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A NOTE ON QUASITILTED ALGEBRAS

BY

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Abstract. We provide a characterization of artin algebras without chains of nonzero homomorphisms between indecomposable finitely generated modules starting with an injective module and ending with a projective module.

Throughout this note, by an algebra we mean an artin algebra over a fixed commutative artin ring K, which we shall assume (without loss of generality) to be basic and indecomposable. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, and by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, E)$, where E is a minimal injective cogenerator in $\operatorname{mod} K$. Moreover, we denote by τ_A and τ_A^{-1} the Auslander–Reiten operators $D\operatorname{Tr}$ and $\operatorname{Tr} D$ on $\operatorname{mod} A$, respectively.

A prominent role in the representation theory of algebras is played by quasitilted algebras introduced by Happel, Reiten and Smalø [4]. Recall that the quasitilted algebras are algebras of the form $\operatorname{End}_{\mathcal{H}}(T)$, where T is a tilting object in a hereditary Ext-finite abelian K-category $\mathcal{H}$ (see [4, Chapter II]). It has been proved in [4, Theorem 2.3] that an algebra A is a quasitilted algebra if and only if the global dimension of A is at most two and, for any indecomposable module X in $\operatorname{mod} A$, either the projective dimension $\operatorname{pd}_A X$ or the injective dimension $\operatorname{id}_A X$ is at most one. By a result of Happel and Reiten [3] (see also [18] for the tame case) the class of quasitilted algebras consists of the tilted algebras [5] (endomorphism algebras of tilting modules over hereditary algebras) and the quasitilted algebras of canonical type [9] (endomorphism algebras of tilting objects in hereditary abelian categories whose derived categories of bounded complexes coincide with the derived categories of bounded complexes over canonical algebras in the sense of Ringel [15, 16]). The quasitilted algebras have been applied to determine the structure of algebras whose module category admits a separating family of Auslander–Reiten components (see [8]–[13], [15], [16]) as well as algebras whose Auslander–Reiten quiver consists of semiregular components

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(see [2], [7]). Moreover, in the representation theory of selfinjective algebras an important role is played by the selfinjective algebras of quasitilted type, that is, the orbit algebras $\hat{B}/G$, where $\hat{B}$ is the repetitive category of a quasitilted algebra B and G is an admissible infinite cyclic group of automorphisms of $\hat{B}$ (see [21]). For example, every nonsimple selfinjective algebra of polynomial growth over an algebraically closed field is a socle deformation of a selfinjective algebra $\hat{B}/G$ given by a quasitilted algebra B with positive semidefinite Euler form (see [17], [20]).

We refer also to [19, 23] for characterizations of algebras with almost all indecomposable modules of projective or injective dimension at most one, invoking quasitilted algebras.

Recently, it has been proved by Jaworska, Malicki and Skowroński [6] that an artin algebra A is a tilted algebra if and only if there exists a sincere module M in $\operatorname{mod} A$ such that, for any indecomposable module X in $\operatorname{mod} A$, we have $\operatorname{Hom}_A(X, M) = 0$ or $\operatorname{Hom}_A(M, \tau_A X) = 0$, providing an affirmative answer to the question raised over twenty years ago in [14].

The aim of this note is to prove the following theorem.

THEOREM. *Let A be an algebra. The following statements are equivalent:*

(i) *For every indecomposable module X in* $\operatorname{mod} A$, *we have*
$$\operatorname{Hom}_A(X, A) = 0 \quad \text{or} \quad \operatorname{Hom}_A(D(A), X) = 0.$$
(ii) *A is a quasitilted algebra of canonical type or a tilted algebra of the form $\operatorname{End}_H(T)$, for a hereditary algebra H and a tilting module T in $\operatorname{mod} H$ such that, for every indecomposable projective direct summand P of T, T has no indecomposable direct summand which is a factor of the indecomposable injective module I with $\operatorname{top} P = \operatorname{soc} I$.*

(iii) *There is no chain $X_0 \to X_1 \to \cdots \to X_n$ of nonzero homomorphisms in $\operatorname{ind} A$ with X_0 an injective module and X_n a projective module.*

Proof. The implication (iii)⇒(i) is obvious.

We will show now that (ii) implies (iii). Assume to the contrary that there is a chain
$$X_0 \to X_1 \to \cdots \to X_n$$
of nonzero homomorphisms in $\operatorname{ind} A$ with X_0 injective and X_n projective. Then A is a tilted algebra. Indeed, if A is a quasitilted algebra of canonical type, then it follows from [9, Theorem 3.4] that Γ_A admits a separating family of semiregular (ray or coray) tubes, and consequently $\operatorname{ind} A$ does not contain a chain of nonzero homomorphisms starting from an injective module and ending in a projective module. Assume A is a tilted algebra of the form $\operatorname{End}_H(T)$, where H is a hereditary algebra and T is a tilting module in $\operatorname{mod} H$ such that, for every indecomposable projective direct summand P of T, T has

