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ORLICZ SEQUENCE SPACES THAT ARE UNIFORMLY ROTUND IN A WEAKLY COMPACT SET OF DIRECTIONS

BY

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Abstract. Necessary and sufficient conditions are given for Orlicz sequence spaces equipped with the Orlicz norm to be uniformly rotund in a weakly compact set of directions, using only conditions on the generating function of the space.

Let X be a Banach space and let $S(X)$ and $B(X)$ be the unit sphere and unit ball of X. Then X is said to be:

- *uniformly rotund in a weakly compact set of directions* (URWC) if $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n + y_n\| \to 2$, and $x_n - y_n \xrightarrow{w} z$ (in the weak topology) imply that $z = 0$ (see [10]);
- *uniformly rotund in every direction* (URED) if $\|x_n\| \to 1$, $\|y_n\| \to 1$, $\|x_n + y_n\| \to 2$, and $x_n - y_n \to z$ (in the norm topology) imply that $z = 0$;
- *uniformly weak* rotund* (W*UR) if $\|x_n\| \to 1$, $\|y_n\| \to 1$, and $\|x_n + y_n\| \to 2$ imply that $x_n - y_n \xrightarrow{w} 0$;
- *rotund* (R) if $\|x\| = 1$, $\|y\| = 1$, and $\|x + y\| = 2$ imply that $x = y$.

Clearly,

$$\mathrm{W^*UR} \Rightarrow \mathrm{URWC} \Rightarrow \mathrm{URED} \Rightarrow \mathrm{R}.$$

Banach spaces with these types of rotundity were studied in [2, 4, 5, 10, 11] and have been applied to fixed point theory. For Orlicz spaces with the Luxemburg norm, W*UR is equivalent to R. But for Orlicz spaces with the Orlicz norm, W*UR and URED have rather different criteria [8, 15, 16]. All known characterizations of URWC for Orlicz spaces with the Orlicz norm involve both elements of the Orlicz space and the generating function M [3, 13, 17]. Up to now, no characterization of URWC by using only conditions on the generating function M has been given for Orlicz sequence spaces. As

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stated in [12], "some new methods and techniques are needed to solve this kind of difficult problem".

In this paper, we give a characterization of URWC by using only conditions on the generating function M for Orlicz sequence spaces, following the solution of this problem for Orlicz function spaces in [9]. The proof of our result is fairly complicated.

A function $M\colon \mathbb{R} \to \mathbb{R}_+$ is called an *N-function* if M is convex and even, $\lim_{u\to 0} M(u)/u = 0$, and $\lim_{u\to\infty} M(u)/u = \infty$. The *complementary function* N of M in the sense of Young is defined by

$$N(v) = \sup_{u\in\mathbb{R}}\{uv - M(u)\}.$$

It is known that if M is an N-function, then so also is N. Let p and q be the right-hand derivatives of M and N, respectively. Then M is said to be:

- *strictly convex* (SC) if
$$M\left(\frac{u+v}{2}\right) < \frac{M(u)+M(v)}{2} \qquad \text{for } u \neq v;$$
- *uniformly convex* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|u-v| \geq \varepsilon \max(|u|,|v|)$, then
$$M\left(\frac{u+v}{2}\right) < (1-\delta)\,\frac{M(u)+M(v)}{2}.$$

Moreover, M is said to satisfy the δ_2 *condition for small* u ($M \in \delta_2$) if for some $u_0 > 0$ there exists $K > 0$ such that $M(u_0) > 0$ and $M(2u) \leq KM(u)$ for all $u \leq u_0$.

For a real scalar sequence $x = \{x(i)\}$, let $\varrho_M(x) = \sum_{i=1}^{\infty} M(x(i))$, called the *modular* of x. The *Orlicz sequence space* l_M generated by M is the Banach space

$$l_M = \{x = \{x(i)\} : \varrho_M(\lambda x) < \infty \text{ for some } \lambda\},$$

equipped with the *Orlicz norm*

$$\|x\|_M = \sup_{\varrho_N(y)\leq 1} \sum_{i=1}^{\infty} x(i)y(i) = \inf_{k>0} \frac{1}{k}(1 + \varrho_M(kx)).$$

See [2, 7] for references on Orlicz spaces.

We first state several lemmas.

Lemma 1 ([14]). *For $x \in l_M$, if $\varrho_M(p(kx)) = 1$, then*

$$\|x\|_M = \sum_{i=1}^{\infty} |x(i)|p(k|x(i)|) = \frac{1}{k}\,(1 + \varrho_M(kx)).$$

LEMMA 2 ([14]). *For $x \in l_M$, there exists $k > 0$ such that*

$$\|x\|_M = \frac{1}{k}(1 + \varrho_M(kx)).$$

LEMMA 3 ([14]). *Let*

$$\|x_n\|_M = \frac{1}{k_n}(1 + \varrho_M(k_n x_n)) \leq 2 \qquad (n = 1, 2, \ldots).$$

If $k_n \to \infty$, then $x_n(i) \to 0$ for all i.

