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THE HERZ–SCHUR MULTIPLIER NORM OF SETS SATISFYING THE LEINERT CONDITION

BY

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Abstract. It is well known that in a free group $\mathbb{F}$, one has $\|\chi_E\|_{M_{cb}A(\mathbb{F})} \leq 2$, where E is the set of all the generators. We show that the (completely) bounded multiplier norm of any set satisfying the Leinert condition depends only on its cardinality. Consequently, based on a result of Wysoczański, we obtain a formula for $\|\chi_E\|_{M_{cb}A(\mathbb{F})}$.

1. Introduction. In this paper we are interested in finding the Herz–Schur multiplier norm of the characteristic function of the set of generators of a free group. It was shown in [3] that the Herz–Schur norm on a group G is the same as the completely bounded multiplier norm of the Fourier algebra $A(G)$. In a group, we actually make use of the notion of a subset satisfying the Leinert condition, which is slightly weaker than freeness. Our main result shows that the multiplier norm of the characteristic function of such a set depends only on its cardinality. Thus, in order to obtain these values we only need to compute them for specific examples. When $n = 2p$, the set of generators and their inverses in $\mathbb{F}_p$ satisfies the Leinert condition and has cardinality n. For all n, the set of generators of $*_{i=1}^{n}\mathbb{Z}/2\mathbb{Z}$ is another example. We will use results of Haagerup, Steenstrup and Szwarc [10] and of Wysoczański [17], which characterize the Herz–Schur multipliers for radial functions on various discrete groups.

We use two methods of proof. One is algebraic, quick, elegant and very general. The other one is combinatorial and we present it only for the groups $\mathbb{F}_2$ and $\mathbb{F}_\infty$. Even if this second method is lengthier, we choose to present it first, as it gives a good insight on what the corresponding matrices look like. In this way, we also collect many small results that are related to the particular question of computing the exact norm of multipliers.

The organization of the paper is the following: In the preliminary section, we set up the general context and give the necessary definitions and results we rely on. In the next section, we give the definition of the Leinert condition

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and describe one of its combinatorial characterizations. In Sections 4 and 5, we present the combinatorial and the algebraic methods for calculating the announced norm. In the last section, we end with some remarks on the bounded multiplier norm.

2. Preliminaries. Our standard reference for details on operator spaces and completely bounded maps is the book of Effros and Ruan [5]. Let G be a discrete group. Denote by $A(G)$ the *Fourier algebra* of G, which consists of all coefficient functions

$$\varphi(s) = \langle \lambda(s)\xi \mid \eta\rangle, \qquad s \in G,\ \xi,\eta \in \ell_2(G),$$

of the left regular representation λ on $\ell_2(G)$. Multiplication in $A(G)$ is pointwise multiplication (hence commutative) and its norm is given by

$$\|\varphi\|_{A(G)} = \inf\{\|\xi\|\|\eta\| \mid \varphi(s) = \langle \lambda(s)\xi \mid \eta\rangle\}.$$

The *Fourier–Stieltjes algebra* $B(G)$ is defined to be the space of all coefficient functions

$$\psi(s) = \langle \pi(s)\xi \mid \eta\rangle, \qquad s \in G,\ \xi,\eta \in H_\pi,$$

where $\pi : G \to B(H_\pi)$ is the universal representation of G. It is known that $B(G)$ is a commutative algebra and its norm is given by

$$\|\psi\|_{B(G)} = \inf\{\|\xi\|\,\|\eta\| \mid \psi(s) = \langle \pi(s)\xi \mid \eta\rangle,\ \pi : G \to B(H_\pi) \text{ a unitary representation}\}.$$

The Fourier algebra $A(G)$ can be isometrically identified with the predual of $\mathrm{VN}(G)$ (the *group von Neumann algebra*) and thus it has a canonical operator space structure which makes $A(G)$ a completely contractive Banach algebra. Let us recall it briefly: we consider G a discrete group and $\mathrm{VN}(G)$ the von Neumann algebra on $\ell_2(G)$ generated by the left translations $\lambda(g)$ for $g \in G$. Furthermore, $\mathrm{VN}(G)$ is a type II_1 von Neumann algebra and its canonical normal faithful trace is given by the evaluation on e, that is, $\tau(\lambda(g)) = \delta_{g,e}$. For any $p \geq 1$, the completion of $\mathrm{VN}(G)$ for the norm $(\tau(|x|^p))^{1/p}$ is the L_p space associated to $\mathrm{VN}(G)$, where $|x| = (x^*x)^{1/2}$ (for $p = \infty$, we recover the norm on $\mathrm{VN}(G)$). Of course, $L_2(G,\tau) = \ell_2(G)$ and $L_1(G,\tau) = A(G)$.

A function $\varphi : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if $\varphi.\psi \in A(G)$ for any $\psi \in A(G)$. Then we consider the multiplication map $m_\varphi : A(G) \to A(G)$ defined as

$$m_\varphi(\psi) = \varphi.\psi.$$

By the closed graph theorem, $m_\varphi : A(G) \to A(G)$ is bounded on $A(G)$. We denote by $MA(G)$ the space of all multipliers of $A(G)$, equipped with the natural norm $\|\varphi\|_{MA(G)} = \|m_\varphi\|$. A multiplier φ is called *completely bounded* if $\|m_\varphi\|_{cb} < \infty$. We denote by $M_{cb}A(G)$ the space of all *completely bounded multipliers* of $A(G)$ equipped with the completely bounded norm. When m_φ is bounded, its adjoint is well defined on $\mathrm{VN}(G)$, weak-$*$ continuous, and is given by $m_\varphi^*\lambda(g) = \varphi(g)\lambda(g)$. Of course, $\|m_\varphi^*\|_{cb(\mathrm{VN}(G),\mathrm{VN}(G))} = \|m_\varphi\|_{cb}$. We refer to [5] for more details.

Let G be an (infinite) discrete group and $A = [A(s,t)]_{s,t\in G}$ and $B = [B(s,t)]_{s,t\in G}$ be two (infinite) matrices indexed by G. Then we can define the *Schur multiplication* of A and B by $[A * B(s,t)] = [A(s,t)\cdot B(s,t)]$ and we call a function $A : G \times G \to \mathbb{C}$ a *Schur multiplier* if there exists $\alpha > 0$ such that $\|A * K\| \le \alpha\|K\|$ for any finite matrix K. Here, the norm $\|\cdot\|$ is the operator norm on $B(\ell_2(G))$. We denote by $|\!|\!|\cdot|\!|\!|$ the Schur multiplier norm defined as follows:

$$|\!|\!|A|\!|\!| = \sup\{\|A * k\| \mid k \in B(\ell_2(G)) \text{ with } \|k\| \le 1\}.$$

Bounded Schur multipliers are automatically continuous for the weak operator topology. Hence by Russo–Dye's theorem, we can restrict the supremum to the finite unitary matrices.

Given a function $\varphi : G \to \mathbb{C}$, we define $M\varphi : G \times G \to \mathbb{C}$ by $M\varphi(s,t) = \varphi(s^{-1}t)$ and we can regard $M\varphi = [M\varphi(s,t)]_{s,t\in G}$ as an (infinite) matrix indexed by G.

