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WHEN DOES AN $AB5^*$ MODULE HAVE FINITE HOLLOW DIMENSION?

BY

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Abstract. Using a lattice-theoretical approach we find characterizations of modules with finite uniform dimension and of modules with finite hollow dimension.

1. Introduction. It is well known that Goldie introduced finite-dimensional modules in [2]. Then the concept of the Goldie dimension of a module was dualized by Varadarajan in [10] and termed dual Goldie dimension. The concepts of the Goldie dimension and dual Goldie dimension of modules can be extended to modular lattices as in [4]. We also note that Goldie dimensions of balanced lattices were studied in [13]. In this paper we give some characterizations of lattices with finite Goldie dimension and, by passing to the opposite lattice, some characterizations of lattices with finite dual Goldie dimension. At the end, we give some applications in module theory. Note that we prefer the terms uniform dimension and hollow dimension instead of the terms Goldie dimension and dual Goldie dimension, respectively. This paper was motivated by and is written in the spirit of [6].

In what follows, R is an associative ring with unit and all modules are unitary right R-modules.

2. Lattices. Let L be a complete modular lattice with least element 0 and greatest element 1. If $a \leq b$ are elements of a lattice L then b/a will denote the set of elements $x \in L$ such that $a \leq x \leq b$. An element e of L is called *essential* if $e \wedge a \neq 0$ for all $0 \neq a \in L$. In particular, 1 is an essential element of L. The set of essential elements of L will be denoted by $E(L)$. Let $e \leq b$ in L. Then it is easy to see that $e \in E(L)$ if and only if $e \in E(b/0)$ and $b \in E(L)$.

An element s of L is called *small* if s is an essential element of the opposite lattice L^o. In other words, s is a small element of L if and only if

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$1 \neq s \vee b$ for all $1 \neq b \in L$. Clearly, 0 is a small element of L. The set of small elements of L will be denoted by $S(L)$. Note that $S(L) = E(L^o)$ and $E(L) = S(L^o)$. Note also the following simple fact about modular lattices.

LEMMA 2.1. *Let a, b and c be elements of a lattice L such that $a \wedge b = 0$ and $(a \vee b) \wedge c = 0$. Then $a \wedge (b \vee c) = 0$.*

Let $\mathbb{N}$ denote the set of natural numbers $1, 2, \ldots$. Given $n \in \mathbb{N}$, a subset $S = \{x_i : 1 \leq i \leq n\}$ of L is called *independent* if $x_i \neq 0$ $(1 \leq i \leq n)$ and

$$x_j \wedge (x_1 \vee \cdots \vee x_{j-1} \vee x_{j+1} \vee \cdots \vee x_n) = 0$$

for all $1 \leq j \leq n$. An arbitrary non-empty subset T of L is called *independent* provided every finite non-empty subset of T is independent. Note the following simple fact.

LEMMA 2.2. *Let S be an independent set in a lattice L and let x be a non-zero element of L such that $x \wedge (\bigvee F) = 0$ for every finite non-empty subset F of S. Then the set $S \cup \{x\}$ is independent.*

Proof. By Lemma 2.1. ∎

Let n be a positive integer and $\{b_1, \ldots, b_n\}$ an independent set of L. Let $a_i \in E(b_i/0)$ $(1 \leq i \leq n)$. Then $a_1 \vee \cdots \vee a_n \in E((b_1 \vee \cdots \vee b_n)/0)$ (see, for example, [8, Proposition 2.6] or [4, Lemma 3]).

The lattice L is called *noetherian* (respectively, *artinian*) provided for any given chain $a_1 \leq a_2 \leq \cdots$ (respectively, $a_1 \geq a_2 \geq \cdots$) of elements of L there exists a positive integer n such that $a_n = a_{n+1} = \cdots$. The lattice L is said to satisfy the *maximal condition* (respectively, *minimal condition*) provided every non-empty subset of L contains a maximal (respectively, minimal) member. Recall that $c \in L$ is a *maximal member* (respectively, *minimal member*) of a non-empty subset S of L in case whenever $c \leq x$ (respectively, $c \geq x$) for some $x \in S$ then $c = x$. It is easy to prove that L is noetherian (respectively, artinian) if and only if L satisfies the maximal (respectively, minimal) condition. It is clear that L is noetherian (respectively, artinian) if and only if the opposite lattice L^o is artinian (respectively, noetherian).

By a *direct set* in the lattice L we mean a non-empty subset S of L such that whenever $a \in S$ and $b \in S$ then there exists $c \in S$ with $a \vee b \leq c$. By an *inverse set* in L we mean a direct set in L^o. Thus T is an inverse set in L if and only if T is non-empty and given $u, v \in T$ there exists $w \in T$ with $w \leq u \wedge v$. We shall call an element f in L *finitely generated* if whenever $f = \bigvee S$ for some direct set S in L, then there exists $x \in S$ such that $f = x$. Note that 0 is always a finitely generated element of L. We shall call an element f of L *strongly finitely generated* provided $f \leq \bigvee S$, for a direct set S in L, implies that $f \leq x$ for some $x \in S$.