LEMMA 4 ([14]). *In a rotund Orlicz sequence space l_M, let*

$$\|x_n\|_M = \frac{1}{k_n}(1 + \varrho_M(k_n x_n)) \to 1, \qquad \|y_n\|_M = \frac{1}{h_n}(1 + \varrho_M(h_n y_n)) \to 1$$

as $n \to \infty$, where $\{k_n\}$ and $\{h_n\}$ are bounded. If $\|x_n + y_n\|_M \to 2$, then $k_n x_n(i) - h_n y_n(i) \to 0$ for all i.

LEMMA 5 ([14]). *Let $\{x_n\}$, $\{y_n\}$, $\{k_n\}$ and $\{h_n\}$ be as in Lemma 4. If $v_n \in l_N$, $\varrho_N(v_n) \leq 1$ and $\sum_{i=1}^{\infty}(x_n(i) + y_n(i))v_n(i) \to 2$, then for all subsets I_n of positive integers, we have uniformly*

$$\lim_{n\to\infty} \sum_{i \in I_n} (k_n x_n(i) - h_n y_n(i)) v_n(i) = \lim_{n\to\infty} \sum_{i \in I_n} (M(k_n x_n(i)) - M(h_n y_n(i))).$$

LEMMA 6 ([6, 9]). *For $0 < \lambda < 1$, the function*

$$\phi(t) = \frac{M(\lambda u + (1-\lambda)t)}{\lambda M(u) + (1-\lambda)M(t)}$$

is increasing in $[0, u]$.

LEMMA 7 ([9]). *For given $\varepsilon \in (0,1)$ and $[\alpha, \beta] \subset (0,1)$,*

$$\lim_{X \to 1} Y = 1$$

where

$$X = X_{1/2}(u) = \frac{M\left(\frac{u + (1-\varepsilon)u}{2}\right)}{\frac{M(u) + M((1-\varepsilon)u)}{2}},$$

$$Y = Y_{\alpha,\beta}(u) = \inf_{\lambda \in [\alpha,\beta]} \frac{M(\lambda u + (1-\lambda)(1-\varepsilon)u)}{\lambda M(u) + (1-\lambda)M((1-\varepsilon)u)}.$$

LEMMA 8 ([9]). *For $u > 0$, if*

$$\frac{\frac{M(u) + M((1-\varepsilon)u)}{2}}{M\left(\frac{u + (1-\varepsilon)u}{2}\right)} \leq 1 + \eta,$$

then there exists $(1 - \varepsilon/2)u \leq t \leq u$ with

$$p\left(\left(1 - \frac{\varepsilon}{2}\right)t\right) \geq \left(1 - 2\eta\,\frac{2-\varepsilon}{\varepsilon}\right)p(t).$$

Lemma 9 ([14]). *Let l_M be an Orlicz sequence space equipped with the Orlicz norm. Then l_M is URED if and only if*

(i) $M \in \mathrm{SC}[0, \pi_M]$, *where* $\pi_M = \inf\{t : N(p(t)) \geq 1\}$;

(ii) *for* $[\alpha, \beta] \subset (0,1)$ *and for* $\varepsilon, \varepsilon' \in (0,1)$, *there exist* $u_0 > 0$, $D = D(\varepsilon, \varepsilon') > 0$, *and* $\gamma = \gamma(\varepsilon, \varepsilon') > 0$ *such that for all* $\lambda \in [\alpha, \beta]$ *and all* u, v *with* $\max\{|u|, |v|\} \leq u_0$, $|u - v| \geq \varepsilon \max\{|u|, |v|\}$ *and* $\lambda M(u) + (1-\lambda)M(v) \leq (1+\gamma)M(\lambda u + (1-\lambda)v)$ *we have*

$$M(u) \leq D(\varepsilon, \varepsilon') \frac{M(\varepsilon' u)}{\varepsilon'}.$$

Moreover, $\lambda M(u) + (1-\lambda)M((1-\varepsilon)u) \leq (1+\gamma)M(\lambda u + (1-\lambda)(1-\varepsilon)u)$, *so*

$$M(u) \leq D(\varepsilon, \varepsilon') \, \frac{M(\varepsilon' u)}{\varepsilon'}.$$

Proof. First, we show that URED implies (ii). In fact, if we suppose (ii) is not true then for some $\varepsilon > 0$ (assume $\varepsilon = \min\{\varepsilon, \varepsilon'\}$) there exist $u_n \searrow 0$ and $\lambda_n \in [\alpha, \beta]$ such that

$$\lambda_n M(u_n) + (1-\lambda_n)M((1-\varepsilon)u_n) \leq \left(1 + \frac{1}{n}\right) M(\lambda_n u_n + (1-\lambda_n)(1-\varepsilon)u_n),$$

and

$$M(u_n) \geq 2^n n \, \frac{M(\varepsilon u_n)}{\varepsilon}.$$

By Lemma 8 there exists $(1-\varepsilon/2)u_n \leq t_n \leq u_n$ with

$$p\left(\left(1 - \frac{\varepsilon}{2}\right) t_n\right) \geq \left(1 - 2\eta \, \frac{2-\varepsilon}{\varepsilon}\right) p(t_n),$$

and

$$\frac{M(\varepsilon t_n)}{\varepsilon M(t_n)} \to 0.$$

Indeed, referring to the argument of §1 of Chapter 1 in [7], we get

$$M(\varepsilon t_n) - \frac{\varepsilon t_n}{u_n} M(u_n) \leq \frac{\varepsilon t_n}{\varepsilon u_n}(M(\varepsilon u_n) - \varepsilon M(u_n)),$$

$$\frac{u_n}{t_n} \frac{M(t_n)}{M(u_n)} \frac{M(\varepsilon t_n)}{\varepsilon M(t_n)} - 1 \leq \frac{M(\varepsilon u_n)}{\varepsilon M(u_n)} - 1 \to -1,$$

and so

$$\frac{u_n}{t_n} \, \frac{M(t_n)}{M(u_n)} \, \frac{M(\varepsilon t_n)}{\varepsilon M(t_n)} \to 0.$$

