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POISSON'S EQUATION AND CHARACTERIZATIONS OF REFLEXIVITY OF BANACH SPACES

BY

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Abstract. Let X be a Banach space with a basis. We prove that X is reflexive if and only if every power-bounded linear operator T satisfies Browder's equality

$$\Big\{x \in X : \sup_n \Big\|\sum_{k=1}^{n} T^k x\Big\| < \infty\Big\} = (I-T)X.$$

We then deduce that X (with a basis) is reflexive if and only if every strongly continuous bounded semigroup $\{T_t : t \geq 0\}$ with generator A satisfies

$$AX = \Big\{x \in X : \sup_{s>0} \Big\|\int_0^s T_t x\,dt\Big\| < \infty\Big\}.$$

The range $(I-T)X$ (respectively, AX for continuous time) is the space of $x \in X$ for which Poisson's equation $(I-T)y = x$ ($Ay = x$ in continuous time) has a solution $y \in X$; the above equalities for the ranges express sufficient (and obviously necessary) conditions for solvability of Poisson's equation.

1. Introduction. Let X be a (real or complex) Banach space. Poisson's equation (which was considered originally for the Laplacian in certain function spaces) has been abstracted to solving the equation $Ay = x$ for a given $x \in X$, where A is the infinitesimal generator of a strongly continuous one-parameter bounded semigroup of linear operators $\{T_t : t \geq 0\}$ (see [9]).

In "discrete time", solving Poisson's equation for a power-bounded linear operator T means solving $(I-T)y = x$ for a given $x \in X$. In ergodic theory, elements of $(I-T)X$ are called *coboundaries*, and it is of interest to find conditions for x to be a coboundary, i.e. for the solvability of Poisson's equation.

Obviously, since $\|n^{-1}\sum_{k=1}^{n} T^k x\| \to 0$ if and only if $x \in \overline{(I-T)X}$ (e.g. [8, p. 73]), for any power-bounded operator T on X we have

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$$(I-T)X \subset \Big\{x \in X : \sup_n \Big\| \sum_{k=1}^n T^k x \Big\| < \infty\Big\} \subset \overline{(I-T)X}.$$

It was proved by F. Browder [2] (and rediscovered in [3]) that if X is reflexive, then for every power-bounded operator T on X we have

$$(I-T)X = \Big\{x \in X : \sup_n \Big\| \sum_{k=1}^n T^k x \Big\| < \infty\Big\}. \tag{1}$$

Browder's equality (1) means that a solution y to Poisson's equation $(I-T)y = x$ exists if (and only if) $\sup_n \|\sum_{k=1}^n T^k x\| < \infty$.

In this paper we prove that if X is a Banach space with a basis such that (1) holds for every power-bounded operator T on X, then X is reflexive. The continuous time analogue of this result is then deduced in §4.

A bounded linear operator T on a (real or complex) Banach space X is called *mean ergodic* if

$$E(T)x := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n T^k x \text{ exists for all } x \in X.$$

The general mean ergodic theorem, proved (independently) by Lorch, by Kakutani and by Yosida, says that if X is a reflexive Banach space, then every power-bounded linear operator T is mean ergodic (e.g. [8, p. 73]). In [5] we proved that *if X is a Banach space with a basis, then mean ergodicity of all power-bounded operators implies reflexivity of X.*

For a power-bounded operator T, mean ergodicity is equivalent to the *ergodic decomposition* $X = F(T) \oplus \overline{(I-T)X}$, where $F(T)$ is the space of fixed points of T. In [10] it was shown that if $(I-T)X$ is closed (without assuming mean ergodicity), then T is mean ergodic, and $\|n^{-1}\sum_{k=1}^n T^k - E(T)\| \to 0$ (i.e. T is *uniformly ergodic*).

We denote

$$G(T) := \Big\{x \in X : \sup_n \Big\| \sum_{k=1}^n T^k x \Big\| < \infty\Big\}.$$

It was shown in [4] that $G(T)$ is closed if and only if $(I-T)X$ is closed, which is equivalent to uniform ergodicity of T. If X is infinite-dimensional and has a basis, then by [5, Corollary 3] it admits a power-bounded operator T which is not uniformly ergodic, so in general $G(T)$ is not closed.