Note that if U is any non-empty subset of L then the collection of elements of L of the form $\bigvee F$, where F runs through the finite non-empty subsets of U, is a direct set in L; we shall denote this set by $P(U)$. Note that $\bigvee U = \bigvee P(U)$. Thus an element f is (strongly) finitely generated if and only if for every non-empty set U with $f = \bigvee U$ ($f \leq \bigvee U$) there exists a finite subset F of U such that $f = \bigvee F$ ($f \leq \bigvee F$). An element f of L will be called an *sfg-element* if whenever $f \leq \bigvee P(U)$ for some independent set U in L then $f \leq z$ for some $z \in P(U)$, that is, $f \leq u_1 \vee \cdots \vee u_n$ for some positive integer n and elements $u_i \in U$ $(1 \leq i \leq n)$.

The lattice L is called *upper continuous* if

$$a \wedge \Big(\bigvee_{i\in I} x_i\Big) = \bigvee_{i\in I}(a \wedge x_i)$$

for every element a and direct set $\{x_i : i \in I\}$ in L. On the other hand, L is called *lower continuous* if

$$a \vee \Big(\bigwedge_{i\in I} y_i\Big) = \bigwedge_{i\in I}(a \vee y_i)$$

for every element a and inverse set $\{y_i : i \in I\}$ in L. Note that L is upper continuous if and only if L^o is lower continuous. For more information about lattice theory we refer the reader to [3], [7] and [8].

Clearly, every strongly finitely generated element of a general lattice L is finitely generated. The converse is true if L is upper continuous but we are not sure if it is true more generally.

LEMMA 2.3. *Let L be an upper continuous (complete modular) lattice. Then every finitely generated element of L is strongly finitely generated.*

Proof. Let a be any finitely generated element of L. Next let $\{x_i : i \in I\}$ be any direct set in L such that $a \leq \bigvee_{i\in I} x_i$. Then

$$a = a \wedge \Big(\bigvee_{i\in I} x_i\Big) = \bigvee_{i\in I}(a \wedge x_i).$$

Because a is finitely generated, there exists $j \in I$ such that $a = a \wedge x_j$ and hence $a \leq x_j$. It follows that a is strongly finitely generated. ∎

Another situation where finitely generated elements are strongly finitely generated is given in the next result.

LEMMA 2.4. *The following statements are equivalent for a lattice L:*

(i) *L is noetherian.*
(ii) *Every element of L is strongly finitely generated.*
(iii) *Every element of L is finitely generated.*

Proof. (i)⇒(ii). Let $a \in L$. Next let $X = \{x_i : i \in I\}$ be any direct set in L such that $a \leq \bigvee_{i\in I} x_i$. Because L satisfies the maximal condition, X contains a maximal member x_j for some $j \in I$. Let $i \in I$. There exists $k \in I$

such that $x_j \le x_i \vee x_j \le x_k$. By the choice of j, we have $x_j = x_k$ and hence $x_i \le x_j$. Thus $x_i \le x_j$ for all $i \in I$ and

$$a \le \bigvee_{i \in I} x_i = x_j.$$

It follows that a is strongly finitely generated.

(ii)$\Rightarrow$(iii). Clear.

(iii)$\Rightarrow$(i). Let $b_1 \le b_2 \le \cdots$ be a chain of elements in L. (Such a chain of elements is called *ascending.*) Let $b = \bigvee_{i \in \mathbb{N}} b_i$. Because b is finitely generated, there exists $n \in \mathbb{N}$ such that $b = b_n$ and hence $b_n = b_{n+1} = \cdots$. It follows that L is noetherian. ■

There are weaker forms of the upper continuous condition that a lattice L can satisfy and we consider these next. We shall denote these conditions by (UC1), (UC2) and (UC3) and define them as follows:

(UC1) For each element $a \in L$ and direct set $\{x_i : i \in I\}$, $a \wedge x_i = 0$ $(i \in I)$ implies that $a \wedge (\bigvee_{i \in I} x_i) = 0$.
(UC2) $a \wedge (\bigvee S) = 0$ for every element a in L and every independent set S in L such that $a \wedge (\bigvee F) = 0$ for every finite subset F of S.
(UC3) For each element $a \in L$ there exists an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in E(L)$.

LEMMA 2.5. *Consider the following conditions on a lattice L.*

(i) *L is upper continuous.*
(ii) *L satisfies* (UC1).
(iii) *L satisfies* (UC2).
(iv) *L satisfies* (UC3).

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv).

