} - t_s,$$

since $r + j \le k + 1$ (because $s \le k + 1$). Also,

$$F^{(k+j)}_{(k+j)+r} = 2^{k+r+j-2} - t_r.$$

Therefore,

$$F^{(k+j)}_{(k+j)+r} - F^{(k)}_{k+(r+j)} = t_s - t_r,$$

and the result follows.

Suppose now that (m, ℓ, k) is a solution of $F_m^{(\ell)} - F_m^{(k)} = c > 0$. Then $\ell > k$, yielding $\ell = k + s$ with $s > 0$. Also, the hypothesis $m > \ell + 2$ implies the existence of integers r and p such that $m = k + r$ and $r = s + p$. Thus

$$F^{(k+s)}_{k+s+p} - F^{(k)}_{k+s+p} = c.$$

On the other hand, since $p \le k - s + 1$ (because $m \le 2k + 1$ implies $k + (s + p) \le 2k + 1$), we can use (3.18) to get

$$F^{(k+s)}_{(k+s)+p} = 2^{k+s+p-2} - t_p \quad \text{and} \quad F^{(k)}_{k+(s+p)} = 2^{k+s+p-2} - t_r,$$

leading to

$$c = F^{(k+s)}_{k+s+p} - F^{(k)}_{k+s+p} = (2^{k+s+p-2} - t_p) - (2^{k+s+p-2} - t_r) = t_r - t_s.$$

The proof is complete. ■

4. The proof of Theorem 1.3. First, we claim that $n < m$. To derive a contradiction, suppose that $n \ge m$. Then (1.6) gives $F_n^{(\ell)} \le F_m^{(k)} + 1$. However, $F_m^{(k)} + 1 < F_m^{(k+1)}$ for $m > k + 2$. In fact, since $(F_n^{(\ell)})_\ell$ is nondecreasing, it suffices to prove this inequality for $m = k + 3$. This holds because

$$F^{(k+1)}_{k+3} = 2^{k+1} - 1 > 2^{k+1} - 2 = F^{(k)}_{k+3} + 1.$$

Thus, we obtain the contradiction

$$F_n^{(\ell)} \le F_m^{(k)} + 1 < F_m^{(k+1)} \le F_n^{(\ell)},$$

where we have used that the sequences $(F_n^{(\ell)})_n$ and $(F_n^{(\ell)})_\ell$ are nondecreasing. Therefore, $m > n$ as claimed and we can follow the proof of Theorem 1.1. Summarizing, the previous theorem (for $c = \pm 1$) ensures that the possible solutions (m, n, k, ℓ) of (1.6) must satisfy

$$m < 8 \cdot 10^{246},$$

where we have used that $c_1 < 5.47$, $c_2 < 2.74$ and so $1.9 \cdot 10^{146} c_2^{24} \log^{27} c_2 < 6.4 \cdot 10^{156}$.

Since this upper bound on $\max\{m, n, \ell, k\}$ is too large, we need to use Lemma 2.2.

We recall that $2 \leq k \leq 1640$, $\ell < 2 \cdot 10^{28}$ and $n < m < 7.1 \cdot 10^{246}$. In order to use Lemma 2.2, we rewrite (3.11) as

$$|e^{\Theta} - 1| < \frac{1}{2^{\ell/4}},$$

where $\Theta := (m-1)\log\alpha - (n-2)\log 2 + \log g(\alpha, k)$. Recall that we have proved that $e^{\Theta} \neq 1$ (the paragraph below (3.11)) and so $\Theta \neq 0$.

If $\Theta > 0$, then $\Theta < e^{\Theta} - 1 < 1/2^{\ell/4}$. In the case of $\Theta < 0$, we use $1 - e^{-|\Theta|} = |e^{\Theta} - 1| < 1/2^{\ell/4}$ to get $e^{|\Theta|} < 1/(1 - 2^{-\ell/4})$. Thus

$$|\Theta| < e^{|\Theta|} - 1 < \frac{2^{-\ell/4}}{1 - 2^{-\ell/4}} < 2^{-\ell/4 + 1.5},$$

where we have used that $1/(1 - 2^{-\ell/4}) < 2^{1.5}$ for $\ell \geq 3$.

The further arguments work for $\Theta > 0$ and $\Theta < 0$ in a very similar way. Thus, to avoid unnecessary repetitions we shall consider only the case $\Theta > 0$. Then

$$0 < (m-1)\log\alpha - (n-2)\log 2 + \log g(\alpha, k) < (\sqrt[4]{2})^{-\ell}$$

and so

$$0 < (m-1)\gamma_k - (n-2) + \mu_k < 1.45 \cdot (\sqrt[4]{2})^{-\ell} \tag{4.1}$$

with $\gamma_k := \log\alpha^{(k)}/\log 2$ and $\mu_k := \log g(\alpha^{(k)}, k)/\log 2$. Here, we added the superscript to α to emphasize its dependence on k.

We claim that γ_k is irrational for any integer $k \geq 2$. In fact, if $\gamma_k = p/q$ for some positive integers p and q, then $2^p = (\alpha^{(k)})^q$ and as before we can conjugate this relation by some automorphism of the Galois group of the splitting field of $\psi_k(x)$ over $\mathbb{Q}$ to get $2^p < |(\alpha_i^{(k)})^q| < 1$ for $i > 1$, which is absurd, since $p \geq 1$. Let $q_{n,k}$ be the denominator of the nth convergent of the continued fraction of γ_k. Taking $M_k := 1.9 \cdot 10^{146} k^{24} \log^{27} k \leq M_{1640} < 7.1 \cdot 10^{246}$, we use *Mathematica* to get

$$\min_{2 \leq k \leq 1640} q_{650,k} > 6 \cdot 10^{308} > 6M_{1640}.$$

Also

$$\max_{2 \leq k \leq 1640} q_{650,k} < 2 \cdot 10^{1112}.$$

Defining $\epsilon_k := \|\mu_k q_{650,k}\| - M_k \|\gamma_k q_{650,k}\|$ for $2 \leq k \leq 1640$, we get

$$\min_{2 \leq k \leq 1640} \epsilon_k > 5.2 \cdot 10^{-169}.$$

Note that the conditions of Lemma 2.2 are fulfilled for $A = 1.45$ and $B = \sqrt[4]{2}$, and hence there is no solution to inequality (4.1) (and then no solution to the Diophantine equation (1.4)) for m and ℓ satisfying

$$m < M_k < 7.1 \cdot 10^{246} \quad \text{and} \quad \ell \geq \frac{\log(Aq_{650,k}/\epsilon_k)}{\log B}.$$

Since $m < M_k$ (for $2 \leq k \leq 1640$), we have

$$\ell < \frac{\log(Aq_{650,k}/\epsilon_k)}{\log B} \leq \frac{\log(1.45 \cdot 2 \cdot 10^{1112}/(5.2 \cdot 10^{-169}))}{\log \sqrt[4]{2}} < 17014.18\ldots.$$

Therefore, $2 \leq k \leq 1640$ and $k < \ell \leq 17014$. Now, by applying (3.6), we obtain $n < m < 6.5 \cdot 10^{51}$.

To deal with these remaining cases, we prepared a Mathematica routine that returns $\{56, \{2, 4\}\}$, which corresponds to the only solution $(m, n, \ell, k) = (10, 8, 4, 2)$. The calculations took roughly 8 days on 2.5 GHz Intel Core i5 4GB Mac OSX. The proof is complete. ∎

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