Browder's equality (1) was proved in [11] for every contraction on $L_1(\mu)$ (and in [1] for certain power-bounded operators of L_1), so this equality in general does not imply mean ergodicity. This result of [11] also shows that having (1) for every contraction is not sufficient to obtain reflexivity; see [6] for an example of a non-reflexive X with a basis and separable dual, such that all contractions of X and all contractions of X^* are mean ergodic and satisfy (1).

2. Preliminary results

PROPOSITION 2.1. *A power-bounded operator T on a Banach space X is mean ergodic if (and only if) $(I-T)\overline{(I-T)X} = (I-T)X$.*

Proof. If T is mean ergodic, then $X = F(T) \oplus \overline{(I-T)X}$, and the condition follows.

Assume now that T is not mean ergodic. Then there exists $x \in X$ such that $n^{-1}\sum_{k=1}^n T^k x$ does not converge; put $y_0 := (I-T)x$. Define $Y = \overline{(I-T)X}$; then Y is T-invariant, and $\|n^{-1}\sum_{k=1}^n T^k y\| \to 0$ for any $y \in Y$, so $\overline{(I-T)Y} = Y$. Hence $(I-T)\overline{(I-T)X} = (I-T)Y$. If $(I-T)X = (I-T)Y$, then there is $y_1 \in Y$ with $(I-T)y_1 = (I-T)x = y_0$, which yields $(I-T)(x-y_1) = 0$. Hence

$$x - y_1 = \frac{1}{n}\sum_{k=1}^n T^k(x-y_1) = \frac{1}{n}\sum_{k=1}^n T^k x - \frac{1}{n}\sum_{k=1}^n T^k y_1 .$$

Since $\|n^{-1}\sum_{k=1}^n T^k y_1\| \to 0$, the above yields $n^{-1}\sum_{k=1}^n T^k x \to x - y_1$, contradicting the choice of x. Hence $(I-T)\overline{(I-T)X} = (I-T)Y \neq (I-T)X$. ∎

Combining Proposition 2.1 with [5, Corollary 2] we obtain our first result (which is also a consequence of Theorem 3.1 below):

THEOREM 2.2. *The following assertions are equivalent for a Banach space X:*

(i) *X is reflexive.*
(ii) *Every power-bounded operator T defined on a closed subspace $Y \subset X$ with $TY \subset Y$ satisfies*

$$(I-T)Y = \Big\{y \in Y : \sup_n \Big\|\sum_{k=1}^n T^k y\Big\| < \infty\Big\}. \tag{2}$$

(iii) *Every mean ergodic power-bounded operator T defined on a closed subspace $Y \subset X$ with $TY \subset Y$ satisfies* (2).

Proof. Assume first that X is reflexive. Then any closed subspace Y is reflexive, and for any power-bounded operator T on a reflexive Banach space Y the equality (2) follows from [2].

Clearly (ii) implies (iii).

Assume (iii). Let S be a power-bounded operator on a closed subspace Z, and put $Y = \overline{(I-S)Z}$. Then Y is S-invariant, and $T = S_{|Y}$ is mean ergodic, with $\overline{(I-T)Y} = Y$. By (iii), (2) holds, so for $z \in Z$ we have

$$(I-S)z \in Y \cap G(S) = G(T) = (I-T)Y.$$

This yields $(I-S)Z = (I-T)Y$, so

$$(I-S)Z = (I-T)Y = (I-T)\overline{(I-S)Z} = (I-S)\overline{(I-S)Z}.$$

Applying Proposition 2.1 to S we conclude that S is mean ergodic on Z.

Thus every power-bounded operator S on a closed subspace $Z \subset X$ is mean ergodic, so by the ergodic characterization of [5, Corollary 2], X is reflexive. ∎

For any power-bounded operator T on a Banach space X we have

$$(I-T)\overline{(I-T)X} \subset (I-T)X \subset \Big\{x \in X : \sup_n \Big\|\sum_{k=1}^n T^k x\Big\| < \infty\Big\}. \tag{3}$$

Equality in the second inclusion does not imply mean ergodicity—equality holds for every contraction T on L_1, even not mean ergodic [11]. It is easy to construct a mean ergodic power-bounded operator T without equality in the second inclusion above [11].