Proof. (i)$\Rightarrow$(ii). Let $a \in L$ and let $\{x_i : i \in I\}$ be a direct set in L such that $a \wedge x_i = 0$ $(i \in I)$. Because L is upper continuous, we have

$$a \wedge \Big(\bigvee_{i \in I} x_i\Big) = \bigvee_{i \in I} (a \wedge x_i) = 0.$$

Thus L satisfies (UC1).

(ii)$\Rightarrow$(iii). Suppose that L satisfies (UC1). Let $a \in L$ and let S be an independent set in L such that $a \wedge (\bigvee F) = 0$ for every finite subset F of S. As we remarked above, the set $P(S)$ is a direct set in L. Moreover, $a \wedge p = 0$ for all $p \in P(S)$. Because L satisfies (UC1), $a \wedge (\bigvee P(S)) = 0$ and it follows that $a \wedge (\bigvee S) = 0$. Thus L satisfies (UC2).

(iii)$\Rightarrow$(iv). Let $a \in L$. If $a \in E(L)$ then $a \wedge 0 = 0$ and $a \vee 0 \in E(L)$. Now suppose that $a \notin E(L)$. There exists a non-zero element u in L with $a \wedge u = 0$. Let $\mathcal{S}$ denote the collection of all independent sets S in L such that $a \wedge (\bigvee S) = 0$. Note that $\{u\} \in \mathcal{S}$. Let T_λ $(\lambda \in \Lambda)$ be any chain in $\mathcal{S}$ and let $T = \bigcup_{\lambda\in\Lambda} T_\lambda$. For any finite subset F of T there exists $\lambda \in \Lambda$ such that $F \subseteq T_\lambda$. It follows that T is an independent set in L and, moreover, $a \wedge (\bigvee F) = 0$ for every finite subset F of T. By (iii), $a \wedge (\bigvee T) = 0$. It follows that $T \in \mathcal{S}$. By Zorn's lemma, $\mathcal{S}$ contains a maximal member U. Note that U is an independent set in L such that $a \wedge (\bigvee U) = 0$. Let $b = \bigvee U$. Let $c \in L$. Suppose that $(a \vee b) \wedge c = 0$. Note that $b \wedge c = 0$ implies that the set $W = U \cup \{c\}$ is independent (Lemma 2.2). Note also that $\bigvee W = b \vee c$. Next, Lemma 2.1 gives $a \wedge (\bigvee W) = a \wedge (b \vee c) = 0$. This means that $W \in \mathcal{S}$. By the choice of U, $W = U$ and hence $c \in U$. In this case, $c \leq (a \vee b) \wedge c = 0$. Thus $c = 0$. It follows that $a \wedge b = 0$ and $a \vee b \in E(L)$. Thus L satisfies (UC3). ∎

3. Uniform dimension of lattices. Let L be a (complete modular) lattice with least element 0 and greatest element 1. The lattice L has *finite uniform dimension* provided L does not contain an infinite independent set (of non-zero elements). On the other hand, the lattice L will be said to have *finite hollow dimension* if the opposite lattice L^o has finite uniform dimension. The theory of lattices with finite uniform dimension is well established and, by taking opposite lattices, there is a corresponding theory of lattices with finite hollow dimension. The lattice L is called *uniform* provided $L \neq \{0\}$ and $a \wedge b \neq 0$ for all non-zero elements a and b in L. On the other hand, L is *hollow* if $L \neq \{1\}$ and $1 \neq a \vee b$ for all elements a and b of L with $a \neq 1$ and $b \neq 1$. Thus L is uniform if and only if L^o is hollow. An element $u \in L$ is called *uniform* if the sublattice $u/0$ is uniform. Note in particular that uniform elements are non-zero. Next an element $h \in L$ is called *hollow* provided $1/h$ is a hollow lattice. Clearly z is a uniform element of L if and only if z is a hollow element of L^o. Note the following fundamental result (see, for example, [4, Theorem 5]).

Lemma 3.1. *A non-zero lattice L has finite uniform dimension if and only if there exists a positive integer n and an independent set of uniform elements u_i $(1 \leq i \leq n)$ such that $u_1 \vee \cdots \vee u_n \in E(L)$. Moreover in this case, the following statements are true:*

(i) *If m is a positive integer and $\{w_i : 1 \leq i \leq m\}$ is an independent set of uniform elements of L such that $\bigvee_{1\leq i\leq m} w_i \in E(L)$, then $m = n$.*
(ii) *Every independent set in L has at most n members.*

The positive integer n in Lemma 3.1 is called the *uniform dimension* of L and will be denoted by $u(L)$. In case L is a zero lattice we shall say that L has *uniform dimension* 0 and write $u(L) = 0$.

