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COMPOSITION OF AXIAL FUNCTIONS OF PRODUCTS OF FINITE SETS

BY

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To the memory of my Father

Abstract. We show that every function $f\colon A\times B\to A\times B$, where $|A|\le 3$ and $|B|<\omega$, can be represented as a composition $f_1\circ f_2\circ f_3\circ f_4$ of four axial functions, where f_1 is a vertical function. We also prove that for every finite set A of cardinality at least 3, there exist a finite set B and a function $f\colon A\times B\to A\times B$ such that $f\ne f_1\circ f_2\circ f_3\circ f_4$ for any axial functions f_1,f_2,f_3,f_4, whenever f_1 is a horizontal function.

A function $f\colon A\times B\to A\times B$ is called *vertical* ($f\in \mathrm{V}$) if there exists a function $f_1\colon A\times B\to A$ such that $f(a,b)=(f_1(a,b),b)$. It is called *horizontal* ($f\in\mathrm{H}$) if $f(a,b)=(a,f_2(a,b))$ for some function $f_2\colon A\times B\to B$. If f is horizontal or vertical then we call it *axial*. A one-to-one function from $A\times B$ onto $A\times B$ is called a *permutation* of $A\times B$. The family of all functions from $A\times B$ into $A\times B$ is denoted by $(A\times B)^{A\times B}$. If $F_1,\dots,F_n\subseteq (A\times B)^{A\times B}$, then we write $F_1F_2\dots F_n$ to denote $\{f_1\circ\dots\circ f_n : f_i\in F_i,\ i=1,\dots,n\}$.

It is convenient, especially when the set $A\times B$ is finite, to use matrices to represent functions from $(A\times B)^{A\times B}$. Given a function $f\in(A\times B)^{A\times B}$ and a matrix $M=[m_{(a,b)}]$ of size $|A|\times|B|$, we define $f[M]=[m_{f(a,b)}]$. If the elements of the matrix $M=[m_{(a,b)}]$ are distinct, then the matrix $f[M]$ uniquely determines the function f. We will often identify the elements (a,b) of $A\times B$ and the corresponding entries $m_{(a,b)}$ of the matrix M. Observe also that if $f,g\in(A\times B)^{A\times B}$, then $f[g[M]]=(g\circ f)[M]$. To see this let $M'=g[M]$ and $M''=f[M']=f[g[M]]$. For $(a,b)\in A\times B$ we have $m''_{(a,b)}=m'_{(c,d)}$ for some $(c,d)\in A\times B$ such that $(c,d)=f(a,b)$. Now, note that $m'_{(c,d)}=m_{g(c,d)}$. Hence $m''_{(a,b)}=m_{g(c,d)}=m_{g(f(a,b))}=m_{(g\circ f)(a,b)}$.

Banach ([M, Problem 47]) asked whether every permutation function of a cartesian product of two infinite countable sets can be represented as a composition of finitely many axial functions. The question was answered

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affirmatively by Nosarzewska [N]. Research on this subject was continued by Ehrenfeucht and Grzegorek [EG, G]. They discussed the smallest possible number of axial functions needed. Also, they considered the case when both sets are finite. In particular, they proved the following.

THEOREM 1. *Let $f, p \in (A \times B)^{A\times B}$ and p be a permutation.*

(i) *Then $p = p_1 \circ p_2 \circ p_3 \circ p_4$, where all p_i are axial permutations of $A \times B$.*
(ii) *If A is finite, then $p = p_1 \circ p_2 \circ p_3$, where all p_i are axial permutations of $A \times B$ and $p_1 \in \mathrm{V}$.*
(iii) *If $A \times B$ is infinite, then $f = f_1 \circ f_2 \circ f_3$, where all f_i are axial functions.*
(iv) *If $A \times B$ is finite, then f can be represented as $f = f_1 \circ \cdots \circ f_6$, where all f_i are axial functions and $f_1 \in \mathrm{V}$.*

Let us mention here that Ehrenfeucht and Grzegorek also showed that it is not possible to decrease the numbers 4 in part (i) and 3 in parts (ii) and (iii) of the above theorem. In addition, in part (i) it cannot be specified that $p_1 \in \mathrm{V}$. However, it was not proved that 6 in part (iii) is the smallest possible. Later, Szyszkowski [S] proved that 6 can be decreased to 5. He also gave an example which showed that 5 cannot be decreased to 3. In his example one of the sets has at least 4 elements and the other one at least 5. Below, we present an example in which both sets have exactly three elements each. It is worth noting that, if one of the sets has at most two elements, then three axial functions are enough (see Remark 5).