THEOREM 2.3. *Let X be a Banach space with a basis. Then X is reflexive if and only if every power-bounded operator T on X satisfies*

$$\Big\{x \in X : \sup_n \Big\|\sum_{k=1}^n T^k x\Big\| < \infty\Big\} = (I-T)\overline{(I-T)X}. \tag{4}$$

Proof. If X is reflexive, then every power-bounded operator T is mean ergodic, so we have $(I-T)\overline{(I-T)X} = (I-T)X$, and (4) holds by applying (1) to T.

Assume now that a power-bounded operator T on X satisfies (4). Then by (3) we have $(I-T)\overline{(I-T)X} = (I-T)X$, and thus T is mean ergodic by Proposition 2.1. If every power-bounded operator T satisfies (4), then every power-bounded operator T is mean ergodic, so X is reflexive by the characterization in [5] for Banach spaces with a basis. ∎

THEOREM 2.4. *Let T be a power-bounded operator on a Banach space X. If $\overline{(I-T)X}$ is reflexive, then T is mean ergodic, and Browder's equality* (1) *holds.*

Proof. Since $Y := \overline{(I-T)X}$ is reflexive and T-invariant, by [2] we have

$$G(T_{|Y}) := \Big\{y \in Y : \sup_n \Big\|\sum_{k=1}^n T^k y\Big\| < \infty\Big\} = (I-T)Y.$$

If T is not mean ergodic, Proposition 2.1 yields

$$(I-T)Y = (I-T)\overline{(I-T)X} \neq (I-T)X \subset Y \cap G(T) = G(T_{|Y}),$$

which is a contradiction. ∎

Remark. Reflexivity of $\overline{(I-T)X}$ is far from being necessary for mean ergodicity of T.

3. The main result. In view of (3), equality (4) implies (1), and our main result below improves Theorem 2.3. It also provides an improvement of Theorem 2.2 when X has a basis.

Theorem 3.1. *The following assertions are equivalent for a (separable) Banach space X with a basis:*

(i) *X is reflexive.*
(ii) *Every power-bounded operator T on X satisfies Browder's equality* (1).
(iii) *Every mean ergodic power-bounded operator T on X satisfies* (1).

When X is reflexive, all power-bounded operators T satisfy (1) by [2], so we only have to show that (iii) implies (i). The proof will use the following simple lemma, suggested by the referee as a substitute to our original use of [4, Theorem 2.3].

Lemma 3.2. *Let U be the closed unit ball of a Banach space X, and T a power-bounded operator on X. Then $\overline{(I-T)U} \subset G(T)$.*

Proof. Obviously $(I-T)U \subset G(T)$. Let $y \in \overline{(I-T)U}$. Then there exists $\{x_j\} \subset U$ with $\|y-(I-T)x_j\| = \epsilon_j \to 0$. Denote $M = \sup_{n\ge 0}\|T^k\|$. Then for $n \ge 1$ we have

$$\Big\|\sum_{k=1}^{n} T^k y\Big\| \le \Big\|\sum_{k=1}^{n} T^k[y-(I-T)x_j]\Big\| + \Big\|\sum_{k=1}^{n} T^k(I-T)x_j\Big\|$$
$$\le nM\epsilon_j + 2M \xrightarrow[j\to\infty]{} 2M. \ \blacksquare$$

To prove the theorem, we follow the strategy of [5]. If X is non-reflexive and has a basis, then by [14] it has a non-shrinking basis. Therefore Theorem 3.1 is a consequence of the following.

Theorem 3.3. *Let X be a Banach space having a non-shrinking finite-dimensional Schauder decomposition. Then there exists a mean ergodic power-bounded operator T such that Browder's equality* (1) *fails.*

The first step is the following lemma of [5].