Corollary 3.2. *Let e be an essential element of a lattice L such that the sublattice $e/0$ has finite uniform dimension. Then L has finite uniform dimension and $u(L) = u(e/0)$. Moreover, $u(a/0) \leq u(L)$ for all $a \in L$.*

Proof. By Lemma 3.1. ■

LEMMA 3.3. *Consider the following statements for a lattice L.*

(i) *L is noetherian.*
(ii) *L has finite uniform dimension.*
(iii) *L satisfies* (UC2) *and* (UC3).

Then (i)⇒(ii)⇒(iii).

Proof. (i)⇒(ii). Suppose that L is noetherian but that L does not have finite uniform dimension. Then there exists an infinite independent set of (non-zero) elements x_n ($n \in \mathbb{N}$). Consider the ascending chain $x_1 \leq x_1 \vee x_2 \leq \cdots$ in L. Because L is noetherian, there exists a positive integer k such that

$$x_1 \vee \cdots \vee x_k = x_1 \vee \cdots \vee x_k \vee x_{k+1}.$$

This implies that $x_{k+1} \leq x_{k+1} \wedge (x_1 \vee \cdots \vee x_k) = 0$ and hence $x_{k+1} = 0$, a contradiction. This shows that (i)⇒(ii).

(ii)⇒(iii). Clearly L satisfies (UC2) because every independent set is finite. By Lemma 2.5, L satisfies (UC3). ■

If R is a ring and M a unital right R-module then $L(M_R)$ will denote the complete modular lattice of submodules of M. Note that if $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ the rational field then $L(\mathbb{Q}_{\mathbb{Z}})$ has finite uniform dimension 1 but is not noetherian. Moreover, if V is an infinite-dimensional vector space over $\mathbb{Q}$ then $L(V_{\mathbb{Q}})$ satisfies (UC2) (and hence also (UC3)) but does not have finite uniform dimension. However, some lattices which satisfy (UC3) do have finite uniform dimension. We shall be interested in the following conditions on essential elements of a lattice L:

(A) For each $e \in E(L)$ there exists $f \in E(L)$ such that $f \leq e$ and f is strongly finitely generated.
(B) For each $e \in E(L)$ there exists $f \in E(L)$ such that $f \leq e$ and f is finitely generated.
(C) For each $e \in E(L)$ there exists $f \in E(L)$ such that $f \leq e$ and f is an sfg-element.

THEOREM 3.4. *A lattice L has finite uniform dimension if and only if L satisfies* (UC3) *and* (C).

Proof. Suppose first that L has finite uniform dimension. Then L satisfies (UC3) by Lemma 3.3 and L satisfies (C) because every independent set in L is finite. Conversely, suppose that L satisfies (UC3) and (C). Suppose that L does not have finite uniform dimension. Then L contains an infinite independent set S of (non-zero) elements x_n ($n \in \mathbb{N}$). Let $x = \bigvee S$. The condition (UC3) for L gives an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in E(L)$. Suppose that $y \neq 0$. By Lemma 2.2, the set $S \cup \{y\}$ is independent. By hypothesis, there exists $f \in E(L)$ such that f is an *sfg*-element of L and $f \leq x \vee y$. Then $f \leq \bigvee_{n\in\mathbb{N}} (x_n \vee y)$ and hence $f \leq x_1 \vee \cdots \vee x_m \vee y$ for some $m \in \mathbb{N}$. Now $\{x_1, \ldots, x_{m+1}, y\}$ is an independent set because $(x_1 \vee \cdots \vee x_{m+1}) \wedge y \leq x \wedge y = 0$ (Lemma 2.2). Therefore

$$f \wedge x_{m+1} \leq (x_1 \vee \cdots \vee x_m \vee y) \wedge x_{m+1} = 0.$$

But this implies that $f \wedge x_{m+1} = 0$ and hence, because $f \in E(L)$, we have $x_{m+1} = 0$, a contradiction. Thus L has finite uniform dimension. Now suppose that $y = 0$. By adapting the above proof we again obtain a contradiction. ■

Corollary 3.5.

(i) *Every lattice which satisfies* (UC3) *and* (A) *has finite uniform dimension.*
(ii) *Every upper continuous lattice which satisfies* (B) *has finite uniform dimension.*

Proof. (i) By Theorem 3.4.
(ii) By Lemmas 2.3 and 2.5 and Theorem 3.4. ■

The converse of Corollary 3.5(i) holds for certain lattices. First we prove a preparatory result.

Lemma 3.6. *Let a and b be strongly finitely generated elements of a lattice L. Then the element $a \vee b$ is also strongly finitely generated.*

Proof. Suppose that $a \vee b \leq \bigvee U$ for some non-empty set U in L. Then $a \leq \bigvee U$ implies that $a \leq \bigvee F$ for some finite subset F of U. Similarly $b \leq \bigvee G$ for some finite subset G of U. It follows that $a \vee b \leq \bigvee (F \cup G)$. Thus $a \vee b$ is strongly finitely generated. ■

Theorem 3.7. *Let L be a lattice such that for each uniform element u there exists a non-zero strongly finitely generated element s with $s \leq u$. Then L has finite uniform dimension if and only if L satisfies* (UC3) *and* (A).