Using an (unpublished) result of Gilbert [7], Bożejko and Fendler [3] showed that $M_{cb}A(G)$ is isometrically isomorphic to $B_2(G)$, the space of all *Herz–Schur multipliers* on G. Then the completely bounded multiplier norm is exactly the Herz–Schur multiplier norm, so it is given by

$$\|\varphi\|_{M_{cb}A(G)} = |\!|\!|M\varphi|\!|\!| = \sup\{\|M\varphi * k\| \mid k \text{ a finite unitary matrix}\} \tag{1}$$

This result also shows that a function $\varphi : G \to \mathbb{C}$ is in $M_{cb}A(G)$ with $\|\varphi\|_{M_{cb}A(G)} \le 1$ if and only if there exist a Hilbert space K and two bounded maps $\alpha, \beta : G \to K$ such that

$$\varphi(s^{-1}t) = \langle \beta(t) \mid \alpha(s)\rangle \quad \forall s,t \in G \tag{2}$$

and

$$\sup_{s\in G}\{\|\alpha(s)\|_K\} \sup_{t\in G}\{\|\beta(t)\|_K\} \le 1. \tag{3}$$

A short and elegant proof of this result can be found in [11] or [15].

For any group G, we have $B(G) \subseteq M_{cb}A(G) \subseteq MA(G)$ and equalities hold if G is amenable. When G is the free group on at least two generators (hence non-amenable), all the above inclusions are strict ([12], [2], [6]).

As usual, we will use the notation χ_E for the characteristic function of a subset E.

The general problem of computing the exact norms of multipliers is difficult to address. In a remarkable work, Haagerup, Steenstrup and Szwarc gave an answer for some multipliers on free groups, that we describe below.

As usual, we denote by $\mathbb{F}_N$ the free group with finitely or infinitely many generators $N \geq 2$, denoted by $g_1, g_2, \ldots$. A function $\varphi : \mathbb{F}_N \to \mathbb{C}$ is said to be *radial* if there exists $\dot{\varphi} : \mathbb{N} \to \mathbb{C}$ such that $\varphi(x) = \dot{\varphi}(|x|)$, where $|\cdot|$ represents the usual length function on the free group. We also define the translation $\tau(A)$ of a matrix $A = [a_{ij}]_{i,j \in \mathbb{N}}$ in the following way:

$$\tau(a)_{ij} = \begin{cases} a_{i-1,j-1} & \text{if } i, j \geq 1, \\ 0 & \text{if } i = 0 \text{ or } j = 0. \end{cases}$$

Of course, τ is a contraction on $B(\ell_2(G))$, so that for any $t \in \mathbb{C}$ with $|t| < 1$, $(1 - t\tau)^{-1}$ is well defined on $B(\ell_2(G))$ as a Neumann series.

The following is Theorem 4.2 of [10]:

Theorem 1. *Consider a free group $\mathbb{F}_N$ with finitely or infinitely many generators $N \geq 2$. Let $\varphi : \mathbb{F}_N \to \mathbb{C}$ be a radial function with its corresponding function $\dot{\varphi} : \mathbb{N} \to \mathbb{C}$. Let $H = [h_{ij}]_{i,j \in \mathbb{N}}$ be the Hankel matrix given by $h_{ij} = \dot{\varphi}(i + j) - \dot{\varphi}(i + j + 2)$ for $i, j \in \mathbb{N}$. The following are equivalent:*

(i) *φ is a completely bounded multiplier of the Fourier algebra of $\mathbb{F}_N$.*
(ii) *H is of trace class.*

If these two equivalent conditions are satisfied, then there exist unique constants $c_1, c_2 \in \mathbb{C}$ and a unique $\dot{\psi} : \mathbb{N} \to \mathbb{C}$ such that

$$\dot{\varphi}(n) = c_1 + c_2(-1)^n + \dot{\psi}(n), \quad n \in \mathbb{N}, \quad \lim_{n \to \infty} \dot{\psi}(n) = 0.$$

Moreover, if we let $q = 2N - 1$, then

$$\|\varphi\|_{M_{cb}A(\mathbb{F})} = \begin{cases} |c_1| + |c_2| + \|H\|_1 & \text{when } q = \infty, \\ |c_1| + |c_2| + (1 - 1/q)\|(I - \tau/q)^{-1}H\|_1 & \text{when } 2 \leq q < \infty. \end{cases}$$

As $E = \{g_1^{\pm 1}, g_2^{\pm 1}, \ldots\}$ is a radial set, this theorem gives a closed formula for the completely bounded norm of χ_E. The main motivation of this paper is to answer a question of M. Bożejko: is there any such formula for $E = \{g_1, g_2, \ldots\}$ consisting only of generators? And also, do we have a good understanding of the shape of the corresponding Schur multipliers that yield this formula?

The above theorem has been extended in several ways. We will need its adaptation by Wysoczański [17] to arbitrary free products. Before stating it, we give the definition of the block length of an element in a group G of the form $G = \ast_{i=1}^n G_i$. Recall that any $g \in G$ can be either the identity or of the form $g = g_1 g_2 \ldots g_n$ for some $g_k \in G_{i_k} \backslash \{e\}$, where $i_1, \ldots, i_n \in \{1, \ldots, n\}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$. The *block length* of an element $g \in G$ is denoted by $\|g\|$ and is defined as:

(i) $\|e\| = 0$ if $g = e$,
(ii) $\|g\| = n$ if g is expressed as above.

Note that this length is different from the one considered above on $\mathbb{F}_N$. A *block radial function* on G is a function $\phi : G \to \mathbb{C}$ so that $\varphi(g) = \dot{\varphi}(\|g\|)$ for some function $\dot{\varphi}$ on $\mathbb{N}$. We will only need a particular example, i.e. the characterization when $G_i = \mathbb{Z}/2\mathbb{Z}$:

THEOREM 2. *Consider the group $G = *_{i=1}^{n}\mathbb{Z}_2$ where $n \geq 2$, and a radial function $\varphi(g) = \dot{\varphi}(\|g\|)$ on G. Then φ is a Herz–Schur multiplier if and only if the Hankel matrix $H = [h_{ij}]_{i,j\in\mathbb{N}}$ given by $h_{ij} = \dot{\varphi}(i+j) - \dot{\varphi}(i+j+2)$ for $i, j \in \mathbb{N}$ is of trace class. Moreover, there exist unique constants $c_1, c_2 \in \mathbb{C}$ and a unique $\dot{\psi} : \mathbb{N} \to \mathbb{C}$ vanishing at infinity such that*

$$\varphi(g) = c_1 + c_2(-1)^{\|g\|} + \psi(g), \qquad n \in \mathbb{N},$$

and

$$\|\varphi\|_{M_{cb}A(G)} = \begin{cases} |c_1| + |c_2| + \dfrac{n-2}{n-1}\left\|\left(I - \dfrac{\tau}{n-1}\right)^{-1} H\right\|_1 & \textit{when } n < \infty, \\ |c_1| + |c_2| + \|H\|_1 & \textit{when } n = \infty. \end{cases}$$

3. Sets which satisfy the Leinert condition

DEFINITION 3. A set $\Lambda \subseteq G$ satisfies the *Leinert condition* if for all $n \in \mathbb{N}$ and for all $\{x_i\}_{i=1}^{2n} \subset \Lambda$ with $x_i \neq x_{i+1}$ we have

$$x_1x_2^{-1}x_3x_4^{-1}\cdots x_{2n-1}x_{2n}^{-1} \neq e.$$

When Λ satisfies the Leinert condition, we call it a *$\mathcal{L}c$-set*.

Note that if $|\Lambda| \geq 2$ then $x_1x_2^{-1}$ ($x_1 \neq x_2 \in \Lambda$) has infinite order, so G is infinite.

DEFINITION 4. A matrix $A = [a_{ij}]$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$, with entries 0 and 1 is called *chainable* if

(a) it has no zero rows or columns, and
(b) for any pair of entries $a_{rt} = 1$ and $a_{pq} = 1$, there exists a sequence of entries $a_{i_1j_1} = \cdots = a_{i_sj_s} = 1$ such that $i_1 = r$, $j_1 = t$, $i_s = p$, $j_s = q$ and $i_k = i_{k+1}$ or $j_k = j_{k+1}$ for $k = 1, \ldots, s-1$.