Noticing that $1 \leq |u_n/t_n| \leq 1/(1-\varepsilon/2)$ and

$$2 \geq \frac{M(t_n)}{\frac{M(u_n)}{2}} \geq \frac{M((1-\varepsilon/2)u_n)}{\frac{M(u_n)}{2}} \geq \frac{M((1-\varepsilon/2)u_n)}{\frac{M(u_n)+M((1-\varepsilon)u_n)}{2}} \to 1,$$

we have $M(\varepsilon t_n)/\varepsilon M(t_n) \to 0$.

Without loss of generality, passing to a subsequence if necessary, assume that

$$M(t_n) > 2^n n \frac{M(\varepsilon t_n)}{\varepsilon}.$$

Then

$$t_n p(t_n) \geq M(t_n) > 2^n n \frac{M(\varepsilon t_n)}{\varepsilon} \geq 2^n n p\left(\frac{\varepsilon t_n}{2}\right)\frac{\varepsilon t_n}{2\varepsilon}$$

i.e.,

$$p(t_n) \geq 2^{n-1} n p(\varepsilon t_n/2).$$

From the proof of the necessity in the Theorem of [14], we can see that l_M is not URED, a contradiction.

On the other hand, by the proof of the sufficiency in the Theorem of [14], (i) and (ii) imply URED. ■

From Lemma 7, we deduce the following

REMARK 1. l_M is URED if and only if

(i) $M \in \mathrm{SC}[0, \pi_M]$;
(ii) for $0 < \varepsilon, \varepsilon' < 1$ there exist $D(\varepsilon, \varepsilon')$ and $u_0 > 0$ and $\gamma = \gamma(\varepsilon') > 0$ so that for all $|u| \leq u_0$ with $M(u) + M((1-\varepsilon)u) \leq (1+\gamma)2M((1-\varepsilon/2)u)$, we have

$$M(u) \leq D(\varepsilon, \varepsilon') \frac{M(\varepsilon' u)}{\varepsilon'}.$$

For convenience, we will understand $D(\varepsilon, \varepsilon')$ to be the smallest one as in the above inequality.

LEMMA 10. *Let l_M be an Orlicz sequence space equipped with the Orlicz norm. If l_M is URWC, then for any $0 < \varepsilon < 1$ there exists $D(\varepsilon) > 0$ such that*

$$D(\varepsilon, \varepsilon') \leq D(\varepsilon)$$

for all $0 < \varepsilon' < 1$, where $D(\varepsilon, \varepsilon')$ is defined as in (ii) *of Remark* 1.

Proof. Define $D(\varepsilon) = \sup_{\varepsilon' \in (0,1)} D(\varepsilon, \varepsilon')$. It is clear that $D(\varepsilon, \varepsilon')$ is decreasing with respect to ε'. Suppose that $D(\varepsilon) = \infty$. Then there exist $\varepsilon_n \searrow 0$ with $D(\varepsilon, \varepsilon_1) < D(\varepsilon, \varepsilon_2) < \cdots < D(\varepsilon, \varepsilon_n) \nearrow \infty$. Let

$$\Re_n = \left\{ u > 0 : u \leq \frac{1}{2^{n+1}}, \frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} < 1 + \frac{1}{n}, \right.$$
$$\left. M(u) > D(\varepsilon, \varepsilon_{n-1}) \frac{M(\varepsilon_n u)}{\varepsilon_n} \right\}$$

where $D(\varepsilon, \varepsilon_0) = 0$. We now show that for all m,

$$\Re_1 \cap \cdots \cap \Re_m \neq \emptyset.$$

In fact, suppose first that $\Re_1 \cap \Re_2 = \emptyset$, i.e., $\Re_1^c \cup \Re_2^c = \mathbb{R}$. Then for all $u \le 1/2^3$ with

$$\frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} < 1 + \frac{1}{2},$$

we get

$$M(u) \le D(\varepsilon, \varepsilon_0)\,\frac{M(\varepsilon_1 u)}{\varepsilon_1} = 0,$$

or

$$M(u) \le D(\varepsilon, \varepsilon_1)\,\frac{M(\varepsilon_2 u)}{\varepsilon_2}.$$