Proof. The sufficiency follows by Corollary 3.5(i). Conversely, suppose that L has finite uniform dimension. By Lemma 3.3, L satisfies (UC3). Let $e \in E(L)$. By Lemma 3.1, there exist a positive integer n and an independent set of uniform elements u_i $(1 \leq i \leq n)$ of L such that $u_1 \vee \cdots \vee u_n \in E(L)$ and $u_1 \vee \cdots \vee u_n \leq e$. By hypothesis, for each $1 \leq i \leq n$ there exists a non-zero strongly finitely generated element a_i of L with $a_i \leq u_i$. Let $a = a_1 \vee \cdots \vee a_n$. By Lemma 3.6, a is a strongly finitely generated element of L. Since each u_i is uniform, for each $i \in \{1, \ldots, n\}$, $a_i \in E(u_i/0)$. Then $a \in E((u_1 \vee \cdots \vee u_n)/0)$. Thus $a \in E(L)$. Clearly $a \leq e$. It follows that L satisfies (A). ■

Corollary 3.8. *Let L be an upper continuous lattice such that for each uniform element u there exists a non-zero finitely generated element s with $s \leq u$. Then L has finite uniform dimension if and only if L satisfies* (B).

Proof. By Lemmas 2.3 and 2.5 and Theorem 3.7. ∎

4. Further conditions. Let L be any lattice, again with least element 0 and greatest element 1. Note that $E(L)$ is the set of elements $e \in L$ such that every element a in L with $e \wedge a = 0$ satisfies $u(a/0) = 0$, because of course $a = 0$ in this case. For every non-negative integer n, we set $E_n(L)$ to be the set of elements $e \in L$ such that every element $a \in L$ with $e \wedge a = 0$ satisfies $u(a/0) \leq n$. Note that $E(L) = E_0(L)$ and that

$$E(L) = E_0(L) \subseteq E_1(L) \subseteq E_2(L) \subseteq \cdots .$$

Next, $E_\infty(L)$ will denote the collection of elements $e \in L$ such that whenever $a \in L$ with $e \wedge a = 0$ then $a/0$ has finite uniform dimension.

Lemma 4.1. *Let L be a lattice. Then:*

(i) *Given a non-negative integer n, $L = E_n(L)$ if and only if L has finite uniform dimension at most n.*
(ii) *$L = E_\infty(L)$ if and only if L has finite uniform dimension.*

Proof. (i) First suppose that $L = E_n(L)$. Then $0 \in E_n(L)$. If $e \in E(L)$ then $e \wedge 0 = 0$ and hence $u(e/0) \leq n$. By Corollary 3.2, the lattice L has uniform dimension at most n. Conversely, suppose that $u(L) \leq n$. Let $x \in L$. For any $y \in L$ with $x \wedge y = 0$, we have $u(y/0) \leq u(L) \leq n$ by Corollary 3.2. It follows that $x \in E_n(L)$. Thus $L = E_n(L)$.

(ii) Similar to (i). ∎

For every non-negative integer n, it is clear that $a \in E_n(L)$ and $a \leq b \in L$ together imply $b \in E_n(L)$. It follows that if n is a non-negative integer and $c \in L \setminus E_n(L)$ then $d \in L \setminus E_n(L)$ for every $d \in L$ with $c \geq d$. There are similar facts for the sets $E_\infty(L)$ and $L \setminus E_\infty(L)$. In this section we shall be interested in the following conditions where n is any non-negative integer:

(E_n) a is a finitely generated element of L for each $a \in L \setminus E_n(L)$.

In addition we shall be interested in the following condition on a lattice L:

(E_∞) a is a finitely generated element of L for each $a \in L \setminus E_\infty(L)$.

Lemma 4.2. *Let n be any non-negative integer or ∞. Then a lattice L satisfies* (E_n) *if and only if $a/0$ is noetherian for every $a \in L \setminus E_n(L)$.*

Proof. Suppose first that L satisfies (E_n). Let $a \in L \setminus E_n(L)$. For any $b \leq a$, the above remarks show that $b \in L \setminus E_n(L)$ and hence b is finitely generated. Thus b is finitely generated for all $b \in a/0$. By Lemma 2.4, $a/0$ is noetherian.

Conversely, if $a/0$ is noetherian for each $a \in L \setminus E_n(L)$ then $a \in a/0$ implies that a is finitely generated for each $a \in L \setminus E_n(L)$, by Lemma 2.4. Thus L satisfies (E_n). ■

THEOREM 4.3. *The following statements are equivalent for a lattice L:*

(i) *L has finite uniform dimension.*
(ii) *L satisfies* (UC3) *and* (E_m) *for some non-negative integer m.*
(iii) *L satisfies* (UC2) *and* (E_∞).