EXAMPLE 2. *The number* 6 *in Theorem* 1(iv) *cannot be reduced to* 3.

Proof. Let

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Define $f \in (A \times B)^{A\times B}$ by

$$f[M] = \begin{bmatrix} i & g & e \\ d & h & b \\ h & a & c \end{bmatrix}.$$

We will justify why $f \neq f_1 \circ f_2 \circ f_3$ for any axial functions f_1, f_2, f_3 such that $f_1 \in \mathrm{V}$ (the case when $f_1 \in \mathrm{H}$ is analogous). Notice that for the equality $f[M] = (f_1 \circ f_2 \circ f_3)[M] = f_3[f_2[f_1[M]]]$ to hold, entries from each column of $f[M]$ would have to appear in different rows of $f_1[M]$. In particular, since the sets $\{i, d, h\}$ and $\{g, h, a\}$ form two columns of $f[M]$, the elements a, d, g

would have to appear in different rows of $f_1[M]$ than the element h. But this is impossible. ∎

The question, which remains open, is whether every function can be obtained as a composition of four axial functions. We give partial answers to this question.

THEOREM 3.

(i) *If* $|A| = 3$, *then* $(A \times B)^{A\times B} = \mathrm{VHVH}$.

(ii) *If* $|A| \geq 3$, *then there exists an integer* m_0 *such that* $(A \times B)^{A\times B} \neq \mathrm{HVHV}$ *whenever* $|B| \geq m_0$.

Theorem 3 implies the following.

COROLLARY 4. *There exist finite sets* A, B *and a function* $f\colon A \times B \to A \times B$ *such that* $f \in \mathrm{VHVH}$ *and* $f \notin \mathrm{HVHV}$.

Proof of Theorem 3. (i) First observe that it suffices to prove the result for functions $f\colon A \times B \to A \times B$ such that the entries in each row of the matrix $f[M]$ are all distinct, that is, $|f(\{a\}\times B)| = |B|$ for every $a \in A$. This is so because for an arbitrary function $f'\colon A \times B \to A \times B$, there exists a function f of the above type and a horizontal function $h\colon A\times B \to A\times B$ such that $f'[M] = h[f[M]] = (f \circ h)[M]$. Since the composition of two horizontal functions is a horizontal function, if $f \in \mathrm{VHVH}$, then also $f' \in \mathrm{VHVH}$. Hence, let f be a function such that the entries in each row of the matrix $f[M]$ are all distinct. It can be easily proved, by induction on $|B|$, that there exists a horizontal permutation $h\colon A \times B \to A \times B$ such that there exists a partition of B into three sets B_1, B_2, B_3 (some of which may be empty) with the following properties:

1. $|(f \circ h)(A \times \{b_1\})| = 1$ for every $b_1 \in B_1$,
2. $|(f \circ h)(A \times \{b_2\})| = 2$ for every $b_2 \in B_2$,
3. $|(f \circ h)^{-1}(\{m\})| \leq 2$ for every $m \in (f \circ h)(A \times B_3)$ and $|\{m \in (f \circ h)(A \times B_3)\colon |(f \circ h)^{-1}(\{m\})| = 2\}| \equiv 0 \bmod 3$,
4. if $(f \circ h)(A \times \{b\}) \cap (f \circ h)(A \times \{b'\}) \neq \emptyset$, then $b, b' \in B_3$.