Thus for all such u, we get

$$M(u) \le D(\varepsilon, \varepsilon_1)\frac{M(\varepsilon_2 u)}{\varepsilon_2},$$

so $D(\varepsilon, \varepsilon_2) \le D(\varepsilon, \varepsilon_1)$, contrary to $D(\varepsilon, \varepsilon_1) < D(\varepsilon, \varepsilon_2)$. Proceeding inductively, assume that $\Re_1 \cap \cdots \cap \Re_n \neq \emptyset$, and suppose that $\Re_1 \cap \cdots \cap \Re_{n+1} = \emptyset$, i.e., $\Re_1^c \cup \cdots \cup \Re_{n+1}^c = \Re$. Then for all $u \le 1/2^{n+1}$ with

$$\frac{\frac{M(u)+M((1-\varepsilon)u)}{2}}{M\left(\frac{u+(1-\varepsilon)u}{2}\right)} < 1 + \frac{1}{n}$$

at least one of the following conditions holds:

$$\begin{aligned} M(u) &\le D(\varepsilon, \varepsilon_1)\frac{M(\varepsilon_2 u)}{\varepsilon_2}, \\ M(u) &\le D(\varepsilon, \varepsilon_2)\frac{M(\varepsilon_3 u)}{\varepsilon_3}, \quad \ldots, \\ M(u) &\le D(\varepsilon, \varepsilon_n)\frac{M(\varepsilon_{n+1} u)}{\varepsilon_{n+1}}, \end{aligned}$$

so we have at least one of the following contradictions:

$$\begin{aligned} D(\varepsilon, \varepsilon_2) < D(\varepsilon, \varepsilon_1) &\le D(\varepsilon, \varepsilon_2), \\ D(\varepsilon, \varepsilon_3) < D(\varepsilon, \varepsilon_2) &\le D(\varepsilon, \varepsilon_3), \quad \ldots, \\ D(\varepsilon, \varepsilon_{n+1}) < D(\varepsilon, \varepsilon_n) &\le D(\varepsilon, \varepsilon_{(n+1)}). \end{aligned}$$

Hence $\Re_1 \cap \cdots \cap \Re_m \neq \emptyset$ for all $m \in \mathbb{N}$ by the induction principle. Take $u_n \in \Re_1 \cap \cdots \cap \Re_n$. Then $u_n \le 1/2^{n+1}$,

$$\frac{\frac{M(u_n)+M((1-\varepsilon)u_n)}{2}}{M\left(\frac{u_n+(1-\varepsilon)u_n}{2}\right)} < 1 + \frac{1}{n},$$

and

$$M(u_n) > D(\varepsilon, \varepsilon_{k-1})\frac{M(\varepsilon_k u_n)}{\varepsilon_k}, \quad 0 \le k \le n. \tag{1}$$

By Lemma 8, there exists $(1-\varepsilon/2)u_n \le t_n \le u_n$ with

$$p\left(\left(1-\frac{\varepsilon}{2}\right)t_n\right) \ge \left(1-\frac{2}{n}\,\frac{2-\varepsilon}{\varepsilon}\right)p(t_n). \tag{2}$$

Without loss of generality, assume that $p(u_1) \le 1$. Take $c > 0$ with $1/2 \ge N(p(c)) > 0$ and take a positive integer m_n so that

$$\frac{1}{2}-\frac{1}{2^n} \le m_n\left(1-\frac{\varepsilon}{2}\right)t_n p\left(\left(1-\frac{\varepsilon}{2}\right)t_n\right) \le \frac{1}{2}.$$

Define

$$\begin{aligned}
d_n &= \inf\{s>0 : N(p(s)) + N(p(c)) + m_n N(p(t_n)) \ge 1\},\\
d_n^\star &= \inf\{s>0 : N(p(s)) + N(p(c)) + m_n N(p((1-\varepsilon/2)t_n)) \ge 1\}.
\end{aligned}$$

Then $d_n \le d_n^\star \le b$ where $N(p(b)) \ge 1$, and by (i), we have

$$\begin{aligned}
&N\left(p\left(\frac{n}{1+n}d_n^\star\right)\right) - N(p(d_n))\\
&\quad\le [1-N(p(c)) - m_n N(p((1-\varepsilon/2)t_n))] - [1-N(p(c)) - m_n N(p(t_n))]\\
&\quad= m_n[N(p(t_n)) - N(p((1-\varepsilon/2)t_n))]\\
&\quad= m_n \int_{p((1-\varepsilon/2)t_n)}^{p(t_n)} q(s)\,ds \le m_n[p(t_n) - p((1-\varepsilon/2)t_n)]q(p(t_n))\\
&\quad\le m_n \frac{2}{n}\,\frac{2-\varepsilon}{\varepsilon}\,p(t_n)t_n\\
&\quad\le \frac{2}{n}\,\frac{2-\varepsilon}{\varepsilon}\,\frac{2}{2-\varepsilon}(1-\varepsilon/2)t_n\,\frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}\,p((1-\varepsilon/2)t_n)m_n\\
&\quad\le \frac{2}{n}\,\frac{2}{\varepsilon}\,\frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}\,\frac{1}{2} \to 0.
\end{aligned}$$