Proof. (i)⇒(ii). Suppose that $u(L) = n$ for some positive integer n. (If $L = \{0\}$ then there is nothing to prove.) By Lemma 3.3, L satisfies (UC3). Moreover, by Lemma 4.1, $L = E_n(L)$ and so L satisfies (E_n) vacuously.

(ii)⇒(i). Suppose that $L \neq \{0\}$. Suppose first that $m = 0$. If $a \wedge b \neq 0$ for all non-zero elements a, b in L then L is uniform and hence $u(L) = 1$. Suppose that $u \wedge w = 0$ for some non-zero elements $u, w \in L$. Because L satisfies (UC3), there exists $z \in L$ such that $u \wedge z = 0$ and $u \vee z \in E(L)$. Clearly $z \neq 0$. Note that neither u nor z belongs to $E(L)$. By hypothesis and Lemma 4.2, both $u/0$ and $z/0$ are noetherian sublattices of L. By Lemma 3.3, both $u/0$ and $z/0$ have finite uniform dimension. But $u \wedge z = 0$. By Lemmas 2.1, 2.2 and 3.1, $(u \vee z)/0$ has finite uniform dimension. It follows that L has finite uniform dimension because $u \vee z \in E(L)$ (Corollary 3.2).

Now suppose that $m \geq 1$. Suppose that L does not have finite uniform dimension. Then there exists an independent set of (non-zero) elements x_i $(1 \leq i \leq m+2)$. Let $x = x_1 \vee \cdots \vee x_{m+1}$. Note that $x \wedge x_{m+2} = 0$ so that $x \notin E(L)$. There exists an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in E(L)$. Note that $y \neq 0$. By Lemma 2.2, the set $\{x_1, \ldots, x_{m+1}, y\}$ is independent. Therefore, for each $1 \leq i \leq m+1$, we have

$$x_i \wedge (x_1 \vee \cdots \vee x_{i-1} \vee x_{i+1} \vee \cdots \vee x_{m+1} \vee y) = 0.$$

By Lemma 3.1, $x_i \notin E_m(L)$ and hence, by Lemma 4.2, $x_i/0$ is noetherian for all $1 \leq i \leq m+1$. Similarly $y/0$ is noetherian. But Lemma 3.3 then gives that each of the lattices $x_i/0$ $(1 \leq i \leq m+1)$ and $y/0$ has finite uniform dimension. Since the set $\{x_1, \ldots, x_{m+1}, y\}$ is independent it follows that the sublattice $(x \vee y)/0$ has finite uniform dimension. Because $x \vee y \in E(L)$, the lattice L has finite uniform dimension, a contradiction. Thus L has finite uniform dimension.

(i)⇒(iii). Suppose that L has finite uniform dimension. Then every independent set of non-zero elements of L is finite and thus L clearly satisfies (UC2). Moreover, by Lemma 4.1, L satisfies (E_∞).

(iii)⇒(i). Suppose that L satisfies (UC2) and (E_∞). Suppose that L does not have finite uniform dimension. Then L contains an independent set S of (non-zero) elements x_n $(n \in \mathbb{N})$. Let I and J be disjoint infinite subsets of $\mathbb{N}$. Let $T = \{x_n : n \in I\}$ and $U = \{x_n : n \in J\}$. Let F be any finite subset of T. Then $(\bigvee F)\wedge(\bigvee G)=0$ for every finite non-empty subset G of U. Because L satisfies (UC2), $(\bigvee F)\wedge u=0$, where $u=\bigvee U$. Now, since $(\bigvee F)\wedge u=0$ for every finite non-empty subset F of T, we see (using (UC2) in L) that $t\wedge u=0$ where $t=\bigvee T$. Next note that L satisfies (UC3) by Lemma 2.5. Thus there exists an element w in L such that $(t\vee u)\wedge w=0$ and $t\vee u\vee w\in E(L)$. Note that $t\wedge(u\vee w)=0$ by Lemma 2.1. Clearly, $t/0$ does not have finite uniform dimension so that $u\vee w\notin E_\infty(L)$. By Lemma 4.2, $(u\vee w)/0$ is noetherian. Similarly the sublattice $t/0$ is noetherian. By the proof (ii)⇒(i), L has finite uniform dimension, a contradiction. ∎

5. Application to modules. Let R be a ring with identity and let M be a unital right R-module. We can apply the above results on lattices to the lattice $L(M)$ of submodules of M and to its opposite lattice $L(M)^o$. The application to $L(M)$ is straightforward. Note that $L(M)$ is an upper continuous complete modular lattice. Corollary 3.8 gives the following well known result.