Lemma 3.4. *Let X be a Banach space with a non-shrinking Schauder decomposition. Then X has a Schauder decomposition $X = \sum_{k=1}^{\infty} X_k$ with the following property: there exist a functional $h \in X^*$ and a sequence $\{e_k\}$ such that for every $k \ge 1$ we have $e_k \in X_k$, $\|e_k\| \le 1$ and $h(e_k) = 1$.*

Furthermore, if the components of the original non-shrinking decomposition are finite-dimensional, so are all the X_k.

The last part of the lemma follows from the construction in [5]—each X_k is a finite sum of components of the original decomposition.

As noted at the beginning of the proof of [5, Theorem 1], we can change the norm to an equivalent one so that in the decomposition obtained in the above lemma the coordinate projections $Q_k : X \to X_k$ and the partial sums projections $P_k : X \to \sum_{j=1}^k X_j$ (defined respectively by $Q_k(\sum_{j=1}^\infty x_j) = x_k$ and $P_k = \sum_{j=1}^k Q_j$) all have norm 1.

The lemma yields a decomposition $X_k = (X_k \cap \ker h) \oplus \operatorname{span}\{e_k\}$.

LEMMA 3.5. *Let* $X = \sum_{k=1}^\infty X_k$ *be the Schauder decomposition with coordinate projections* Q_k*, obtained in Lemma 3.4, let* $e_0 = 0$*, and put* $u_n = e_n - e_{n-1}$ *for* $n \geq 1$*. For* $k \geq 1$ *define* $E_{2k} = \operatorname{span}\{u_k\}$ *and* $E_{2k-1} = X_k \cap \ker h$*. Then* $X = \sum_{m=1}^\infty E_m$ *is a Schauder decomposition of* X *with coordinate projections* $\bar{Q}_m$ *given by*

- $\bar{Q}_{2k-1} = R_k Q_k$*, where* $R_k : X_k \to E_{2k-1}$ *is defined by* $R_k x_k = x_k - h(x_k)e_k$*.*
- $\bar{Q}_{2k}x = (h - \sum_{j=0}^{k-1} Q_j^* h)(x)u_k$*, where* $Q_0 = 0$*.*

Proof. For $x \in X_k$ we have $x - h(x)e_k \in E_{2k-1}$, and $\sum_{j=1}^k u_j = e_k$. Hence $\sum_{m=1}^{2n} E_m = \sum_{k=1}^n X_k$, so $\operatorname{span}\{\bigcup_{m\geq 1} E_m\}$ is dense in X.

We first show that each $\bar{Q}_m$ as defined is a projection onto E_m which vanishes on E_l for $l \neq m$.

It is easily checked that R_k is a projection of X_k onto E_{2k-1}, for any $k \geq 1$, so $R_k Q_k R_k Q_k = R_k R_k Q_k = R_k Q_k$, and thus $\bar{Q}_{2k-1}$ is a projection onto E_{2k-1}. Since $Q_k X_j = \{0\}$ for $j \neq k$, we have $\bar{Q}_{2k-1} E_{2j-1} = \{0\}$ for $j \neq k$.

Since $u_l \in X_{l-1} \oplus X_l$, we have $Q_k E_{2l} = \{0\}$ when $k < l-1$ or $k > l$. For $l = k$ we have $Q_k u_l = e_k$ and $R_k Q_k u_l = R_k e_k = 0$ since $h(e_k) = 1$. For $l = k+1$ we have $Q_k u_l = -e_k$ and $R_k Q_k u_l = 0$. Thus $\bar{Q}_{2k-1} E_m = \{0\}$ for $m \neq 2k-1$.

We now look at $\bar{Q}_{2k}$. By definition it takes X into E_{2k}, so to show it is a projection it is enough to check that $\bar{Q}_{2k} u_k = u_k$. We compute

$$\begin{aligned}\bar{Q}_{2k} u_k &= \Big(h(u_k) - \sum_{j=0}^{k-1} h(Q_j u_k)\Big) u_k = (h(e_k) - h(e_{k-1}) - h(Q_{k-1} u_k)) u_k \\ &= (h(e_k) - h(e_{k-1}) + h(e_{k-1})) u_k = h(e_k) u_k = u_k.\end{aligned}$$

For $x \in E_{2l-1}$ we have $h(x) = 0$, and $Q_j x = 0$ for $j \neq l$, $h(Q_l x) = h(x) = 0$. Hence $\bar{Q}_{2k} E_{2l-1} = \{0\}$.