Also by (i), we have $\frac{n}{1+n}d_n^\star - d_n \to 0$, so $d_n^\star - d_n \to 0$. Take $p_-(d_n^\star) \le \theta_n \le p(d_n^\star)$ such that

$$N(\theta_n) + N(p(c)) + m_n N(p((1-\varepsilon/2)t_n)) = 1$$

where $p_-(s) = \lim_{t\to 0^-} p(s+t)$. Next define

$$\begin{aligned}
k_n &= 1 + M(d_n) + M(c) + m_n M(t_n),\\
h_n &= 1 + M(d_n^\star)) + M(c) + m_n M((1-\varepsilon/2)t_n),\\
x_n &= \frac{1}{k_n}\,(d_n, c, \dots, \overbrace{t_n, \dots, t_n}^{m_n}, 0, \dots),\\
y_n &= \frac{1}{h_n}\,(d_n^\star, c, \dots, \overbrace{(1-\varepsilon/2)t_n, \dots, (1-\varepsilon/2)t_n}^{m_n}, 0, \dots),
\end{aligned}$$

$$v_n = (\theta_n, p(c), \overbrace{p((1-\varepsilon/2)t_n), \dots, p((1-\varepsilon/2)t_n)}^{m_n}, 0, \dots).$$

Then

$$\begin{aligned}\varrho_n(v_n) &= 1,\\ h_n \le k_n &\le 1 + M(c) + M(b) + t_n p(t_n m_n)\\ &\le 1 + M(c) + M(b) + \frac{2}{2-\varepsilon}\,\frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}\,\frac{1}{2},\\ k_n - h_n &\ge \frac{\varepsilon}{2}\,t_n p((1-\varepsilon/2)t_n)m_n \ge \frac{\varepsilon}{8}.\end{aligned}$$

On the other hand, by the Hölder inequality,

$$\|y_n\|_M = \langle v_n, y_n\rangle = 1.$$

From

$$\|x_n\|_M \le \frac{1}{k_n}\,(1 + \varrho_M(k_n x_n)) = 1,$$

and by the uniform continuity of M on bounded closed intervals, we get

$$\begin{aligned}\langle v_n, k_n x_n\rangle &= d_n[\theta_n] + cp(c) + m_n t_n p((1-\varepsilon/2)t_n)\\ &\le (d_n^\star - d_n)p(b) + d_n^\star\theta_n + cp(c) + \frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}\,m_n t_n p(t_n)\\ &\le (d_n^\star - d_n)p(b) + \frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}\,[1 + M(d_n^\star) + M(c) + m_n M(t_n)]\\ &\le (d_n^\star - d_n)p(b) + \frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}\,[M(d_n^\star) - M(d_n)]\\ &\quad + \frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}\,[1 + M(d_n) + M(c) + m_n M(t_n)]\\ &\le (d_n^\star - d_n)p(b) + \frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}\,[M(d_n^\star) - M(d_n)] + \frac{1}{1-\frac{2}{n}\frac{2-\varepsilon}{\varepsilon}}k_n\\ &\to k_n,\end{aligned}$$

hence $\langle v_n, x_n\rangle \to 1$, and consequently $\|x_n + y_n\|_M \to 2$. Noticing that

$$M(u_n)m_n \le D(\varepsilon,\varepsilon)\,\frac{M(\varepsilon u_n)}{\varepsilon}\,m_n \le D(\varepsilon,\varepsilon)\,\frac{M((1-\varepsilon/2)t_n)}{\varepsilon}\,m_n \le \frac{D(\varepsilon,\varepsilon)}{2\varepsilon},$$

without loss of generality, we assume $\varepsilon > \varepsilon_1$. So for arbitrary $\tau > 0$, take I with $1/D(\varepsilon,\varepsilon_I) < \tau\cdot 2\varepsilon/D(\varepsilon,\varepsilon)$, and take k_0 such that for all $k \ge k_0$ with $\sup_{1\le i\le I} M(\varepsilon_k u_i)m_i/\varepsilon_k < \tau$, we have

$$\begin{aligned}&\sup_{i\ge 1}\frac{M(\varepsilon_k u_i)m_i}{\varepsilon_k}\\ &\qquad = \max\left\{\sup_{1\le i\le I}\frac{M(\varepsilon_k u_i)m_i}{\varepsilon_k}, \sup_{I<i<k}\frac{M(\varepsilon_k u_i)m_i}{\varepsilon_k}, \sup_{i\ge k}\frac{M(\varepsilon_k u_i)m_i}{\varepsilon_k}\right\}\end{aligned}$$

$$\le \max\left\{\sup_{1\le i\le I}\frac{M(\varepsilon_k u_i)m_i}{\varepsilon_k},\ \sup_{I<i<k}\frac{M(\varepsilon_i u_i)m_i}{\varepsilon_i},\sup_{i\ge k}\frac{M(u_i)m_i}{D(\varepsilon,\varepsilon_{k-1})}\right\}$$

$$\le \max\left\{\sup_{1\le i\le I}\frac{M(\varepsilon_k u_i)m_i}{\varepsilon_k},\ \sup_{I<i<k}\frac{M(u_i)m_i}{D(\varepsilon,\varepsilon_{i-1})},\sup_{i\ge k}\frac{M(u_i)m_i}{D(\varepsilon,\varepsilon_{k-1})}\right\} < \tau.$$