no indecomposable direct summand which is a factor of an indecomposable injective module I with $\operatorname{soc} I = \operatorname{top} P$. Since the projective module X_n belongs to the torsion-free class $\mathcal{Y}(T) = \{Y \in \operatorname{mod} A \mid \operatorname{Tor}_1^A(Y,T) = 0\}$ of $\operatorname{mod} A$ induced by the left A-module T and T is a splitting tilting module (see [1, Corollary VI.5.7]), we conclude that all modules $X_0, X_1, \dots, X_n$ belong to $\mathcal{Y}(T)$. Further, by the tilting theorem of Brenner and Butler (see [1, Theorem VI.3.8]), the functor $\operatorname{Hom}_H(T,-) : \operatorname{mod} H \to \operatorname{mod} A$ induces an equivalence $\mathcal{T}(T) \to \mathcal{Y}(T)$, where $\mathcal{T}(T) = \{M \in \operatorname{mod} H \mid \operatorname{Ext}_H^1(T,M) = 0\}$ is the torsion class of $\operatorname{mod} H$ induced by T. In particular, there exist indecomposable modules $M_0, M_1, \dots, M_n$ in $\mathcal{T}(T)$ such that $X_i = \operatorname{Hom}_H(T, M_i)$ for any $i \in \{0, 1, \dots, n\}$. Observe that M_n is a direct summand of T, because $X_n = \operatorname{Hom}_H(T, M_n)$ is projective in $\operatorname{mod} A$. Moreover, since $X_0 = \operatorname{Hom}_H(T, M_0)$ is injective in $\operatorname{mod} A$ and belongs to $\mathcal{Y}(T)$, we conclude from [1, Theorem VI.5.8] that M_0 is an injective module in $\operatorname{mod} H$ such that $\operatorname{soc} M_0 = \operatorname{top} P_0$, for an indecomposable projective direct summand P_0 of T. Invoking the equivalence $\operatorname{Hom}_H(T,-) : \mathcal{T}(T) \to \mathcal{Y}(T)$ again, we deduce that there is a chain of nonzero homomorphisms

$$M_0 \to M_1 \to \cdots \to M_n$$

in $\operatorname{ind} H$. Since H is a hereditary algebra, $M_0, M_1, \dots, M_n$ are indecomposable H-modules and M_0 is injective, we conclude that the above chain consists of epimorphisms between indecomposable injective modules (see [22, Theorems I.9.2 and I.9.3]). Consequently, M_n is a quotient of M_0 which contradicts the assumption imposed on T. Therefore, (ii) implies (iii).

Finally, we prove that (i) implies (ii). Assume that (i) holds. We prove first that A is a quasitilted algebra. Consider an arbitrary almost split sequence

$$0 \to Y \xrightarrow{\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix}} \bigoplus_{i=1}^{r} E_i \xrightarrow{[g_1 \dots g_r]} X \to 0$$

in $\operatorname{mod} A$. We claim that $\operatorname{Hom}_A(D(A), Y) = 0$ or $\operatorname{Hom}_A(X, A) = 0$. Suppose to the contrary that $\operatorname{Hom}_A(D(A), Y) \neq 0$ and $\operatorname{Hom}_A(X, A) \neq 0$. Then there exists $i \in \{1, \dots, r\}$ such that $\operatorname{Hom}_A(D(A), E_i) \neq 0$, because

$$\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix} : Y \to \bigoplus_{i=1}^{r} E_i$$

is a monomorphism. Further, because $g_i : E_i \to X$ is an irreducible homomorphism, we deduce that g_i is either a proper monomorphism or a proper epimorphism. If g_i is a proper monomorphism, then $\operatorname{Hom}_A(D(A), X) \neq 0$, contradicting (i). Similarly, if g_i is a proper epimorphism, then $\mathrm{Hom}_A(E_i, A) \neq 0$, and we obtain a contradiction with (i) again. Therefore, we have proved that $\mathrm{Hom}_A(D(A), Y) = 0$ or $\mathrm{Hom}_A(X, A) = 0$, for every almost split sequence $0 \to Y \to E \to X \to 0$ in $\operatorname{mod} A$. Applying [1, Lemma IV.2.7], we conclude that, for every module X in $\operatorname{ind} A$, we have $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A \tau_A X \leq 1$.

Now, we prove that $\mathrm{gl.dim}\, A \leq 2$. Let S be a simple module in $\operatorname{mod} A$ and $\pi : P \to S$ be a projective cover of S in $\operatorname{mod} A$. Then P is an indecomposable projective module and $\operatorname{Ker} \pi = \operatorname{rad} P$. If $\operatorname{rad} P$ is a projective module, then $\mathrm{pd}_A S \leq 1$. Assume that $\operatorname{rad} P$ is not projective, and let X be an indecomposable nonprojective direct summand of $\operatorname{rad} P$. Then $\tau_A X \neq 0$ and $\mathrm{Hom}_A(\tau_A^{-1}(\tau_A X), A) = \mathrm{Hom}_A(X, A) \neq 0$. Applying [1, Lemma IV.2.7] again, we conclude that $\mathrm{id}_A \tau_A X \geq 2$. Therefore, we have $\mathrm{pd}_A X \leq 1$. Consequently, $\mathrm{pd}_A \operatorname{rad} P \leq 1$, and hence $\mathrm{pd}_A S \leq 2$. It follows that $\mathrm{gl.dim}\, A \leq 2$.