For $k = 1$ we have $\bar{Q}_2 x = h(x)u_1 = h(x)e_1$ so for $l > 1$ we obtain $\bar{Q}_2 u_l = h(u_l)u_1 = 0$. For $k > 1$ and $l \neq k$ we have

$$\begin{aligned}\bar{Q}_{2k}u_l &= \Big(h(u_l) - \sum_{j=1}^{k-1} h(Q_j u_l)\Big)u_k \\ &= \Big(h(e_l) - h(e_{l-1}) - \sum_{j=1}^{k-1}[h(Q_j e_l) - h(Q_j e_{l-1})]\Big)u_k.\end{aligned}$$

This is 0 for $l > k$ since all terms in the sum are 0. For $l \leq k-1$ we have in the sum only $h(e_l) - h(e_{l-1}) = 0$, so $\bar{Q}_{2k}u_l = 0$ for $l \neq k$.

We thus see that each $\bar{Q}_m$ is a projection onto E_m with $\bar{Q}_m E_j = \{0\}$ for $j \neq m$. This also yields $E_m \cap E_j = \{0\}$ for $j \neq m$.

Claim. *Put* $\bar{P}_n = \sum_{j=1}^n \bar{Q}_j$. *Then* $\sup_n \|\bar{P}_n\| < \infty$.

We denote $P_n = \sum_{j=1}^n Q_j$. Since $\{X_n\}$ is a Schauder decomposition of X, we have $\sup_n \|P_n\| < \infty$.

Fix n and let $m > n$. Using $Q_j x = R_j Q_j x + h(Q_j x)e_j$, for $x \in \sum_{k=1}^m X_k$ we obtain

$$\begin{aligned}\bar{P}_{2n}x &= \sum_{j=1}^{2n} \bar{Q}_j x = \sum_{k=1}^{n} R_k Q_k x + \sum_{k=1}^{n}\Big(h(x) - \sum_{j=0}^{k-1} h(Q_j x)\Big)(e_k - e_{k-1}) \\ &= \sum_{k=1}^{n} R_k Q_k x + \sum_{j=0}^{n-1} h(Q_j x)e_j + \Big(h(x) - \sum_{j=0}^{n-1} h(Q_j x)\Big)e_n \\ &= \sum_{k=1}^{n} Q_k x + \Big(h(x) - \sum_{j=0}^{n} h(Q_j x)\Big)e_n \\ &= P_n x + \Big(h - \sum_{j=0}^{n} Q_j^* h\Big)(x)e_n = P_n x + (h - P_n^* h)(x)e_n.\end{aligned}$$

Since $\|e_n\| = 1$, we obtain $\|\bar{P}_{2n}x\| \leq \|P_n\| \cdot \|x\| + \|I - P_n^*\| \cdot \|h\| \cdot \|x\|$, so

$$\sup_n \|\bar{P}_{2n}\| \leq \sup_n \|P_n\| + \|h\|(1 + \sup_n \|P_n\|).$$

We now have $\bar{P}_{2n+1} = \bar{P}_{2n} + \bar{Q}_{2n+1}$, so the above yields

$$\bar{P}_{2n+1} = P_n x + (h - P_n^* h)(x)e_n + R_{n+1}Q_{n+1}x.$$

But $\|R_{n+1}Q_{n+1}x\| \leq \|Q_{n+1}x\| + \|h\| \cdot \|Q_{n+1}x\|$, and $\sup_n \|Q_n\| < \infty$, so we obtain $\sup_n \|\bar{P}_{2n+1}\| < \infty$, and the Claim is proved.