By [1], $\{\widetilde{u}_n\}_{n=1}^{\infty}$ is relatively weakly compact, where $\widetilde{u}_n = (0, 0, \overbrace{u_n, \ldots, u_n}^{m_n}, 0, \ldots)$. But, for $\chi_{e_2} = (0, 1, 0, \ldots)$,

$$\langle \chi_{e_2}, x_n - y_n\rangle = \left(\frac{1}{k_n} - \frac{1}{h_n}\right)c \not\to 0 \quad (n\to\infty),$$

a contradiction with URWC of l_M. ■

Theorem 1. *Let l_M be an Orlicz sequence space equipped with the Orlicz norm. Then l_M is URWC if and only if*

(i) $M\in \mathrm{SC}[0,\pi_M]$;

(ii) *for $[\alpha,\beta]\subset(0,1)$ and $0<\varepsilon<1$ there exist $D=D(\varepsilon)$ and $u_0>0$ such that for all $0<\varepsilon'<1$, we can find $\gamma=\gamma(\varepsilon')>0$ so that for all $\lambda\in[\alpha,\beta]$ and all u,v satisfying* $\max\{|u|,|v|\}\le u_0$, $|u-v|\ge \varepsilon\max\{|u|,|v|\}$, $\lambda M(u)+(1-\lambda)M(v)\le(1+\gamma)M(\lambda u+(1-\lambda)v)$, *we have*

$$M(u)\le D\,\frac{M(\varepsilon' u)}{\varepsilon'}.$$

Proof. Necessity. Since URWC implies rotundity, we get (i) (see [2]). By Lemma 10, (ii) follows.

Sufficiency. If we suppose that l_M is not URWC, there exist sequences $\{x_n\}$ and $\{y_n\}$ satisfying $\|x_n\|_M=\frac{1}{k_n}(1+\varrho_M(k_nx_n))\to 1$, $\|y_n\|_M=\frac{1}{h_n}(1+\varrho_M(h_ny_n))\to 1$ $(n\to\infty)$, $\|x_n+y_n\|_M\to 2$ but $x_n-y_n\doteq z_n\xrightarrow{x} z\ne 0$. If for all i, $x_n(i)\to 0$ and $y_n(i)\to 0$, set $x_n'=x_n-z_n/4$, $y_n'=x_n-3z_n/4$. It is easy to see that $\|x_n'\|_M\to 1$, $\|y_n'\|_M\to 1$, $\|x_n'+y_n'\|_M\to 2$ and $x_n'(i)-y_n'(i)\doteq z_n'(i)=(z_n/2)(i)\to(z/2)(i)\ne 0$. Thus $z_n'=z_n/2\xrightarrow{w} z/2\ne 0$ $(n\to\infty)$. Clearly $x_n'(i)\overset{w}{\not\to} 0$. So we assume that $x_n\overset{w}{\not\to} 0$ and $y_n\overset{w}{\not\to} 0$ if necessary replacing $\{x_n\}$ and $\{y_n\}$ by $\{x_n'\}$ and $\{y_n'\}$. By Lemma 3, $\{k_n\}$ and $\{h_n\}$ are bounded, and we may assume that $k_n\to k$, $h_n\to h$, passing to a subsequence if necessary. By Lemma 4, this implies that for all i, $k_nx_n(i)-h_ny_n(i)\to 0$, i.e. $(k_n-h_n)x_n(i)-h_nz_n(i)\to 0$. If $k=h$, it follows that $z_n(i)\to 0$, so $z_n\xrightarrow{w^*} 0$, contradicting $z\ne 0$. Hence $k\ne h$; we assume that $k>h$ and $k_n>h_n$, passing to a subsequence if necessary. We can do the same in the case of $k<h$. Define $\lambda_n=h_n/(k_n+h_n)\le 1/2$. Since $\{k_n\}$ and $\{h_n\}$ are bounded we deduce that $\lambda_n\in[\alpha,\beta]$ for some $[\alpha,\beta]\subset(0,1)$.

Since $z\ne 0$, take a subset $I_0=\{1,2,\ldots,I\}$ such that

$$z|_{I_0}\ne 0. \tag{3}$$

Take an arbitrary $\varepsilon > 0$. Since $\{z_n\}$ is weakly compact, by [1], $\{z_n\}$ is l_N-weakly compact. Take $0 < \varepsilon' < 1$ such that

$$\frac{\varrho_M(\varepsilon' 2kz_n)}{\varepsilon'} < \frac{\varepsilon^2}{4D}. \tag{4}$$

By (ii), there is $\gamma > 0$ such that for all $\lambda \in [\alpha, \beta]$ and all u, v satisfying $\max(|u|, |v|) \geq u_0$, $|u - v| \geq \varepsilon \max(|u|, |v|)$ with $\lambda M(u) + (1-\lambda)M(v) \leq (1+\gamma)M(\lambda u + (1-\lambda)v)$ we have

$$M(u) \leq D\, \frac{M(\varepsilon'\varepsilon u)}{\varepsilon'\varepsilon}. \tag{5}$$