For the matrix $h[f[M]]$ this means that each column with index in B_1 has all entries equal, each column with index in B_2 has only two different entries, and the number of entries appearing twice in the part of $h[f[M]]$ corresponding to B_3 is divisible by 3. In addition, columns with indices in B_3 are the only columns which can share entries with other columns. We have

$$h[f[M]] = \left[\begin{array}{ccc|ccc|cccc} a & b & \dots & p & q & \dots & v & x & y & \dots \\ a & b & \dots & p & s & \dots & v & w & z & \dots \\ a & b & \dots & r & s & \dots & w & x & z & \dots \end{array}\right].$$
$$\qquad\qquad\qquad \underbrace{\qquad\quad}_{B_1}\ \underbrace{\qquad\quad}_{B_2}\ \underbrace{\qquad\qquad}_{B_3}$$

So, if we can prove that $f \circ h \in \mathrm{VHVH}$, then, since h is a horizontal permutation, we will also prove that $f \in \mathrm{VHVH}$. Hence, without loss of generality, we can assume that f itself satisfies the above four conditions.

Now, let us partition the set $(A \times B) \setminus f(A \times B)$ into sets E_1, E_2, E_3 such that $|E_1| = 2|B_1|$ and $|E_2| = |B_2|$. Next define the partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4 \cup \mathcal{P}_5$ of $A \times B$ into sets of size 3 as follows:

- $\{m_1, m_2, m_3\} \in \mathcal{P}_1$ if $m_1 \in f(A \times B_1)$, $m_2, m_3 \in E_1$,
- $\{m_1, m_2, m_3\} \in \mathcal{P}_2$ if $m_1, m_2 \in f(A \times \{b_2\})$ for some $b_2 \in B_2$, $m_3 \in E_2$,
- $\{m_1, m_2, m_3\} \in \mathcal{P}_3$ if $m_1, m_2, m_3 \in E_3$,
- $\{m_1, m_2, m_3\} \in \mathcal{P}_4$ if $m_i \in f(A \times B_3)$, $|f^{-1}(\{m_i\})| = 2$ for $i = 1, 2, 3$,
- $\{m_1, m_2, m_3\} \in \mathcal{P}_5$ if $m_i \in f(A \times B_3)$, $|f^{-1}(\{m_i\})| = 1$ for $i = 1, 2, 3$.

The existence of this partition follows from conditions 1–4. Note that these conditions also imply that $|\mathcal{P}_3| = |\mathcal{P}_4|$.

By Theorem 1(ii), there exists a vertical permutation $f_1 \in \mathrm{V}$ such that for each set $P \in \mathcal{P}$, the elements of P appear in different rows of $f_1[M]$ (see the argument in Example 2). Now observe that there exists a horizontal permutation $f_2 \in \mathrm{H}$ such that the columns of $f_2[f_1[M]]$ are the sets of the partition $\mathcal{P}$ and the sets from $\mathcal{P}_1 \cup \mathcal{P}_2$ correspond to columns with indices in $B_1 \cup B_2$. The function f_2 can be modified, so the elements from the sets in $\mathcal{P}_3$ are replaced by the elements from the sets in $\mathcal{P}_4$ and the latter appear twice in the matrix $f_2[f_1[M]]$. Notice that the parts of the matrices $f[M]$ and $f_2[f_1[M]]$ corresponding to $A \times B_3$ are permutations of each other, so by Theorem 1(ii) one can be obtained from the other by performing three axial permutations (note that if both sets A and B are finite, then Theorem 1(ii) with $p_1 \in \mathrm{V}$ replaced by $p_1 \in \mathrm{H}$ also holds). Additionally, the columns in $f_2[f_1[M]]$ with indices in $B_1 \cup B_2$ can be made identical to the columns of $f[M]$ by performing one vertical operation. Consequently, there exist three axial functions $f_3 \in \mathrm{H}$, $f_4 \in \mathrm{V}$, $f_5 \in \mathrm{H}$ such that $f[M] = f_5[f_4[f_3[f_2[f_1[M]]]]]$. Hence $f = f_1 \circ \cdots \circ f_5$. Since $f_2 \in \mathrm{H}$ and $f_3 \in \mathrm{H}$, we have $f_2 \circ f_3 \in \mathrm{H}$ and so $f \in \mathrm{VHVH}$.

(ii) Denote $|A|$ by k. Define

$$n = k(k-1) + 1 \quad \text{and} \quad m_0 = \left\lceil \frac{k\binom{n}{k} - n}{k-2} \right\rceil.$$

Let $m \geq m_0$ be an integer and M be a $k \times m$ matrix with all entries distinct and such that the first n entries in the first row are $a_1, \ldots, a_n$.