This sequence of elements is called a *chain*. A closed chain is called a *cycle*.
The *length* of a chain between (r, t) and (p, q) is the smallest integer s so that (b) holds.

Let us note that all the cycles have even lengths.
We also have the following property:

PROPERTY 5. *If an $m\times n$ matrix A with only 0 and 1 entries has neither zero rows nor zero columns, then there exist permutation matrices P and Q*

such that

$$PAQ = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{bmatrix},$$

where $A_1, , \ldots, A_N$ are chainable matrices.

If we regard A as the adjacency matrix of some bipartite graph, the submatrices $A_1,\ldots,A_N$ correspond to the connected components of the graph.

To make a link between these notions, assume that A is an $n \times n$ submatrix of $M\chi_\Lambda$ that contains a cycle of length $2k$. Then there are indices $(s_i)_{i=1}^{2k}$ and $(t_i)_{i=1}^{2k}$ so that $a_{s_1,t_1} = a_{s_2,t_1} = a_{s_2,t_2} = \cdots = a_{s_1,t_{2k}} = 1$. This exactly means that the words $x_1 = s_1^{-1}t_1$, $x_2 = s_2^{-1}t_1$, $x_3 = s_2^{-1}t_2 \ldots$, $x_{2k} = s_1^{-1}t_{2k}$ belong to Λ. But then

$$x_1x_2^{-1}x_3x_4^{-1}\cdots x_{2n-1}x_{2n}^{-1} = s_1^{-1}t_1(s_2^{-1}t_1)^{-1}\ldots(s_1^{-1}t_{2k})^{-1} = e.$$

The definition of length implies that $x_i \neq x_{i+1}$. Hence, if Λ is a $\mathcal{L}c$-subset of G then there is no cycle in any submatrix of $M\chi_\Lambda$.

The following lemma is a part of Theorem 8.3 in [14].

Lemma 6. *Let G be a group and let $\Lambda \subseteq G$ with $|\Lambda| \geq 2$. Then the following are equivalent:*

(i) *Λ satisfies the Leinert condition.*
(ii) *No $n \times m$ submatrix A of $M\chi_\Lambda$ contains a cycle.*

Proof. We have just explained (i)⇒(ii). Now suppose that Λ is not a $\mathcal{L}c$-set. Then we can find an even number of elements, say $\{x_i\}_{i=1}^{2k} \in \Lambda$ with $x_i \neq x_{i+1}$, such that

$$x_1x_2^{-1}x_3\ldots x_{2k-1}x_{2k}^{-1} = e.$$

We can assume that k is the smallest integer with this property. Then, for $i = 1, \ldots, k$, let

$$s_i = (x_1x_2^{-1})\cdots(x_{2i-1}x_{2i}^{-1}) \quad \text{and} \quad t_i = (x_1x_2^{-1})\cdots(x_{2i-3}x_{2i-2}^{-1})x_{2i-1}.$$

First, if $s_i = s_j$ with $i < j$, then $(x_{2i+1}x_{2i+2}^{-1})\cdots(x_{2j-1}x_{2j}^{-1}) = e$, which contradicts the minimality of k. The same holds for the t_i's. Our hypothesis is that $s_k = e$.

We compute

$$\forall i=1,\ldots,k, \quad s_i^{-1}t_i = x_{2i}, \quad \forall i=1,\ldots,k-1, \quad s_i^{-1}t_{i+1} = x_{2i+1}, \quad s_k^{-1}t_1 = x_1.$$

Hence, we constructed $s_1, \ldots, s_k, t_1, \ldots, t_k$ such that the matrix indexed by them has a cycle. This contradicts our hypothesis, therefore (ii) implies (i). ∎

Remark 7. It was shown in Theorem 8.1 in [14] that an $n \times m$ matrix A does not contain any cycle if and only if any $k \times l$ submatrix of A has at most $k + l - 1$ non-zero entries.

4. The Herz–Schur norm calculation—the combinatorial method. Let $\mathbb{F}_N$ be the free group with a finite or infinite number N of generators and let E be the set of all these generators. We are interested in finding the exact value of the completely bounded multiplier norm of the characteristic function χ_E, or equivalently, the Herz–Schur multiplier norm of χ_E. In this section, we use a combinatorial approach to deal with the two extreme cases: when N is 2 or ∞. For a subset A of a group G, we have

$$\begin{aligned}\|\chi_A\|_{M_{cb}A(G)} &= \|M\chi_A\| = \sup\{\|M\chi_A * K\| \mid K \text{ a finite unitary matrix}\} \\ &= \lim_n(\sup\{\|M\chi_A * K\| \mid K \text{ a finite } n\times n \text{ unitary matrix}\}).\end{aligned}$$

To prove that the completely bounded multiplier norms of χ_A and χ_B coincide if $|A| = |B|$, it suffices to prove that $M\chi_A$ and $M\chi_B$ have the same submatrices. This is the core of the combinatorial argument.

We start with the case $G = \mathbb{F}_2$, the free group with two generators g_1 and g_2. Then $E = \{g_1, g_2\}$. We restrict our attention to a sequence of submatrices $A_n = [a_{st}] \in \mathbb{M}_n$ of $M\chi_E$ obtained by selecting the columns and rows indexed respectively by

$$\{(g_1^{-1}g_2)^k \mid k = 0,\dots,n-1\}, \quad \{(g_1^{-1}g_2)^k g_1^{-1} \mid k = 0,\dots,n-1\}.$$

These matrices are of the following form:

$$a_{st} = \begin{cases} 1 & \text{if } s = t \text{ or } t = s+1,\\ 0 & \text{otherwise,}\end{cases} \quad \text{that is,} \quad A_n = \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1\end{bmatrix}_{n\times n}.$$

Note that if $\Lambda = \{a_1, a_2\}$ satisfies the Leinert condition, then as $a_1a_2^{-1}$ has infinite order, A_n is a submatrix of $M\chi_\Lambda$.

LEMMA 8. *Let A_n be the $n\times n$ matrix defined above. Then*

$$\lim_{n\to\infty} \|A_n\| = \frac{4}{\pi}.$$

Proof. We first show that $\lim_{n\to\infty} \|A_n\| \ge 4/\pi$. Note that

$$\|A_n\| = \sup_{UU^*=1} \|A_n * U\| \ge \sup_{UU^*=1} \frac{|\mathrm{tr}(A_n * U)|}{n} = \frac{\mathrm{tr}\,|A_n|}{n}$$

where $|A_n| = (A_n^*A_n)^{1/2}$. Observe that $A_n^*A_n$ is the tridiagonal matrix

$$\begin{bmatrix} 1 & 1 & & \\ 1 & 2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2\end{bmatrix}$$

and its eigenvalues are given by

$$\lambda_k = 2 + 2\cos\left(\frac{2k\pi}{2n+1}\right) = 4\cos^2\left(\frac{k\pi}{2n+1}\right) \quad \text{for } k = 1, \dots, n.$$

Thus

$$\operatorname{tr}|A_n| = \operatorname{tr}((A_n^* A_n)^{1/2}) = \sum_{k=1}^{n} \sqrt{\lambda_k} = \sum_{k=1}^{n} 2\cos\left(\frac{k\pi}{2n+1}\right) = -1 + \frac{\sin\left(\frac{2n+1}{4n+2}\pi\right)}{\sin\left(\frac{\pi}{4n+2}\right)}.$$

Passing to the limit, we get

$$\lim_{n\to\infty} \|A_n\| \geq \lim_{n\to\infty} \frac{\operatorname{tr}|A_n|}{n} = \frac{4}{\pi}.$$

Next, we show that $\lim_{n\to\infty} \|A_n\| \leq 4/\pi$. The idea of the proof is to twiddle A_n by throwing in an extra 1 at the lower left corner. More precisely, we will construct for each n a new matrix $B_n = [b_{st}] \in \mathbb{M}_n$ which is defined through

$$b_{st} = \begin{cases} 1 & \text{if } s = t \text{ or } s = t+1, \\ 1 & s = n \text{ and } t = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{that is,} \qquad B_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Let us observe that

$$B_{n+1} = \begin{bmatrix} & & & 0 \\ & A_n & & \vdots \\ & & & 1 \\ 1 & 0 & \cdots & 1 \end{bmatrix}.$$

It is a very general fact on multipliers that if A is a submatrix of B then the multiplier norm of A is smaller than the one of B.