By (3),

$$\varrho_M\left(\frac{hz}{k-h}\bigg|_{I_0}\right) > 0. \tag{6}$$

For each n, we split the set $\mathbb{N}$ into the following parts:

$$\begin{aligned}
A_n &= \{i \in \mathbb{N} \setminus I_0 : \max(|k_n x_n(i)|, |h_n y_n(i)|) < \varepsilon\},\\
B_n &= \{i \in \mathbb{N} \setminus I_0 \setminus A_n : |k_n x_n(i) - h_n y(i)| < \varepsilon \max(|k_n x_n(i)|, |h_n y_n(i)|)\},\\
H_n &= \bigg\{i \in \mathbb{N} \setminus I_0 \setminus A_n \setminus B_n : (1+\gamma) M\bigg(\frac{k_n h_n}{k_n + h_n}\,(x_n(i) + y_n(i))\bigg)\\
&\qquad\qquad < \frac{h_n}{k_n + h_n}\, M(k_n x_n(i)) + \frac{k_n}{k_n + h_n}\, M(h_n x_n(i))\bigg\},\\
I_n &= \{i \in \mathbb{N} \setminus I_0 \setminus A_n \setminus B_n \setminus H_n : |x_n(i)| < |y_n(i)|\},\\
Q_n &= \{i \in \mathbb{N} \setminus I_0 \setminus A_n \setminus B_n \setminus H_n \setminus I_n : |z_n(i)| < \varepsilon |x_n(i)|\},\\
T_n &= \mathbb{N} \setminus I_0 \setminus A_n \setminus B_n \setminus H_n \setminus I_n \setminus Q_n\\
&= \bigg\{i \in \mathbb{N} \setminus I_0 : \max(|k_n x_n(i)|, |h_n y_n(i)|) \geq \varepsilon,\\
&\qquad |k_n x_n(i) - h_n y(i)| \geq \varepsilon \max(|k_n x_n(i)|, |h_n y_n(i)|),\\
&\qquad (1+\gamma) M\bigg(\frac{k_n h_n}{k_n + h_n}\,(x_n(i) + y_n(i))\bigg)\\
&\qquad \geq \frac{h_n}{k_n + h_n}\, M(k_n x_n(i)) + \frac{k_n}{k_n + h_n}\, M(h_n x_n(i))\\
&\qquad |z_n(i)| \geq \varepsilon |x_n(i)| \text{ and } |x_n(i)| \geq |y_n(i)|\bigg\}.
\end{aligned}$$

Pick $v_n \in B(l_N)$ such that $[x_n(i) + y_n(i)]v_n(i) \geq 0$ and

$$\langle v_n, x_n + y_n \rangle \to 2.$$

Then

$$\langle v_n, x_n \rangle \to 1, \qquad \langle v_n, y_n \rangle \to 1.$$

Thus

$$k - h = \lim_n (k_n - h_n) = \lim_n \sum_{i=1}^{\infty} [k_n x_n(i) - h_n y_n(i)] v_n(i).$$

In the following, we estimate the sums over the above subsets.

(a) Since $k_n x_n(i) - h_n y_n(i) \to 0$ uniformly on I_0, for n large enough, we get

$$\sum_{i \in I_0} |(k_n x_n(i) - h_n y_n(i)) v_n(i)| < \varepsilon.$$

(b) Clearly, by Hölder's inequality,

$$\sum_{i \in A_n} |(k_n x_n(i) - h_n y_n(i)) v_n(i)| < 2\varepsilon.$$

(c) We also have

$$\begin{aligned} \sum_{i \in B_n} |(k_n x_n(i) - h_n y_n(i)) v_n(i)| &\le \varepsilon \sum_{i \in B_n} (|k_n x_n(i)| + |h_n y_n(i)|) |v_n(i)| \\ &\le \varepsilon (k_n + h_n). \end{aligned}$$

(d) Noticing that

$$\begin{aligned} \frac{\gamma}{1+\gamma} \left[\frac{h_n}{k_n + h_n} \varrho_M(k_n x_n|_{H_n}) + \frac{k_n}{k_n + h_n} \varrho_M(h_n y_n|_{H_n}) \right] \\ \le 2 - \|x_n + y_n\|_M \to 0, \end{aligned}$$

by Lemma 5, we see that for n large enough,

$$\sum_{i \in H_n} |(k_n x_n(i) - h_n y_n(i)) v_n(i)| < \varepsilon.$$

(e) Let $i \in I_n$, that is, $|x_n(i)| < |y_n(i)|$.

If $x_n(i) y_n(i) \ge 0$, as $[x_n(i) + y_n(i)] v_n(i) \ge 0$, we have $x_n(i) v_n(i) \ge 0$ and $y_n(i) v_n(i) \ge 0$, so $x_n(i) z_n(i) = x_n(i)[x_n(i) - y_n(i)] < 0$, thus $z_n(i) v_n(i) \le 0$. Hence

$$\begin{aligned} [k_n x_n(i) - h_n y_n(i)] v_n(i) &= (k_n - h_n) x_n(i) v_n(i) + h_n [x_n(i) - y_n(i)] v_n(i) \\ &= (k_n - h_n) x_n(i) v_n(i) + h_n z_n(i) v_n(i) \\ &\le (k_n - h_n) x_n(i) v_n(i). \end{aligned}$$

If $x_n(i) y_n(i) < 0$, from $|x_n(i)| < |y_n(i)|$, we have $y_n(i) v_n(i) \ge 0$ and $x_n(i) v_n(i) \le 0$. Since $z_n(i) = x_n(i) - y_n(i)$, we have $z_n(i) v_n(i) \le 0$. Hence

$$\begin{aligned} [k_n x_n(i) - h_n y_n(i)] v_n(i) &= (k_n - h_n) x_n(i) v_n(i) + h_n [x_n(i) - y_n(i)] v_n(i) \\ &= (k_n - h_n) x_n(i) v_n(i) + h_n z_n(i) v_n(i) \\ &\le (k_n - h_n) x_n(i) v_n(i). \end{aligned}$$