Since $\lim \bar{P}_m x = x$ on a dense subset, the Claim implies that $\bar{P}_m x \to x$ on all of X and $\sum_{m=1}^\infty E_m$ is a Schauder decomposition. ∎

Proposition 3.6. *Let* $X = \sum_{k=1}^{\infty} X_k$ *be a Schauder decomposition of* X *with coordinate projections* Q_k. *For a sequence* $a := \{a_j\}_{j=1}^{\infty}$ *with* $a_j > 0$ *for* $j \geq 1$ *and* $\sum_{j=1}^{\infty} a_j = 1$ *put* $s_k = \sum_{j=1}^{k} a_j$. *Then for every* $x \in X$ *the series* $\sum_{k=1}^{\infty} s_k Q_k x$ *converges in norm, and the operator* $T_a x := \sum_{k=1}^{\infty} s_k Q_k x$ *is power-bounded on* X.

Proof. The proposition follows from the computations on pages 150–151 of [5] (with $h = 0$). In those computations it is assumed that the coordinate projections Q_k and the partial sums $P_k = \sum_{j=1}^{k} Q_j$ all have norm 1 (and then $\sup_n \|T_a^n\| \leq 2$); the assumption is achieved by a change to an equivalent norm.

The referee noted that the proposition has been known for some time. For example, its proof can be found essentially in [13, Lemma 2.4]. ∎

Proof of Theorem 3.3. Let $X = \sum_{k=1}^{\infty} E_k$ be the Schauder decomposition of X obtained in Lemma 3.5 from the non-shrinking Schauder decomposition $X = \sum_k X_k$ with finite-dimensional components. By the definitions, also all the E_k are finite-dimensional; let $\bar{Q}_k$ be the coordinate projection onto E_k.

Choose $a = \{a_j\}_{j=1}^{\infty}$ with $a_j > 0$ and $\sum_{j=1}^{\infty} a_j = 1$ such that the tails $b_k = \sum_{j=k+1}^{\infty} a_j$ satisfy $\sum_{k=1}^{\infty} b_k < \infty$ (e.g. $a_j = 2^{-j}$), and put $Tx = T_a x = \sum_{k=1}^{\infty} s_k \bar{Q}_k x$. By the proposition above, T is power-bounded. By the definitions $(I - T)x = \sum_{m=1}^{\infty} b_m \bar{Q}_m x$, so $I - T$ is a compact operator since the E_m are finite-dimensional. Since each E_m is T-invariant finite-dimensional and T is power-bounded, T is mean ergodic on X.

We assert that (1) fails, i.e. $(I - T)X \neq G(T)$. Towards a contradiction, assume that $(I - T)X = G(T)$. From Lemma 3.2 we deduce that the unit ball U of X satisfies $\overline{(I - T)U} \subset G(T) = (I - T)X$.

By the construction in Lemma 3.5, $\|\sum_{i=1}^{n} u_i\| = \|e_n\| \leq 1$ for every n, so compactness of $I - T$ implies that there is a subsequence $\{n_p\}$ with $(I - T)e_{n_p} = (I - T)(\sum_{i=1}^{n_p} u_i) \to z \in \overline{(I - T)U}$. Hence $z \in (I - T)X$ by the assumptions, so there is $x \in X$ with $(I - T)x = z$.

Since $\bar{Q}_k \bar{Q}_j = \delta_{j,k} \bar{Q}_k$, for every k we have $T\bar{Q}_k = \bar{Q}_k T$ and $\bar{Q}_k(I - T) = (1 - s_k)\bar{Q}_k$. For $m = 2k - 1$ we have $\bar{Q}_m e_n = 0$ for all n by the definitions, so

$$(1 - s_m)\bar{Q}_m x = \bar{Q}_m(I - T)x = \bar{Q}_m z = \lim_{n_p \to \infty} (I - T)\bar{Q}_m e_{n_p} = 0.$$