In what follows, we will show that $\lim_{n\to\infty} \|B_n\| = 4/\pi$. We mention that this calculation has also been done by Donsig and Davidson in [4], using the result [13, Theorem 2.6] of Mathias, but we include it here for the sake of completeness. Let $G = \mathbb{Z}/n\mathbb{Z}$; then $B_n = M\chi_{\{\dot{0},\dot{1}\}}$. As G is abelian, $\|M\chi_{\{\dot{0},\dot{1}\}}\| = \|\widehat{\chi_{\{\dot{0},\dot{1}\}}}\|_{L_1(\hat{G})}$, so

$$\begin{aligned} \|B_n\| &= \frac{1}{n} \sum_{k=0}^{n-1} |1 + e^{i2k\pi/n}| \\ &= 4\frac{\sin\left(\lfloor \frac{n-1}{2} \rfloor \frac{\pi}{n} + \frac{\pi}{2n}\right) \sin\left(\frac{\pi}{2n}\right)}{n\left(1 - \cos\left(\frac{\pi}{n}\right)\right)} \xrightarrow{n\to\infty} \int_0^1 |1 + e^{i2\pi t}|\, dt = \frac{4}{\pi}. \end{aligned}$$

And finally

$$\lim_{n\to\infty} \|A_n\| = \frac{4}{\pi}.$$

This concludes the proof of the lemma. ∎

REMARK 9. Let $\Lambda = \{a_1, a_2\} \subset G$ be a $\mathcal{L}c$-set. Note that in the full matrix representation of $M\chi_{\{a_1,a_2\}}$, there are exactly two 1's in each column and each row.

LEMMA 10. *Let $\Lambda = \{a_1, a_2\} \subset G$ be a $\mathcal{L}c$-set. If a matrix $A \in \mathbb{M}_{n,m}$ is a submatrix of $M\chi_{\{a_1,a_2\}}$ and is chainable then $|m - n| \le 1$, and up to row and column operations, A can be recovered from A_{n+1} by taking the first n rows and m columns or the last n rows and m columns. Hence $\|A_n\| \le \|A\| \le \|A_{n+1}\|$.*

Proof. Note that by Lemma 6, A has no cycle. Moreover, by the previous remark, there are also at most two 1's in each column and row.

Consider the longest chain in A, say $a_{i_1,j_1} = a_{i_2,j_2} = \cdots = a_{i_k,j_k}$ with $(i_l, j_l) \neq (i_{l+1}, j_{l+1})$ but $i_l = i_{l+1}$ or $j_l = j_{l+1}$.

Up to permutations, we can assume that $j_1 = i_1 = 1$, and also that $i_2 = i_1$. Up to a row permutation, we have $j_2 = 2$. As there are at most two 1's in each row and column, we cannot have $i_3 = 1$, so say $i_3 = 2$ and then $j_3 = 2$. Then, as there are two 1's in the second column, we must have $j_4 \neq 2$, so $i_4 = 2$. Next, as there is no cycle we must have $i_4 \neq 1$, so say $i_4 = 3$. Continuing this procedure, we find that the chain has the same shape as the one from the first l rows and p columns of the matrix A_{l+1} with $|p - l| \le 1$ and it goes from $(1, 1)$ to (l, p). We then look at the other possible (i, j) with $a_{i,j} = 1$. First, $i \le l$ and $j > p$ is impossible as it would contradict either the maximality of the chain (if $i = l$) or the fact that there are at most two 1's in each row. Secondly, it cannot be $(i, 1)$ or (i, p) with $i > l$ as the chain starts at $(1, 1)$ and we choose it maximal. Finally, because of the number of 1's or the cycle condition, we must have $i > l$ and $j > p$, which is impossible because A is chainable. Hence, we have exhausted all the 1's in A, and by chainability, we must have $m = p$ and $n = l$ and we are done. ∎

We are now ready to prove a particular case of the main result of this paper:

THEOREM 11. *Assume $\Lambda = \{a_1, a_2\} \subset G$ satisfies the Leinert condition. Then*

$$\|\chi_{\{a_1,a_2\}}\|_{M_{cb}A(G)} = \|M\chi_{\{g_1,g_2\}}\|_{M_{cb}A(\mathbb{F}_2)} = \frac{4}{\pi}.$$

Proof. Let A be a submatrix of $M\chi_{\{a_1,a_2\}}$. Then by Property 5, A is, up to permutations, a block diagonal matrix whose blocks are chainable. Since each of these blocks must be of the form in Lemma 10, this gives us $4/\pi$ as the upper bound (recall that the Schur norm of a block diagonal matrix is the supremum of the Schur norm of the blocks). To show the reverse inequality, it is enough to note that each A_n, whose norm goes to $4/\pi$, is a submatrix of $M\chi_{\{a_1,a_2\}}$. ∎

In what follows, we present the combinatorial method for finding the Herz–Schur norm of χ_E, where E is the infinite set of all the generators of $\mathbb{F}_\infty$.

Lemma 12. *Let $\Lambda \subseteq G$ be an infinite set which satisfies the Leinert condition. Then the following are equivalent:*

(i) *A is an $n \times m$ submatrix of $M\chi_\Lambda$.*
(ii) *A contains no cycle.*

Proof. Note that (i)⇒(ii) is exactly (i)⇒(ii) in Lemma 6. To prove (ii)⇒(i), we let $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ be an infinite set and consider a matrix $A = [a_{ij}]_{i,j=1}^{n,m}$ with no cycle. We want to construct $\{s_1, \ldots, s_n\}$ and $\{t_1, \ldots, t_m\}$ with $s_i, t_j \in G$ for any $i, j \in \{1, \ldots, n\} \times \{1, \ldots, m\}$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } s_i^{-1} t_j \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Step 1. Suppose that A is chainable. To find the index sets $\{s_1, \ldots, s_n\}$ and $\{t_1, \ldots, t_m\}$, we will first transform A into a more manageable matrix, by permuting rows and columns if necessary. In what follows, we will give a block form of A. To fix notation, we decompose A into block matrices as $A = [X^{i,j}_{y_i,x_j}]$, where (i, j) denotes the position of the block $X^{i,j}_{y_i,x_j}$ in A and (y_i, x_j) represents its size (if one of them is 0, this means that there is no block). Also, we will use the notation C for a matrix that has exactly one 1 in each of its columns and 0's elsewhere, R for the matrix that has exactly one 1 in each of its rows and 0's elsewhere, and X for a general matrix with 0's and 1's.

For simplicity, we assume $n > m$, otherwise we can add rows to A.