Therefore

$$\sum_{i\in I_n}[k_nx_n(i)-h_ny_n(i)]v_n(i)\le\sum_{i\in I_n}(k_n-h_n)x_n(i)v_n(i).$$

(f) Let $i\in Q_n$, that is, $|z_n(i)|\le\varepsilon|x_n(i)|$. From $|y_n(i)|\le|x_n(i)|$ and $[x_n(i)+y_n(i)]v_n(i)\ge 0$, it follows that $x_n(i)v_n(i)\ge 0$, $z_n(i)v_n(i)\ge 0$, and

$$\begin{aligned}[k_nx_n(i)-h_ny_n(i)]v_n(i)&=(k_n-h_n)x_n(i)v_n(i)+h_nz_n(i)v_n(i)\\&\le(k_n-h_n)x_n(i)v_n(i)+\varepsilon h_nx_n(i)v_n(i).\end{aligned}$$

Thus

$$\sum_{i\in Q_n}[k_nx_n(i)-h_ny_n(i)]v_n(i)\le(k_n-h_n+\varepsilon h_n)\sum_{i\in Q_n}x_n(i)v_n(i).$$

(g) Let $i\in T_n$, that is, $\max\{|k_nx_n(i)|,|h_ny_n(i)|\}\ge\varepsilon$, $|k_nx_n(i)-h_ny_n(i)|\ge\varepsilon\max\{k_n|x_n(i)|,|h_ny_n(i)|\}$,

$$\frac{\lambda_nM(k_nx_n(i))+(1-\lambda_n)M(h_ny_n(i))}{M(\lambda_nk_nx_n(i)+(1-\lambda_n)h_ny_n(i))}\le 1+\gamma.$$

Since $\varepsilon|x_n(i)|\le|z_n(i)|$, from (5), it follows that for $i\in T_n$,

$$M(k_nx_n(i))\le D\,\frac{M(\varepsilon'\varepsilon k_nx_n(i))}{\varepsilon'\varepsilon}\le D\,\frac{M(\varepsilon'2kz_n(i))}{\varepsilon'\varepsilon}.$$

Hence, by (4),

$$\varrho_M(k_nx_n|_{T_n})\le D\,\frac{\varrho_M(\varepsilon'2kz_n|_{T_n})}{\varepsilon'\varepsilon}\le\frac{D\varepsilon^2}{D\varepsilon}=\varepsilon.$$

Since $|x_n(i)|>|y_n(i)|$ and $k_n>h_n$, we know that $|k_nx_n(i)|>|h_ny_n(i)|$, so $\varrho_M(h_ny_n|_{T_n})\le\varepsilon$.

Noticing that

$$\frac{1}{k_n}\,\varrho_M(k_nx_n|_{I_0})\ge\varrho_M(x_n|_{I_0})\to\varrho_M\left(\frac{h}{k-h}z|_{I_0}\right)$$

and

$$\begin{aligned}1\leftarrow\|x_n\|_M&=\frac{1}{k_n}\,[1+\varrho_M(k_nx_n|_{I_0})+\varrho_M(k_nx_n|_{\mathbb{N}\setminus I_0})]\\&\ge\|x_n|_{\mathbb{N}\setminus I_0}\|_M+\frac{1}{k_n}\,\varrho_M(k_nx_n|_{I_0})\\&\ge\sum_{i\in\mathbb{N}\setminus I_0}|x_n(i)v_n(i)|+\varrho_M\left(\frac{h}{k-h}z|_{I_0}\right),\end{aligned}$$

we find that for n large enough,

$$\sum_{i\in\mathbb{N}\setminus I_0}|x_n(i)v_n(i)|\le 1-\varrho_M\left(\frac{h}{k-h}z|_{I_0}\right).$$

Moreover,

$$\sum_{i\in I_n\cup Q_n}|x_n(i)v_n(i)|\le\sum_{i\in\mathbb{N}\setminus I_0}|x_n(i)v_n(i)|\le 1-\varrho_M\Big(\frac{h}{k-h}z|_{I_0}\Big).$$

Since $\varepsilon>0$ is arbitrary, (a)–(g) lead to a contradiction:

$$k-h\le(k-h)\Big[1-\varrho_M\Big(\frac{h}{k-h}z|_{I_0}\Big)\Big]<k-h. \ \blacksquare$$

By Lemma 7, we have the following

REMARK 2. l_M is URWC if and only if

(i) $M\in\mathrm{SC}[0,\pi_M]$;
(ii) for $0<\varepsilon<1$ there exist $D=D(\varepsilon)$ and $u_0>0$ such that for all ε', $0<\varepsilon'<1$, we can find $\gamma=\gamma(\varepsilon')>0$ so that for all $|u|\le u_0$ with $M(u)+M((1-\varepsilon)u)\le(1+\gamma)2M((1-\varepsilon/2)u)$, we have

$$M(u)\le D\,\frac{M(\varepsilon' u)}{\varepsilon'}.$$

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