Hence $\bar{Q}_m x = 0$ for m odd. For $m = 2k$ and $n \geq k$ the definition of $\bar{Q}_m$ yields $\bar{Q}_m e_n = u_k$. Hence

$$\begin{aligned}(1 - s_m)\bar{Q}_m x &= \bar{Q}_m(I - T)x = \bar{Q}_m z \\ &= \lim_{n_p \to \infty} \bar{Q}_m(I - T)e_{n_p} = \lim_{n_p \to \infty} (1 - s_m)\bar{Q}_m e_{n_p} = (1 - s_m)u_k,\end{aligned}$$

which yields $\bar{Q}_{2k}x = u_k$. Thus

$$x = \sum_{m=1}^{\infty} \bar{Q}_m x = \sum_{k=1}^{\infty} \bar{Q}_{2k} x = \sum_{k=1}^{\infty} u_k = \lim_{n\to\infty} e_n.$$

This implies that all the coordinate projections of the original Schauder decomposition $X = \sum_{j=1}^{\infty} X_j$ satisfy $Q_j x = 0$, hence $e_n \to 0$, a contradiction to $h(e_n) = 1$ for every n. Thus $(I-T)X = G(T)$ cannot be true. This proves Theorem 3.3. ∎

Remark. The authors are grateful to the referee for simplifying their original proof that (1) fails; the above proof follows the referee's suggestions.

4. On Poisson's equation for one-parameter semigroups. Originally, Poisson's equation was considered for the Laplacian. This has been abstracted to solving the equation $Ay = x$ for a given $x \in X$, where A is the infinitesimal generator of a strongly continuous one-parameter bounded semigroup of linear operators $\{T_t : t \geq 0\}$ (see [9]). We use Theorem 3.1 to obtain a characterization of reflexivity by a condition for solvability of Poisson's equation, for all infinitesimal generators of bounded strongly continuous semigroups.

Theorem 4.1. *The following assertions are equivalent for a Banach space X with a basis:*

(i) *X is reflexive.*
(ii) *Every strongly continuous bounded semigroup $\{T_t : t \geq 0\}$ with generator A satisfies*

$$AX = \Big\{x \in X : \sup_{s>0} \Big\| \int_0^s T_t x\, dt \Big\| < \infty \Big\}. \tag{5}$$

(iii) *Every uniformly continuous bounded semigroup $\{T_t : t \geq 0\}$ with generator A satisfies* (5).

Proof. (i) implies (ii) by Theorem 2.6 of [9] (since the dual semigroup is also strongly continuous, by reflexivity and [7, Theorem 10.6.5]).

Obviously (ii) implies (iii). We show that (iii) implies (i).

Assume that X (with a basis) is not reflexive. By Theorem 3.1 there exists a power-bounded operator T such that (1) fails, which means that for some $x \notin (I-T)X$ we have $\sup_n \|\sum_{k=1}^n T^k x\| < \infty$. We may assume, by changing the norm to an equivalent one, that $\|T\| = 1$. For $t \geq 0$ put $S_t = e^{t(T-I)}$. Then $\{S_t\}$ is a uniformly continuous semigroup, with infinitesimal generator $A = T - I$. The power series expansion yields

$$\|S_t\| = e^{-t}\|e^{tT}\| \leq e^{-t}e^{t\|T\|} = 1.$$

Since $\sup_n \|\sum_{k=1}^n T^k x\| < \infty$, Theorem 5 of [11] yields the existence of some $y^{**} \in X^{**}$ such that $(I - T^{**})y^{**} = x$; hence $x \in A^{**}X^{**}$ (we have identified X with its canonical image in X^{**}). The uniform continuity of $\{S_t\}$ implies that of $\{S_t^{**}\}$, with generator $A^{**} = T^{**} - I$, and for $s > 0$ we obtain

$$\Big\|\int_0^s S_t x\,dt\Big\| = \Big\|\int_0^s S_t^{**} x\,dt\Big\| = \|-S_s^{**}y^{**} + y^{**}\| \le 2\|y^{**}\|.$$

Since $x \notin (I - T)X = AX$, the contraction semigroup $\{S_t\}$ does not satisfy (5). Hence X is reflexive when (iii) holds. ∎

Remark. The idea of using the semigroup $e^{t(T-I)}$ is due to Rainer Nagel, in the context of characterizing reflexivity by mean ergodicity of all bounded semigroups [12].

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