As a first step, we use permutations to put a 1 at the $(1, 1)$ position. We let $x_1 = y_1 = 1$ and we obtain

$$A \sim \begin{bmatrix} 1 & X^{1,2}_{y_1,m-x_1} \\ X^{2,1}_{n-y_1,x_1} & X^{2,2}_{n-y_1,m-x_1} \end{bmatrix}.$$

Here, $\sim$ stands for equality up to a permutation of rows and columns.

We start by separating the columns of $X^{1,2}_{y_1,m-x_1}$ containing one 1 from those with 0's and put them on the left hand side. Let us suppose that there are k columns, each with exactly one 1.

Similarly, for $X^{2,1}_{n-y_1,x_1}$, we separate the rows containing a 1 and we put them in the top part of the matrix. Let us suppose that there are l rows, each with exactly one 1. We obtain

$$A \sim \begin{bmatrix} 1 & C^{1,2}_{y_1,k} & O \\ R^{2,1}_{l,x_1} & X^{2,2}_{l,k} & X^{2,3}_{l,m-x_1-k} \\ O & X^{3,2}_{n-y_1-l,k} & X^{3,3}_{n-y_1-l,m-x_1-l} \end{bmatrix} \tag{4}$$

where O represents the zero matrix of the appropriate size.

Observe now that k and l cannot be 0 simultaneously. Suppose that $k = l = 0$. Then

$$A \sim \begin{bmatrix} 1 & O_{1,m-1} \\ O_{n-1,1} & * \end{bmatrix}.$$

which is not chainable (unless $m = n = 1$). Hence we can assume that $k > 0$ (by permuting rows and columns in all arguments if necessary). We set $x_2 = k$ and $y_2 = l$. The next step is to notice that $X^{2,2}_{l,k}$ in (4) must be the zero matrix (or, if $x_2 y_2 = 0$, the matrix will not appear in the decomposition), otherwise we again have a cycle. Hence,

$$A \sim \begin{bmatrix} 1 & C^{1,2}_{y_1,x_2} & O \\ R^{2,1}_{y_2,x_1} & O & X^{2,3}_{y_2,m-x_1-x_2} \\ O & X^{3,2}_{n-y_1-y_2,x_2} & X^{3,3}_{n-y_1-y_2,m-d-x_2} \end{bmatrix}. \tag{5}$$

We continue the procedure by looking at $X^{2,3}$, and since we are not allowing cycles in the matrix, we see that each column has at most one 1. By the same argument, $X^{3,2}$ has at most one 1 in each row. The blocks $X^{2,3}$ and $X^{3,2}$ cannot both be zero. We keep decomposing A into blocks of type C and R, until we exhaust all the 1's. Eventually,

$$A \sim \begin{bmatrix} 1 & C^{1,2}_{y_1,x_2} & O & O & O & \cdots & O \\ R^{2,1}_{y_2,x_1} & O & C^{2,3}_{y_2,x_3} & O & O & \cdots & O \\ O & R^{3,2}_{y_3,x_2} & O & C^{3,4}_{y_3,x_4} & O & \cdots & O \\ O & O & \ddots & \ddots & \ddots & \cdots & O \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & O \\ \cdots & \cdots & \cdots & & & O & C^{k-1,k}_{y_{k-1},x_k} \\ \cdots & \cdots & \cdots & & & R^{k,k-1}_{y_k,x_{k-1}} & O \end{bmatrix} \tag{6}$$

where $x_i, y_i \geq 0$, $x_i + y_i > 0$, $\sum_{i=1}^k y_i = n$ and $\sum_{i=1}^k x_i = m$.

Next, we will select columns t_j and rows s_i from $M\chi_\Lambda$ to get A. For convenience, we use double indices to denote the elements of Λ, more precisely $\Lambda = \{\lambda_{i,j} \mid i \geq 1,\ j \in \mathbb{Z}\}$.

We begin with the 1 in the top left corner of $A = (a_{i,j})$, which is indexed by $t_1 = e$ and $s_1 = \lambda_{1,1}^{-1}$. Obviously $\chi_\Lambda(s_i^{-1}t_j) = a_{i,j}$ for $i \le y_1$ and $j \le x_1$.

Let us assume that $t_1, \dots, t_{x_1+\cdots+x_l}$ and $s_1, \dots, s_{y_1+\cdots+y_l}$ have been constructed. Then, for the next x_{l+1} columns, we choose $t_{x_1+\cdots+x_l+k} = s_{y_1+\cdots+y_{l-1}+\alpha_k}\lambda_{l+1,k}$ for $k = 1, \dots, x_{l+1}$ where α_k is the unique index such that the block $(l, l+1)$ of A has a 1 at position (α_k, k). For the next y_{l+1} rows, set $s_{y_1+\cdots+y_l+k} = t_{x_1+\cdots+x_{l-1}+\beta_k}\lambda_{l+1,-k}^{-1}$ for $k = 1, \dots, y_{l+1}$ where β_k is such that the block $(l+1, l)$ of A has a 1 at position (k, β_k).

By construction, $s_i^{-1}t_j \in \Lambda$ if $a_{i,j} = 1$. Conversely, $s_i^{-1}t_j$ is a product of the form $\lambda_{\alpha_1}\lambda_{\alpha_2}^{-1}\cdots\lambda_{\alpha_{2l+1}}$. According to the Leinert condition, this can reduce to some λ_p if and only if we have trivial simplifications (i.e. $\alpha_1 = p$ and $\alpha_{2u} = \alpha_{2u+1}$, or $\alpha_{2l+1} = p$ and $\alpha_{2u} = \alpha_{2u-1}$). One can easily check that this occurs only when $a_{i,j} = 1$. In order to illustrate this technique, we give an example (where the index of each 1 stands for the number of the new generator used):

$$\left[\begin{array}{c|ccc|cc}
1_1 & 1_2 & 1_3 & 1_4 & & \\
\hline
1_5 & & & & & 1_8 \\
1_6 & & & & 1_7 & \\
\hline
 & & 1_9 & & & \\
 & & & 1_{10} & &
\end{array}\right].$$

The indexing sets are, for the columns:

$$\{e, \lambda_1^{-1}\lambda_2, \lambda_1^{-1}\lambda_3, \lambda_1^{-1}\lambda_4, \lambda_6^{-1}\lambda_7, \lambda_5^{-1}\lambda_8\},$$

and for the rows:

$$\{\lambda_1^{-1}, \lambda_5^{-1}, \lambda_6^{-1}, \lambda_1^{-1}\lambda_3\lambda_9^{-1}, \lambda_1^{-1}\lambda_4\lambda_{10}^{-1}\}.$$

STEP 2. If A is not chainable, then by Property 5 there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & & & \\ & \ddots & & \\ & & A_N & \\ & & & O \end{bmatrix},$$

where $A_1, \dots, A_N$ are chainable matrices and O is a rectangular zero matrix.

We apply the argument from Step 1 to the chainable matrices $A_1, \dots, A_N$ with indices coming from different disjoint subsets of Λ and choose other different elements from Λ for row and columns of O. The conclusion then follows. ∎

To sum up, when $\Lambda \subset G$ is an infinite $\mathcal{L}c$-set, then M_{χ_Λ} and M_{χ_E} have the same submatrices. The formula (1) implies the following:

Corollary 13. *If $\Lambda \subset G$ is infinite and satisfies the Leinert condition, then*

$$\|\chi_\Lambda\|_{M_{cb}A(G)} = \|\chi_E\|_{M_{cb}A(\mathbb{F})}.$$

Thus to estimate the multiplier norm of the characteristic function of an infinite $\mathcal{L}c$-set, one only needs to compute it for one concrete example. Actually, $E = \{g_1^{\pm 1}, \ldots, g_n^{\pm 1}, \ldots\}$ is such an example and to calculate its norm, we may rely on Theorem 4.2 of [10], which is Theorem 1 above. Note that the radial multiplier χ_E corresponds to the projection onto words of length one. We obtain $H = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and the following corollary:

Corollary 14. *Let $\Lambda \subset G$ be an infinite $\mathcal{L}c$-set. Then*

$$\|\chi_\Lambda\|_{M_{cb}A(G)} = 2.$$

For completeness, we will explain how to obtain this value directly. We first show that the norm is at most 2, by using the pattern of Lemma 12. Note that a similar argument can be found in [9]. Then we show this inequality is in fact an equality.