THEOREM 5.1. *A module M has finite uniform dimension if and only if every essential submodule of M contains a finitely generated essential submodule of M.*

On the other hand, Lemma 2.5 and Theorem 4.3 give the following result:

THEOREM 5.2. *Let M be a module and let $\mathcal{S}$ denote the collection of submodules N of M such that every submodule L of M with $N\cap L=0$ has finite uniform dimension. Then M has finite uniform dimension if and only if every submodule N not in $\mathcal{S}$ is finitely generated.*

The module M is called an *$AB5^*$-module* in case $L(M)$ is lower continuous or, in other words, $L(M)^o$ is upper continuous. Modules M such that $L(M)^o$ satisfies (UC3) are called *weakly supplemented*. That is, M is weakly supplemented if and only if for each submodule N of M there exists a submodule L of M such that $M=N+L$ and $N\cap L$ is a small submodule of M (see [11] and [12]). A submodule N of M is called *proper* provided $N\neq M$ or, in other words, N is a non-zero element of $L(M)^o$. A finite collection of submodules N_i $(1\le i\le k)$, for some positive integer k, is called *coindependent* in case N_i is a proper submodule of M for each $1\le i\le k$ and

$$M=N_i+(N_1\cap\dots\cap N_{i-1}\cap N_{i+1}\cap\dots\cap N_k)$$

for all $1\le i\le k$. An arbitrary family of submodules of M will be called *coindependent* provided every finite subfamily is coindependent. The module M has *finite hollow dimension* provided it does not contain an infinite coindependent family (of proper submodules). Thus

$$\begin{aligned} M \text{ has finite hollow dimension } &\Leftrightarrow L(M) \text{ has finite hollow dimension} \\ &\Leftrightarrow L(M)^o \text{ has finite uniform dimension,} \end{aligned}$$

and in this case the hollow dimension, $h(M)$, of M is the uniform dimension of the lattice $L(M)^o$.

Given a submodule L of M, the factor module M/L is called (*strongly*) *finitely cogenerated* in case L is a (strongly) finitely generated element of the lattice $L(M)^0$. Thus, as usual, M/L is a finitely cogenerated module if and only if the fact that $L = \bigcap_{i\in I} N_i$ for some non-empty collection of submodules N_i $(i \in I)$ of M implies that $L = \bigcap_{j\in J} N_j$ for some finite subset J of I. On the other hand, M/L is a strongly finitely cogenerated module if and only if the fact that $L \supseteq \bigcap_{i\in I} N_i$ for some non-empty collection of submodules N_i $(i \in I)$ of M implies that $L \supseteq \bigcap_{j\in J} N_j$ for some finite subset J of I. For a given submodule L of M, the module M/L will be called an *sfc-module* in case L is an sfg-element of $L(M)^o$. Thus M/L is an sfc-module if and only if for any given coindependent set of proper submodules N_i $(i \in I)$ of M such that $\bigcap_{i\in I} N_i \subseteq L$, there exists a finite subset J of I such that $\bigcap_{j\in J} N_j \subseteq L$.

Theorem 3.4 when applied to the lattice $L(M)^o$ gives the next result.

THEOREM 5.3. *A module M has finite hollow dimension if and only if M is weakly supplemented and for every small submodule S of M there exists a small submodule T of M with $S \subseteq T$ and M/T an sfc-module.*

Next, Corollary 3.5 gives:

PROPOSITION 5.4. *Let M be an $AB5^*$-module (respectively, a weakly supplemented module) such that for each small submodule S of M there exists a small submodule T of M with $S \subseteq T$ and M/T (respectively, strongly) finitely cogenerated. Then M has finite hollow dimension.*