Moreover, observe that, if M is a module in $\operatorname{mod} A$ with $\mathrm{pd}_A M \leq 1$, then for every submodule N of M, we have $\mathrm{pd}_A N \leq 1$. Indeed, let $u : N \to M$ be a canonical inclusion in $\operatorname{mod} A$. Then, there is a short exact sequence in $\operatorname{mod} A$ of the form

$$0 \to N \xrightarrow{u} M \xrightarrow{p} V \to 0,$$

where $V = \operatorname{Coker} u$ and p is a canonical epimorphism. Since $\mathrm{gl.dim}\, A \leq 2$, for every module Z in $\operatorname{mod} A$, the above short exact sequence induces the following exact sequence in $\operatorname{mod} K$

$$\mathrm{Ext}_A^2(V, Z) \to \mathrm{Ext}_A^2(M, Z) \to \mathrm{Ext}_A^2(N, Z) \to 0.$$

But $\mathrm{pd}_A M \leq 1$ implies $\mathrm{Ext}_A^2(M, Z) = 0$, and consequently $\mathrm{Ext}_A^2(N, Z) = 0$. It follows that $\mathrm{Ext}_A^2(N, -) = 0$, and hence $\mathrm{pd}_A N \leq 1$.

Now, let X be an arbitrary indecomposable module in $\operatorname{mod} A$. We claim that $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$. Suppose that $\mathrm{id}_A X \geq 2$. Then we have $\mathrm{Hom}_A(\tau_A^{-1} X, A) \neq 0$, by [1, Lemma IV.2.7]. Moreover, by the above part of the proof, we have $\mathrm{pd}_A \tau_A^{-1} X \leq 1$. Since X is noninjective, there exists an almost split sequence in $\operatorname{mod} A$ of the form

$$0 \to X \xrightarrow{\begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix}} \bigoplus_{j=1}^{s} F_j \xrightarrow{[h_1 \ldots h_s]} \tau_A^{-1} X \to 0.$$

Fix $j \in \{1, \ldots, s\}$. We claim that $\mathrm{pd}_A F_j \leq 1$. We may assume that F_j is not projective. If h_j is a monomorphism, then F_j is isomorphic to a submodule of $\tau_A^{-1} X$, and hence $\mathrm{pd}_A \tau_A^{-1} X \leq 1$ implies $\mathrm{pd}_A F_j \leq 1$. Assume that h_j is an epimorphism. Then $\mathrm{Hom}_A(F_j, A) \neq 0$, because $\mathrm{Hom}_A(\tau_A^{-1} X, A) \neq 0$. Hence $\mathrm{Hom}_A(\tau_A^{-1}(\tau_A F_j), A) = \mathrm{Hom}_A(F_j, A) \neq 0$ implies $\mathrm{id}_A \tau_A F_j \geq 2$, again by [1, Lemma IV.2.7]. Then it follows from the above part of the proof that

$\mathrm{pd}_A F_j \leq 1$. Concluding, we obtain $\mathrm{pd}_A \bigoplus_{j=1}^{s} F_j \leq 1$. This implies that $\mathrm{pd}_A X \leq 1$, because X is isomorphic to a submodule of $\bigoplus_{j=1}^{s} F_j$.

Summing up, we have proved that $\mathrm{gl.dim}\, A \leq 2$ and, for every indecomposable module X in $\mathrm{mod}\, A$, we have $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$, or equivalently, A is a quasitilted algebra. It follows from [3, Theorem 3.5] that A is a tilted algebra or a quasitilted algebra of canonical type. Assume that A is a tilted algebra. Then $A = \mathrm{End}_H(T)$ for a hereditary algebra H and a tilting module T in $\mathrm{mod}\, H$. Suppose that there exist indecomposable direct summands P and T_0 of T such that P is projective and T_0 is a factor of an indecomposable injective module I with $\mathrm{top}\, P = \mathrm{soc}\, I$. Applying [1, Proposition VI.5.8], we conclude that $\mathrm{Hom}_H(T, I)$ is an indecomposable injective module in $\mathrm{mod}\, A$. Moreover, an epimorphism $h : I \to T_0$ induces a nonzero homomorphism $\mathrm{Hom}_H(T, h) : \mathrm{Hom}_H(T, I) \to \mathrm{Hom}_H(T, T_0)$, because the functor $\mathrm{Hom}_H(T, -)$ induces an equivalence of categories $\mathcal{T}(T) \to \mathcal{Y}(T)$, and both I and T_0 belong to $\mathcal{T}(T)$. Hence we have a nonzero homomorphism in $\mathrm{mod}\, A$ from the indecomposable injective module $\mathrm{Hom}_H(T, I)$ to the indecomposable projective module $\mathrm{Hom}_H(T, T_0)$, contradicting (i). Therefore, indeed (i) implies (ii). ∎

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