Proof. Let A be any submatrix of $M\chi_\Lambda$. Since the Schur norm of a matrix with blocks on the diagonal is the maximum of the Schur norms of the blocks, we can focus on the case where A is chainable. By the proof of Lemma 12, A is of the form (6). We can write $A = X + Y$, where X contains only the blocks of A of type R and Y those of type C and a 1 at the top left corner. Since both matrices Y and X have at most one 1 in each column and at most one 1 in each row respectively, we see that the Schur norm of each matrix is less than 1. Hence,

$$\|\chi_\Lambda\|_{M_{cb}A(G)} \le 2.$$

To prove equality, we use duality and the corresponding multiplier on the group algebra. Since we can choose a particular $\mathcal{L}c$-set, we take $\Lambda = \{g_i^{\pm 1} \mid i \ge 1\} \subset \mathbb{F}_\infty$.

Let $\mathcal{O}_n$ be the Cuntz algebra with n generators, which is the universal C^*-algebra generated by isometries $(u_i)_{i=1}^n$ satisfying the relation $\sum_{i=1}^n u_i u_i^* = 1$.

Fix $n \ge 1$, and consider $S \in \mathcal{O}_n \otimes \mathrm{VN}(\mathbb{F}_\infty)$ given by

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n u_i \otimes \lambda(g_i).$$

This is an isometry as $S_n^* S_n = 1$, but it is not unitary since $\mathrm{Id} \otimes \tau(S_n S_n^*) = (1/n)1$ (with τ the trace on $\mathrm{VN}(\mathbb{F}_\infty)$). Let $U_n = S_n(1 - S_n S_n^*) + (1 - S_n S_n^*) S_n^*$. It is easily checked that U_n is self-adjoint and U_n^2 is a projection. Hence $\|U_n\|_{\mathcal{O}_n \otimes \mathrm{VN}(\mathbb{F}_\infty)} = 1$.

Next, we compute $\mathrm{Id}\otimes\chi_E(U_n)$. First,

$$\begin{aligned} S_nS_nS_n^* &= \frac{1}{n^{3/2}}\sum_{i,j,k=1}^{n} u_iu_ju_k^*\otimes\lambda(g_ig_jg_k^{-1}) \\ &= \frac{1}{n^{3/2}}\sum_{i=1}^{n} 1\otimes\lambda(g_i)+\frac{1}{n^{3/2}}\sum_{\substack{i,j,k=1\\ j\neq k}}^{n} u_iu_ju_k^*\otimes\lambda(g_ig_jg_k^{-1}), \end{aligned}$$

which gives

$$\mathrm{Id}\otimes\chi_E(S_nS_nS_n^*)=\frac{1}{n^{3/2}}\sum_{i=1}^{n}1\otimes\lambda(g_i).$$

Hence $\|\mathrm{Id}\otimes\chi_E(S_nS_nS_n^*)\|\le 1/\sqrt{n}$ and similarly for its adjoint. Of course, $\mathrm{Id}\otimes\chi_E(S_n+S_n^*)=S_n+S_n^*$. To conclude, recall that for any non-unitary isometry, $\|S_n+S_n^*\|=2$, which implies

$$\|\mathrm{Id}\otimes\chi_E(U_n)\|\ge\|S_n+S_n^*\|-\frac{2}{\sqrt{n}}=2-\frac{2}{\sqrt{n}},$$

and we are done. ∎

Actually, using all the S_k's one can easily derive the lower bound in Theorem 1 (for $n=\infty$).

We briefly indicate the shape of matrices A on which the Schur multiplier norm is almost achieved. For d, $l>0$, let $C_{d,l}=(c_{(i_1,\dots,i_{l-1}),(j_1,\dots,j_l)})$ be the $d^{l-1}\times d^l$ matrix with $c_{(i_1,\dots,i_{l-1}),(j_1,\dots,j_l)}=\delta_{(i_1,\dots,i_{l-1}),(j_1,\dots,j_{l-1})}$ and $R_{d,l}=C_{d,l}^*$. With the notation (6), one can choose $k=d$, $C^{i,i+1}=C_{d,i}$, $R^{i+1,i}=R_{d,i}$ and let $d\to\infty$ (one does not need the 1 at the $(1,1)$ position). In terms of Fock spaces, the description of the shift S_d is not as nice as the one above (unless we go to the completely bounded side).

One can extend this combinatorial method to finite $\mathcal{L}c$-sets, but the statements and their proofs become really heavy.

5. The Herz–Schur norm calculation—the algebraic method. We start with the following well known properties of the Herz–Schur multiplier norm:

Proposition 15. *Let G be a discrete group.*

(i) *If $E\subseteq G$ and $x\in G$, then $\|\chi_E\|_{M_{cb}A(G)}=\|\chi_{xE}\|_{M_{cb}A(G)}$.*
(ii) *If H is a subgroup of G and $E\subseteq H$, then $\|\chi_E\|_{M_{cb}A(G)}=\|\chi_E\|_{M_{cb}A(H)}$.*

Proof. For x in G let τ_x be translation by x on $A(G)$; it is a complete isometry. Then (i) follows from $\tau_x\chi_E\tau_{x^{-1}}=\chi_{xE}$.

It is well known (see [5]) that for any subgroup H of G, the extension map

$$\iota : A(H) \to A(G), \qquad \iota(\phi)(g) = \begin{cases} \phi(g) & \text{if } g \in H, \\ 0 & \text{otherwise,} \end{cases}$$

is a complete embedding. Moreover, the restriction map ρ is a left inverse for ι and is a complete contraction. Then $\chi_E^G = \rho\chi_E^H\iota$, which gives the estimates. ∎

Now, let $\mathbb{F}_2$ be the free group on two generators g_1 and g_2. Then it is trivial to see that the set $\{e, g_1^{-1}g_2\}$ satisfies the Leinert condition. (In fact, it can be verified that any set of cardinality 2 satisfies the Leinert condition if $g_1^{-1}g_2$ has infinite order.) Let $\mathbb{Z}$ be the group generated by $g_1^{-1}g_2$. We have

$$\begin{aligned} \|\chi_{\{g_1,g_2\}}\|_{M_{cb}A(\mathbb{F}_2)} &= \|\chi_{g_1\{e,g_1^{-1}g_2\}}\|_{M_{cb}A(\mathbb{F}_2)} = \|\chi_{\{e,g_1^{-1}g_2\}}\|_{M_{cb}A(\mathbb{F}_2)} \\ &= \|\chi_{\{e,g_1^{-1}g_2\}}\|_{M_{cb}A(\mathbb{Z})} = \|\chi_{\{e,g_1^{-1}g_2\}}\|_{B(\mathbb{Z})} \\ &= \|\chi_{\{e,g_1^{-1}g_2\}}\|_{A(\mathbb{Z})} = \|\widehat{\chi_{\{e,g_1^{-1}g_2\}}}\|_{L_1(\hat{\mathbb{Z}})} \\ &= \|\widehat{\chi_{\{e,g_1^{-1}g_2\}}}\|_{L_1(\mathbb{T})}, \end{aligned}$$

where $\widehat{\chi}_{\{e,g_1^{-1}g_2\}}(z) = 1 + z$. Hence

$$\|\chi_{\{g_1,g_2\}}\|_{M_{cb}A(\mathbb{F}_2)} = \frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\theta}|\, d\theta = \frac{4}{\pi}.$$