EXAMPLE 5.5. Let $\mathfrak{P}$ be the set of all prime integers. Let $M = \bigoplus_{p\in\mathfrak{P}} M_p$ with $M_p \cong \mathbb{Z}/p\mathbb{Z}$ for all $p \in \mathfrak{P}$. Note that if A is a submodule of M and $x = \sum_{i=1}^{k} x_{p_i} \in A$ with $0 \neq x_{p_i} \in M_{p_i}$ for every $1 \leq i \leq k$, then $M_{p_i} \subseteq A$ for every $1 \leq i \leq k$. In fact, let $1 \leq i_0 \leq k$ and let $\alpha = \prod_{i\neq i_0} p_i$. Then $\alpha x = \alpha x_{p_{i_0}} \in M_{p_{i_0}} - \{0\}$. Hence $M_{p_{i_0}} = \mathbb{Z}(\alpha x) \leq A$. It follows easily that every submodule of M has the form $N = \bigoplus_{q\in I} M_q$ with $I \subseteq \mathfrak{P}$. Let $(A_\lambda)_{\lambda\in\Lambda}$ be an inverse family of submodules of M and let N be a submodule of M. Clearly, $N + \bigcap_{\lambda\in\Lambda} A_\lambda \subseteq \bigcap_{\lambda\in\Lambda}(N + A_\lambda)$. On the other hand, since M is semisimple, there exists a subset $\mathfrak{L} \subseteq \mathfrak{P}$ such that $\bigcap_{\lambda\in\Lambda}(N + A_\lambda) = N \oplus \bigoplus_{l\in\mathfrak{L}} M_l$. Let $l_0 \in \mathfrak{L}$. Let $x_{l_0} \in M_{l_0} - \{0\}$. Let $\lambda \in \Lambda$. Since $x_{l_0} \in N + A_\lambda$, there exist $n_0 \in N$ and $a_\lambda \in A_\lambda$ such that $x_{l_0} = n_0 + a_\lambda$. Thus $a_\lambda = -n_0 + x_{l_0} \in A_\lambda$. By a previous remark, we get $M_{l_0} \leq A_\lambda$. Therefore $M_{l_0} \subseteq \bigcap_{\lambda\in\Lambda} A_\lambda$. It follows that $M_{l_0} \subseteq N+\bigcap_{\lambda\in\Lambda} A_\lambda$. So $\bigoplus_{l\in\mathfrak{L}} M_l \subseteq N+\bigcap_{\lambda\in\Lambda} A_\lambda$. Hence $\bigcap_{\lambda\in\Lambda}(N+A_\lambda) \subseteq N + \bigcap_{\lambda\in\Lambda} A_\lambda$. Consequently, $\bigcap_{\lambda\in\Lambda}(N + A_\lambda) = N + \bigcap_{\lambda\in\Lambda} A_\lambda$. Then M is $AB5^*$. On the other hand, it is clear that M does not have finite hollow dimension. Therefore M contains a small submodule S such that for every small submodule T containing S, M/T is not finitely cogenerated (Proposition 5.4).

Let M be any module. We say that M is a *generalized Hopfian* module if every epimorphism from M to M has a small kernel (see [1]). By [5, remark iv), p. 28], if a module M has finite hollow dimension such that the hollow dimension of M is equal to the hollow dimension of M/N for a submodule N of M, then N is small in M. Therefore every module with finite hollow dimension is generalized Hopfian.

We say that a module M has the $(*)$ *property* if it satisfies the following condition:

$(*)$ For every small submodule N of M, there exists an epimorphism $f : M/N \to M$.

Now we give an application of Proposition 5.4:

THEOREM 5.6. *Let M be a finitely cogenerated $AB5^*$ module with the $(*)$ property. Then M is generalized Hopfian if and only if M has finite hollow dimension.*

Proof. $(\Leftarrow)$ This is clear.

$(\Rightarrow)$ Let L be a small submodule of M. By the $(*)$ property, there exists an epimorphism $f : M/L \to M$. Let T be a submodule of M such that $L \subseteq T$ and $\operatorname{Ker} f = T/L$. Therefore M/T is finitely cogenerated since $M/T \cong M$. Now consider the natural epimorphism $\pi : M \to M/L$. So we have the epimorphism $f\pi : M \to M$. Since M is generalized Hopfian, $\operatorname{Ker}(f\pi) = T$ is small in M. The result follows from Proposition 5.4. ∎

Note that the above theorem is dual to [9, Lemma 3.2]. The following examples show that the assumption "M has the $(*)$ property" in Theorem 5.6 is sufficient but not necessary.

EXAMPLE 5.7. Let M be an artinian module which is not semisimple (e.g. we can take M to be the module R_R, where R denotes the ring of all upper triangular 2×2 matrices with entries in a field F or we can take M to be the Prüfer group $\mathbb{Z}_{p^\infty}$, where p is a prime number). Since M is artinian, M has finite hollow dimension and M is a finitely cogenerated and $AB5^*$ module. On the other hand, note that $M/\operatorname{Rad}(M)$ is semisimple. Therefore there is no epimorphism from $M/\operatorname{Rad}(M)$ to M. This implies that the module M does not have the $(*)$ property.

Given a non-negative integer n, $H_n(M)$ will denote the collection of submodules N of M such that every submodule L which satisfies $M = N + L$ also satisfies $h(M/L) \leq n$. Next, $H_\infty(M)$ will denote the collection of submodules N of M such that whenever L is a submodule of M with $M = N+L$ then M/L has finite hollow dimension. Theorem 4.3 then gives:

THEOREM 5.8. *The following statements are equivalent for a module M:*

(i) *M has finite hollow dimension.*
(ii) *M is a weakly supplemented module such that, for some non-negative integer n, M/N is finitely cogenerated for every submodule N of M with $N \notin H_n(M)$.*
(iii) *M satisfies the following two conditions:*
  (a) *$A+\bigcap_{i\in I} A_i = M$ for every submodule A of M and every coindependent set $\{A_i\}_{i\in I}$ of M such that $A+\bigcap_{i\in F} A_i = M$ for every finite subset F of I,*
  (b) *M/N is finitely cogenerated for every submodule N of M with $N \notin H_\infty(M)$.*

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