We define a function $f\colon A \times B \to A \times B$ ($|B| = m$) by defining $f[M]$. The entries from the "bottom" $k-2$ rows of M do not appear in $f[M]$. The first $\binom{n}{|A|} = \binom{n}{k}$ columns of $f[M]$ are formed by all k-subsets of $\{a_1, \ldots, a_n\}$. The remaining $(m - \binom{n}{k})$ columns of $f[M]$ are formed using all the entries from the first two rows of M except $a_1, \ldots, a_n$ (some of them may need to appear more than once). Note that this is possible because the number $2m - n$ of entries in the first two rows of M except $a_1, \dots, a_n$ is not greater than the number $k(m - \binom{n}{k})$ of positions in $m - \binom{n}{k}$ columns of $f[M]$. Indeed,

$$m \geq m_0 = \left\lceil \frac{k\binom{n}{k} - n}{k-2} \right\rceil \geq \frac{k\binom{n}{k} - n}{k-2}.$$

Consequently, $m(k-2) \geq k\binom{n}{k} - n$ and $k\big(m - \binom{n}{k}\big) \geq 2m - n$.

$$M = \underbrace{\begin{bmatrix} a_1 & a_3 & \cdots & & a_n & \cdots \\ \cdot & \cdot & & & & \\ \vdots & \vdots & & & \vdots & \\ \cdot & \cdot & & & & \end{bmatrix}}_{m \text{ columns}}$$

$$f[M] = \left[\;\underbrace{\begin{matrix} a_{i_1} & a_{j_1} & \cdots \\ a_{i_2} & a_{j_2} & \cdots \\ \vdots & \vdots & \vdots \\ a_{i_k} & a_{j_k} & \cdots \end{matrix}}_{\substack{\binom{n}{k} \text{ columns} \\ \text{all } k\text{-subsets of} \\ \{a_1,\dots,a_n\}}} \quad \underbrace{\begin{matrix} \qquad\qquad \cdots \qquad\qquad \\ \cdots \\ \vdots \\ \cdots \end{matrix}}_{\substack{m - \binom{n}{k} \text{ columns} \\ \text{all entries from the first two} \\ \text{rows of } M \text{ except } a_1,\dots,a_n}}\;\right]$$

We will show that $f \notin \mathrm{HVHV}$. Assume that this is not the case and there exist axial functions $f_1, \dots, f_4$ such that $f_1 \in \mathrm{H}$ and $f = f_1 \circ f_2 \circ f_3 \circ f_4$, i.e. $f[M] = f_4[f_3[f_2[f_1[M]]]]$.

Note that there are k elements out of $a_1, \dots, a_n$ such that there is a row of the matrix $f_2[f_1[M]]$ which does not contain any of these k elements. Since all the elements of M from the first two rows appear in the final matrix $f[M]$, we see that f_1 is a permutation on each of the first two rows of M. Let us denote by $b_1, \dots, b_n$ the elements from the second row of $f_1[M]$ that appear in the same columns as $a_1, \dots, a_n$ by $b_1, \dots, b_n$, respectively. Since $n = k(k-1)$ and $|A| = k$, by the Pigeonhole Principle, there exists a row in $f_2[f_1[M]]$ which contains at least k elements out of $b_1, \dots, b_n$. This row does not contain at least k elements out of $a_1, \dots, a_n$, say $a_{i_1}, \dots, a_{i_k}$.

Applying the functions $f_3 \in \mathrm{H}$ and $f_4 \in \mathrm{V}$ to the matrix $f_2[f_1[M]]]]$ will not result in a matrix containing a column whose entries are $a_{i_1}, \dots, a_{i_k}$. Hence $f[M] \neq f_4[f_3[f_2[f_1[M]]]]$, a contradiction. ∎

Let us mention here that, using a similar technique to the one used in the proof of Theorem 3(i), we can show the following.

REMARK 5. *Let* $|A| = 2$. *Then* $(A \times B)^{A\times B} = \mathrm{HVH}$. *In addition, if* $|B| \geq 3$, *then* $(A \times B)^{A\times B} \neq \mathrm{VHV}$.

A counterexample justifying the second part of the above remark can be found in [S, p. 36].

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