A key ingredient in this section is the following characterization of the Leinert condition:

Proposition 16. *Let G be a discrete group. Then $\Lambda \subseteq G$ satisfies the Leinert condition if and only if $u\Lambda$ is free in $G * \mathbb{Z}$, where u is a generator of $\mathbb{Z}$.*

Proof. Let $\Lambda = \{a_1, a_2, \ldots\}$ be a $\mathcal{L}c$-subset of G. We have to show that $u\Lambda = \{ua_1, ua_2, \ldots\}$ is free. To this end, we analyze the form of a product

$$w = (ua_{i_1})^{\epsilon_1}(ua_{i_2})^{\epsilon_2}\cdots(ua_{i_n})^{\epsilon_n}$$

with $\epsilon_k = \pm 1$ and we assume that there is no trivial simplification, that is, $\epsilon_k\epsilon_{k+1} \neq -1$ or $i_k \neq i_{k+1}$.

We describe the resulting word w in letters a_i's and u after all possible cancellations of u's. We will write E (resp. $\overline{E}$) for a word of the form $a_{i_l}a_{i_{l+1}}^{-1}\cdots a_{i_{l+2(p-1)}}a_{i_{l+2p-1}}^{-1}$ (resp. $a_{i_l}^{-1}a_{i_{l+1}}\cdots a_{i_{l+2(p-1)}}^{-1}a_{i_{l+2p-1}}$). Note that our assumption is exactly that such elements in G are different from the identity. Similarly for odd products, we write O (resp. $\overline{O}$) for a word of the form $a_{i_l}a_{i_{l+1}}^{-1}\cdots a_{i_{l+2(p-1)}}$ (resp. $a_{i_l}^{-1}a_{i_{l+1}}\cdots a_{i_{l+2(p-1)}}^{-1}$). Hence $(uO)^k$ means a word of the form $uO_1\cdots uO_k$. If $\epsilon_1 = 1$, after all possible cancellations of

u's, w will have one of the following forms:

$$w_1 = \prod_{s=1}^{p} \big((uO)^{k_s}(uEu^{-1})(\bar{O}u^{-1})^{l_s}\bar{E}\big) \qquad \text{with } p \geq 1,$$

$$w_2 = \Big(\prod_{s=1}^{p-1} \big((uO)^{k_s}(uEu^{-1})(\bar{O}u^{-1})^{l_s}\bar{E}\big)\Big)(uO)^{k_p} \qquad \text{with } p \geq 1,\ k_p \geq 1,$$

$$w_3 = \Big(\prod_{s=1}^{p-1} \big((uO)^{k_s}(uEu^{-1})(\bar{O}u^{-1})^{l_s}\bar{E}\big)\Big)(uO)^{k_p}(uEu^{-1})(\bar{O}u^{-1})^{l_p} \qquad \text{with } p \geq 1,\ k_p, l_p \geq 0.$$

This can be checked by induction on n. When $n = 1$, $w = ua_{i_1}$ so it is of the form w_2. Then we have the following rules for multiplication on the right:

$$w_1 ua_{i_{n+1}} \to w_2, \quad w_1(ua_{i_{n+1}})^{-1} \to w_3, \quad w_2 ua_{i_{n+1}} \to w_2,$$
$$w_2(ua_{i_{n+1}})^{-1} \to w_3, \quad w_3 ua_{i_{n+1}} \to w_2 \quad \text{if } l_p = 0,$$
$$w_3 ua_{i_{n+1}} \to w_1 \quad \text{if } l_p > 0, \quad w_3(ua_{i_{n+1}})^{-1} \to w_3.$$

Now assume that w is the identity. Then, as all u must cancel, $\sum \epsilon_i = 0$, in particular n is even. As $xy = 1$ implies $yx = 1$, conjugating w if necessary, we may assume that $\epsilon_1 = 1$. Taking into account that some words of the form O may cancel, but those of the form E are different from e, we see from the above description that w has block length greater than 3. Consequently, $w \neq e$ and $u\Lambda$ is free.

To prove the other implication, we note that $u\Lambda$ being free implies that $u\Lambda$ satisfies the Leinert condition, which further implies that $\Lambda = u^{-1}(u\Lambda)$ also does. ∎

Let G be any discrete group and $\Lambda \subset H$ a $\mathcal{L}c$-subset, say $\Lambda = \{a_1, a_2, \ldots\}$. Let $\Gamma = G * \mathbb{Z}$ with u a generator of $\mathbb{Z}$. By the above proposition, the set $\{ua_1, ua_2, \ldots\}$ is the set of generators of a copy of the free group $\mathbb{F}_{|\Lambda|}$ in Γ. As usual, we denote by g_i the generators of the free group. Now, by Proposition 15,

$$\|\chi_{\{a_1\ldots,a_{|\Lambda|}\}}\|_{M_{cb}A(G)} = \|\chi_{\{a_1,\ldots,a_{|\Lambda|}\}}\|_{M_{cb}A(\Gamma)} = \|\chi_{\{ua_1,\ldots,ua_{|\Lambda|}\}}\|_{M_{cb}A(\Gamma)}$$
$$= \|\chi_{\{ua_1,\ldots,ua_{|\Lambda|}\}}\|_{M_{cb}A(\mathbb{F}_{|\Lambda|})} = \|\chi_{\{g_1,\ldots,g_{|\Lambda|}\}}\|_{M_{cb}A(\mathbb{F}_{|\Lambda|})}.$$

This leads us to the following theorem:

THEOREM 17. *Let G be a discrete group and Λ be a $\mathcal{L}c$-subset. Then*

$$\|\chi_\Lambda\|_{M_{cb}A(G)} = \|\chi_{\{g_1,\ldots,g_{|\Lambda|}\}}\|_{M_{cb}A(\mathbb{F}_{|\Lambda|})}.$$

Hence, the norm $\|\chi_\Lambda\|_{M_{cb}A(G)}$ depends only on $|\Lambda|$. To compute it, we only need to do so on a particular example. But the set of generators of $*_{i=1}^{n}\mathbb{Z}_2$ is an $\mathcal{L}c$-set of cardinality n; according to Theorem 2, these values can be evaluated using the following:

COROLLARY 18. *Let $\mathbb{F}_n$ be the free group with n generators and E be a subset of all its generators. Then*

$$\|\chi_E\|_{M_{cb}A(\mathbb{F}_n)} = \frac{n-2}{n-1}\left\|\left(I - \frac{\tau}{n-1}\right)^{-1} H\right\|_1,$$

where $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

6. Remarks on the multiplier norm. In this section we will focus on the multiplier norm. The algebraic method also works in this case:

THEOREM 19. *Let G be a discrete group and Λ be a subset satisfying the Leinert condition. Then*

$$\|\chi_\Lambda\|_{MA(G)} = \|\chi_{\{g_1,\dots,g_{|\Lambda|}\}}\|_{MA(\mathbb{F}_{|\Lambda|})}.$$

It is obvious that $\|\chi_{\{g_1,\dots,g_{|\Lambda|}\}}\|_{MA(\mathbb{F}_{|\Lambda|})} \le \|\chi_{\{g_1,\dots,g_{|\Lambda|}\}}\|_{M_{cb}A(\mathbb{F}_{|\Lambda|})} \le 2$. Unlike in the completely bounded case, we have no nice formula in this setting. We can however prove the following estimate:

THEOREM 20. *Let E be an infinite set of generators of $\mathbb{F}$. Then*

$$\frac{16}{3\pi} \le \|\chi_E\|_{MA(\mathbb{F})} < 2.$$

Consequently, the bounded and completely bounded norms of χ_E on $A(\mathbb{F})$ are different.

For convenience, we turn to the dual notion and look at the multiplier norm on $\mathrm{VN}(\mathbb{F})$.

Let H be the subspace of $\mathrm{VN}(\mathbb{F})$ generated by the $\lambda(g_i)$'s. It has been shown in [1] that on H, the L_2 and L_∞ norms are equivalent. More precisely, for $\alpha_1, \dots, \alpha_n \in \mathbb{C}$,

$$\Big(\sum_{i=1}^n |\alpha_i|^2\Big)^{1/2} \le \Big\|\sum_{i=1}^n \alpha_i\lambda(g_i)\Big\|_{\mathrm{VN}(\mathbb{F})}$$
$$= \min\Big\{2t + \sum_{i=1}^n (\sqrt{t^2+|\alpha_i|^2} - t) \;\Big|\; t \ge 0\Big\} \le 2\Big(\sum_{i=1}^n |\alpha_i|^2\Big)^{1/2}$$

and 2 is the best possible constant. Moreover, this formula holds for any family satisfying the Leinert condition.

LEMMA 21. *We have*

$$\|\chi_E\|_{MA(\mathbb{F})} \le \sup_{x\in H,\, \|x\|_2=1} \|x\|_1\|x\|_\infty.$$

Proof. This is a very general fact as χ_E is an orthogonal projection on L_2. Take $y \in \mathrm{VN}(\mathbb{F})$; then $y = \lambda x + z$ with x in H, $\|x\|_2 = 1$ and z orthogonal to x and $\|\chi_E(y)\|_\infty = |\lambda| \cdot \|x\|_\infty$. But thanks to Hölder's inequality we have

$$|\lambda| = |\tau(yx^*)| \leq \|x\|_1 \|y\|_\infty. \blacksquare$$

Proof of the upper bound in Theorem 20. It suffices to prove that $\sup_{x\in H,\, \|x\|_2=1} \|x\|_1\|x\|_\infty < 2$. Assume that this is not true. Let $\varepsilon > 0$ and $x \in H$ with $\|x\|_2 = 1$ so that $\|x\|_1\|x\|_\infty > 2-\varepsilon$. Using the Akemann–Ostrand estimate, we have $2\tau(|x|) \geq 2-\varepsilon$ and $\|x\|_\infty \geq 2-\epsilon$ as $\|x\|_1 \leq \|x\|_2 = 1$. Hence,

$$\big\| |x| - 1\big\|_2^2 = \tau(x^*x) - 2\tau(|x|) + 1 \leq \varepsilon.$$

On the one hand,

$$\big\| |x|^2\big\|_2 \leq \big\| |x| \big\|_2 + \big\| |x|(|x|-1)\big\|_2 \leq 1 + \|x\|_\infty \big\| |x| - 1\big\|_2 \leq 1 + 2\sqrt{\varepsilon}.$$

But with $x = \sum_{i=1}^n \alpha_i \lambda(g_i)$,

$$|x|^2 = x^*x = \Big(\sum_{i=1}^n |\alpha_i|^2\Big)\lambda(e) + \sum_{i\neq j=1}^n \overline{\alpha_i}\alpha_j \lambda(g_i^{-1}g_j).$$

Hence

$$\big\| |x|^2\big\|_2^2 = \Big(\sum_{i=1}^n |\alpha_i|^2\Big)^2 + \sum_{i\neq j=1}^n |\alpha_i|^2|\alpha_j|^2 = 2\Big(\sum_{i=1}^n |\alpha_i|^2\Big)^2 - \sum_{i=1}^n |\alpha_i|^4.$$

Putting everything together, as $\sum_{i=1}^n |\alpha_i|^2 = 1$, with $\delta = (1+2\sqrt{\varepsilon})^2 - 1$ we get

$$\max_i |\alpha_i|^2 \geq \sum_{i=1}^n |\alpha_i|^4 \geq 1 - \delta.$$

For simplicity, we can assume that $|\alpha_1|^2 \geq 1-\delta$. Taking $t = 1/2$ in the Akemann–Ostrand formula and using $\sqrt{1/4+u^2} - 1/2 \leq u^2$, we have

$$2-\epsilon \leq \|x\|_\infty \leq 1 + \sqrt{\frac{1}{4} + |\alpha_1|^2} - \frac{1}{2} + \sum_{i=2}^n |\alpha_i|^2 = -|\alpha_1|^2 + \sqrt{\frac{1}{4} + |\alpha_1|^2} + \frac{3}{2}.$$

As $\varepsilon \to 0$, $|\alpha_1|$ goes to 1, and we obtain $3/2 \leq \sqrt{5}/2$ which is false. This completes the proof of the upper bound. $\blacksquare$

Proof of the lower bound in Theorem 20. First, we know that we can replace E by any infinite $\mathcal{L}c$-set. We choose E to be the set of all the generators and their inverses. Fix an integer $n \geq 0$ and let

$$s_n = \frac{1}{\sqrt{2n}} \sum_{i=1}^n (\lambda(g_i) + \lambda(g_i^{-1})).$$

This is a self-adjoint element which generates a commutative von Neumann algebra M_n in $\mathrm{VN}(\mathbb{F})$. It is easy to see that s_n is the only element in $\mathrm{VN}(\mathbb{F}_n) \cap H$ that is invariant under the automorphisms of $\mathrm{VN}(\mathbb{F}_n) \cap H$ corresponding to permutations of the generators and their inverses (i.e. $g_i \mapsto g_i^{-1}$). Consequently, for any element $x \in M_n$, we have $\chi_E(x) \in \mathbb{C}s_n$. Recall that on L_2, χ_E is an orthogonal projection. Restricting to M_n, χ_E is the rank one operator p given by $p(x) = \tau(xs_n).s_n$. So $\|\chi_E\|_{M_n} = \|s_n\|_1\|s_n\|_\infty$.

We note that the density of s_n with respect to the Lebesgue measure can be computed explicitly using the R-transform (see [16]). However, our proof does not make use of this formula. Next, by the non-commutative central limit theorem, s_n tends to a semicircle random variable s in moments and in distribution. Hence,

$$\|s_n\|_1 \to \frac{1}{2\pi}\int_{-2}^{2} |t|\sqrt{4-t^2}\,dt = \frac{8}{3\pi}.$$

Also $\|s_n\|_\infty \to 2 = \|s\|_\infty$ by [1]. Finally

$$\|\chi_E\|_{MA(\mathbb{F})} \geq \|s\|_1\|s\|_\infty = \frac{16}{3\pi}. \ \blacksquare$$

It has been shown in [1] that $\sup_{x\in H\cap \mathrm{VN}(\mathbb{F}_n),\, \|x\|_2=1} \|x\|_\infty$ is achieved for $x = s_n$. It is likely that this is also true for $\sup_{x\in H\cap \mathrm{VN}(\mathbb{F}_n),\, \|x\|_2=1} \|x\|_1\|x\|_\infty$, but we do not know how to prove it. For a general $x = \sum_{i=1}^n \alpha_i\lambda(g_i)$, its 1-norm coincides with that of $\sum_{i=1}^n \alpha_i\varepsilon_i$, where ε_i are free Bernoulli variables. Unfortunately, the R-transform of this type of random variable is hard to invert (except when all α_i's are equal), and it is difficult to compute its 1-norm.

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